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Solutions of a System of Two Higher-Order Difference Equations in Terms of Lucas Sequence

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Abstract

In this paper we give some theoretical explanations related to the representation for the general solution of the system of the higher-order rational difference equations

$$x_{n+1}=\frac{5y_{n-k}-5}{y_{n-k}}, \qquad y_{n+1}=\frac{5x_{n-k}-5}{x_{n-k}}, \qquad n,k\in\mathbb{N}_0,$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and the initial conditions $x_{-k}, x_{-k+1},\ldots, x_0, y_{-k}, y_{-k+1},\ldots, y_0$ are non zero real numbers such that their solutions are associated to Lucas numbers.
We also study the stability character and asymptotic behavior of this system.

1. Introduction

Giving theoretical explanations related to the exact solutions of most systems of the higher-order rational difference equations is sophisticated sometimes. Therefore, some of the recent papers give formulas for solutions to systems of difference equations and prove them by using only the method of induction.
The prime purpose of this work is to give some theoretical explanations related to the general solution of the system of the higher-order rational difference equations

$$x_{n+1}=\frac{5y_{n-k}-5}{y_{n-k}}, \qquad y_{n+1}=\frac{5x_{n-k}-5}{x_{n-k}}, \qquad n,k\in\mathbb{N}_0, \tag{1.1}$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and the initial conditions $x_{-k}, x_{-k+1},\ldots, x_0, y_{-k}, y_{-k+1},\ldots, y_0$ are non zero real numbers. The solutions of (1.1) are expressed using the famous Fibonacci and Lucas numbers.
The idea is establish the solution form of system (1.1) using appropriate transformation reducing the system into a system of linear type difference equations.
In [30], the authors give formulas for solutions of the equation

$$y_{n+1}=\frac{1+y_{n-1}}{y_ny_{n-1}}, \qquad n\in\mathbb{N}_0,$$

and prove them by using only the method of induction. However, the formulas are not justified by some theoretical explanations.
Halim et al. [8] gave the solutions of the systems of difference equations

$$x_{n+1}=\frac{1}{\pm 1\pm y_{n-k}}, \quad y_{n+1}=\frac{1}{\pm 1\pm x_{n-k}}, \qquad n,k\in\mathbb{N}_0,$$

such that their solutions are associated to Fibonacci numbers.

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Also, in [7] Halim et al. establish the solution form of equation

$$y_{n+1} = \frac{\alpha + \beta y_{n-1}}{\delta y_n y_{n-1}}, \qquad n \in \mathbb{N}_0,$$

using appropriate transformation reducing the equation into a linear type difference equation, such that their solutions are associated to generalized Padovan numbers.
In [19], Stevic gave a theoretical explanation for the formula of solutions of the following difference equation

$$y_{n+1} = \frac{\alpha y_n + \beta}{\gamma y_n + \delta}, \qquad n \in \mathbb{N}_0,$$

where parameters $\alpha, \beta, \gamma, \delta$ and initial value y_0 are real numbers, such that their solutions are associated to generalized Fibonacci numbers. More details on this aspect can be simply found in refs. [1]-[3],[9]-[13], [19], [22]-[28], [30],[31].

2. Preliminaries

2.1. Linearized stability of the higher-order systems

Let f and g be two continuously differentiable functions:

$$f:\ I^{k+1} \times J^{k+1} \longrightarrow I, \quad g:\ I^{k+1} \times J^{k+1} \longrightarrow J,$$

where I, J are some interval of real numbers. For $n \in \mathbb{N}_0$, consider the system of difference equations

$$\left\{ \begin{array}{ll} x_{n+1} = & f\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}\right) \\ y_{n+1} = & g\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}\right) \end{array} \right. \tag{2.1}$$

where $n, k \in \mathbb{N}_0$, $(x_{-k}, x_{-k+1}, \ldots, x_0) \in I^{k+1}$ and $(y_{-k}, y_{-k+1}, \ldots, y_0) \in J^{k+1}$.
Define the map $H:\ I^{k+1} \times J^{k+1} \longrightarrow I^{k+1} \times J^{k+1}$ by

$$H(W) = (f_0(W), f_1(W), \ldots, f_k(W), g_0(W), g_1(W), \ldots, g_k(W))$$

where

$$W = (u_0, u_1, \ldots, u_k, v_0, v_1, \ldots, v_k)^T,$$

$$f_0(W) = f(W), f_1(W) = u_0, \ldots, f_k(W) = u_{k-1},$$

$$g_0(W) = g(W), g_1(W) = v_0, \ldots, g_k(W) = v_{k-1}.$$

Let

$$W_n = [x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}]^T.$$

Then, we can easily see that system (2.1) is equivalent to the following system written in vector form

$$W_{n+1} = H(W_n), \qquad n \in \mathbb{N}_0, \tag{2.2}$$

that is

$$\left\{ \begin{array}{rr} x_{n+1} = & f\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}\right), \\ x_n = & x_n, \\ \vdots & \\ x_{n-k+1} = & x_{n-k+1}, \\ y_{n+1} = & g\left(x_n, x_{n-1}, \ldots, x_{n-k}, y_n, y_{n-1}, \ldots, y_{n-k}\right), \\ y_n = & y_n, \\ \vdots & \\ y_{n-k+1} = & y_{n-k+1}. \end{array} \right.$$

Definition 2.1 (Equilibrium point)**.** *An equilibrium point $(\bar{x}, \bar{y}) \in I \times J$ of system (2.1) is a solution of the system*

$$\left\{ \begin{array}{l} x = f\left(x, x, \ldots, x, y, y, \ldots, y\right), \\ y = g\left(x, x, \ldots, x, y, y, \ldots, y\right). \end{array} \right.$$

Furthermore, an equilibrium point $\overline{W} \in I^{k+1} \times J^{k+1}$ of system (2.2) is a solution of the system

$$W = H(W).$$

Definition 2.2 (Stability)**.** *Let $\overline{W}$ be an equilibrium point of system (2.2) and $\| \,.\, \|$ be any norm (e.g. the Euclidean norm).*

1. *The equilibrium point $\overline{W}$ is called stable (or locally stable) if for every $\varepsilon > 0$ there exist δ such that $\|W_0 - \overline{W}\| < \delta$ implies $\|W_n - \overline{W}\| < \varepsilon$ for $n \geq 0$.*
2. *The equilibrium point $\overline{W}$ is called asymptotically stable (or locally asymptotically stable) if it is stable and there exist $\gamma > 0$ such that $\|W_0 - \overline{W}\| < \gamma$ implies*
$$\lim_{n\to+\infty} W_n = \overline{W}.$$
3. *The equilibrium point $\overline{W}$ is said to be global attractor (respectively global attractor with basin of attraction a set $G \subseteq I^{k+1} \times J^{k+1}$, if for every W_0 (respectively for every $W_0 \in G$)*
$$\lim_{n\to+\infty} W_n = \overline{W}.$$
4. *The equilibrium point $\overline{W}$ is called globally asymptotically stable (respectively globally asymptotically stable relative to G) if it is asymptotically stable, and if for every W_0 (respectively for every $W_0 \in G$),*
$$\lim_{n\to+\infty} W_n = \overline{W}.$$
5. *The equilibrium point $\overline{W}$ is called unstable if it is not stable.*

Remark 2.3. *Clearly, $(\bar{x},\bar{y}) \in I \times J$ is an equilibrium point for system (2.1) if and only if $\overline{W} = (\bar{x},\bar{x},\cdots,\bar{x},\bar{y},\bar{y},\cdots,\bar{y}) \in I^{k+1} \times J^{k+1}$ is an equilibrium point of system (2.2).*

From here on, by the stability of the equilibrium points of system (2.1), we mean the stability of the corresponding equilibrium points of the equivalent system (2.2). The linearized system, associated to system (2.2), about the equilibrium point

$$\overline{W} = (\bar{x},\bar{x},\cdots,\bar{x},\bar{y},\bar{y},\cdots,\bar{y}),$$

is given by

$$W_{n+1} = AW_n, \qquad n \in \mathbb{N}_0,$$

where A is the Jacobian matrix of the map H at the equilibrium point $\overline{W}$ given by

$$A = \begin{pmatrix}
\frac{\partial f_0}{\partial u_0}(\overline{W}) & \frac{\partial f_0}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_0}{\partial u_k}(\overline{W}) & \frac{\partial f_0}{\partial v_0}(\overline{W}) & \frac{\partial f_0}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_0}{\partial v_k}(\overline{W}) \\
\frac{\partial f_1}{\partial u_0}(\overline{W}) & \frac{\partial f_1}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_1}{\partial u_k}(\overline{W}) & \frac{\partial f_1}{\partial v_0}(\overline{W}) & \frac{\partial f_1}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_1}{\partial v_k}(\overline{W}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial f_k}{\partial u_0}(\overline{W}) & \frac{\partial f_k}{\partial u_1}(\overline{W}) & \dots & \frac{\partial f_k}{\partial u_k}(\overline{W}) & \frac{\partial f_k}{\partial v_0}(\overline{W}) & \frac{\partial f_k}{\partial v_1}(\overline{W}) & \dots & \frac{\partial f_k}{\partial v_k}(\overline{W}) \\
\frac{\partial g_0}{\partial u_0}(\overline{W}) & \frac{\partial g_0}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_0}{\partial u_k}(\overline{W}) & \frac{\partial g_0}{\partial v_0}(\overline{W}) & \frac{\partial g}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_0}{\partial v_k}(\overline{W}) \\
\frac{\partial g_1}{\partial u_0}(\overline{W}) & \frac{\partial g_1}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_1}{\partial u_k}(\overline{W}) & \frac{\partial g_1}{\partial v_0}(\overline{W}) & \frac{\partial g_1}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_1}{\partial v_k}(\overline{W}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\frac{\partial g_k}{\partial u_0}(\overline{W}) & \frac{\partial g_k}{\partial u_1}(\overline{W}) & \dots & \frac{\partial g_k}{\partial u_k}(\overline{W}) & \frac{\partial g_k}{\partial v_0}(\overline{W}) & \frac{\partial g_k}{\partial v_1}(\overline{W}) & \dots & \frac{\partial g_k}{\partial v_k}(\overline{W})
\end{pmatrix}.$$

Theorem 2.4. *(Linearized stability)*

1. *If all the eigenvalues of the Jacobian matrix A lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{W}$ of system (2.2) is asymptotically stable.*
2. *If at least one eigenvalue of the Jacobian matrix A have absolute value greater than one, then the equilibrium point $\overline{W}$ of system (2.2) is unstable.*

2.2. Lucas sequence

The integer sequence defined by the recurrence relation

$$L_{n+1} = L_n + L_{n-1}, \quad n \in \mathbb{N},$$

with the initial conditions $L_0 = 2$ and $L_1 = 1$, is known as the Lucas numbers and was named after François Edouard Anatole Lucas (1842-91). This is the same recurrence relation as for the Fibonacci sequence, but with different initial conditions ($F_0 = 0, F_1 = 1$). The first few terms of the recurrence sequence are $2,1,3,4,7,11,18,29,47,76,\dots$. The Binet's formula for this recurrence sequence can easily be obtained and is given by

$$L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} (the\ so-called\ golden\ number), \quad \beta = \frac{1-\sqrt{5}}{2}.$$

That is,

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \alpha.$$

For more informations associated with Lucas sequence, see [15] and [29].

3. Closed-Form solutions and stability of system (1.1)

For the rest of our discussion we assume L_n, the n-th Lucas number, to satisfy the recurrence equation

$$L_{n+1} = L_n + L_{n-1}, \quad n \in \mathbb{N}_0,$$

with initial conditions $L_0 = 2$ and $L_1 = 1$.

3.1. Linear second order differences equations with constants coefficients.

As is well-known, the equation

$$z_{n+1} + 5z_n + 5z_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{3.1}$$

(the homogeneous linear second order difference equation with constant coefficients), where z_0 and $z_{-1} \in \mathbb{R}$, is usually solved by using the characteristic roots λ_1 and λ_2 of the characteristic polynomial $\lambda^2 + 5\lambda + 5 = 0$, so

$$\lambda_1 = \sqrt{5}\beta, \qquad \lambda_2 = -\sqrt{5}\alpha,$$

and the formulas of general solution is

$$x_n = c_1\lambda_1^n + c_2\lambda_2^n.$$

Using the initial conditions z_0 and z_{-1} with some calculations we get

$$c_1 = -\sqrt{5}\left(z_{-1} - \frac{z_0}{5}\lambda_1\right),$$

$$c_2 = -\sqrt{5}\left(\frac{z_0}{5}\lambda_2 - z_{-1}\right).$$

So,

$$z_n = \sqrt{5}\left(z_{-1}\left[\lambda_2^n - \lambda_1^n\right] - \frac{z_0}{5}\left[\lambda_2^{n+1} - \lambda_1^{n+1}\right]\right),$$

$$= \sqrt{5}\left(z_{-1}(\sqrt{5})^n\left[(-1)^n\alpha^n - \beta^n\right] - \frac{z_0(\sqrt{5})^{n+1}}{(\sqrt{5})^2}\left[(-1)^{n+1}\alpha^{n+1} - \beta^{n+1}\right]\right),$$

by put

$$N_n = \left((-1)^n\alpha^n - \beta^n\right),$$

is obtained that the general solution of equation (3.1) is

$$z_n = (\sqrt{5})^n\left[z_{-1}\sqrt{5}N_n - z_0N_{n+1}\right]. \tag{3.2}$$

Similarly, let

$$z_{n+1} - 5z_n + 5z_{n-1} = 0, \quad n \in \mathbb{N}_0, \tag{3.3}$$

so, by put

$$M_n = \left(\alpha^n - (-1)^n\beta^n\right),$$

is obtained that the general solution of equation (3.3) is

$$z_n = -(\sqrt{5})^n\left[\sqrt{5}z_{-1}M_n - z_0M_{n+1}\right]. \tag{3.4}$$

3.2. Linear system of second order difference equations with constant coefficients.

Let the linear system of second order difference equations

$$u_{n+1} = 5v_n - 5u_{n-1}, \quad v_{n+1} = 5u_n - 5v_{n-1}, \quad n \in \mathbb{N}_0. \tag{3.5}$$

From (3.5) we get

$$v_n = \frac{1}{5}\left(u_{n+1} + 5u_{n-1}\right). \tag{3.6}$$

We replace (3.6) in the second equation of the system (3.5), we get

$$\frac{1}{5}u_{n+2} - 3u_n + 5u_{n-2} = 0,$$

which can be written both as

$$\underbrace{(u_{n+2} - 5u_{n+1} + 5u_n)}_{s_{n+1}} + 5\underbrace{(u_{n+1} - 5u_n + 5u_{n-1})}_{s_n} + 5\underbrace{(u_n - 5u_{n-1} + 5u_{n-2})}_{s_{n-1}} = 0, \quad n \in \mathbb{N},$$

which is in the form of equation (3.1) and as

$$\underbrace{(u_{n+2}+5u_{n+1}+5u_n)}_{k_{n+1}}-5\underbrace{(u_{n+1}+5u_n+5u_{n-1})}_{k_n}+5\underbrace{(u_n+5u_{n-1}+5u_{n-2})}_{k_{n-1}}=0, \quad n\in\mathbb{N}, \tag{3.7}$$

which is in the form of equation (3.3). Form (3.4) and (3.2) we can write

$$\begin{cases} s_{2n}= & (\sqrt{5})^{2n}\left[5s_{-1}F_{2n}+s_0L_{2n+1}\right], \\ s_{2n+1}= & (\sqrt{5})^{2n+2}\left[s_{-1}L_{2n+1}+s_0F_{2n+2}\right]. \end{cases}$$

Hence

$$u_{2n+1}-5u_{2n}+5u_{2n-1}=(\sqrt{5})^{2n}[5(u_0-5u_{-1}+5u_{-2})F_{2n}+(u_1-5u_0+5u_{-1})L_{2n+1}], \tag{3.8}$$

and

$$u_{2n+2}-5u_{2n+1}+5u_{2n}=(\sqrt{5})^{2n+2}[(u_0-5u_{-1}+5u_{-2})L_{2n+1}+(u_1-5u_0+5u_{-1})F_{2n+2}].$$

Similarly, form (3.3) and (3.7) we can write

$$\begin{cases} k_{2n}= & -(\sqrt{5})^{2n}\left[5k_{-1}F_{2n}-k_0L_{2n+1}\right], \\ k_{2n+1}= & -(\sqrt{5})^{2n+2}\left[k_{-1}L_{2n+1}-k_0F_{2n+2}\right]. \end{cases}$$

Hence

$$u_{2n+1}+5u_{2n}+5u_{2n-1}=-(\sqrt{5})^{2n}\left[5(u_0+5u_{-1}+5u_{-2})F_{2n}-(u_1+5u_0+5u_{-1})L_{2n+1}\right], \tag{3.9}$$

and

$$u_{2n+2}-5u_{2n+1}+5u_{2n}=-(\sqrt{5})^{2n+2}\left[(u_0+5u_{-1}+5u_{-2})L_{2n+1}-(u_1+5u_0+5u_{-1})F_{2n+2}\right].$$

Now, by subtracting equation (3.9) from equation (3.8), we obtain

$$u_{2n}=-(\sqrt{5})^{2n}[5v_{-1}F_{2n}-u_0L_{2n+1}]. \tag{3.10}$$

Also, by equation (3.9) and equation (3.8), we obtain

$$v_{2n}=-(\sqrt{5})^{2n}\left[5u_{-1}F_{2n}-v_0L_{2n+1}\right]. \tag{3.11}$$

By a similar calculation, we obtain

$$u_{2n+1}=-(\sqrt{5})^{2n+2}[u_{-1}L_{2n+1}-v_0F_{2n+2}], \tag{3.12}$$

and

$$v_{2n+1}=-(\sqrt{5})^{2n}[v_{-1}L_{2n+1}-u_0F_{2n+2}]. \tag{3.13}$$

Now we consider the system of two first-order difference equations

$$z_{n+1}=\frac{5w_n-5}{w_n}, \qquad w_{n+1}=\frac{5z_n-5}{z_n}, \qquad n\in\mathbb{N}_0. \tag{3.14}$$

where the initial conditions z_0 and w_0 are non zero real numbers.
Through an analytical approach. We put

$$z_n=\frac{u_n}{v_{n-1}}, \quad w_n=\frac{v_n}{u_{n-1}}.$$

Hence we have the system

$$u_{n+1}=5v_n-5u_{n-1}, \quad v_{n+1}=5u_n-5v_{n-1}. \tag{3.15}$$

by formulas, (3.5), (3.2), (3.4), (3.10), (3.11), (3.12) and (3.13) is obtained that the general solution of system (3.15) is

$$\begin{cases} u_{2n} &= -(\sqrt{5})^{2n}\left[5v_{-1}F_{2n}-u_0L_{2n+1}\right], \\ u_{2n+1} &= -(\sqrt{5})^{2n+2}\left[u_{-1}L_{2n+1}-v_0F_{2n+2}\right], \\ v_{2n} &= -(\sqrt{5})^{2n}\left[5u_{-1}F_{2n}-v_0L_{2n+1}\right], \\ v_{2n+1} &= -(\sqrt{5})^{2n+2}\left[v_{-1}L_{2n+1}-u_0F_{2n+2}\right]. \end{cases}$$

From all above mentioned we see that the following theorem holds.

Theorem 3.1. *Let $\{z_n, w_n\}_{n\geq -1}$ be a solution of (3.14). Then, for $n = 2,3,\ldots,$*

$$\begin{cases} z_{2n} &= \dfrac{5F_{2n} - z_0 L_{2n+1}}{L_{2n-1} - z_0 F_{2n}}, \\ z_{2n+1} &= \dfrac{5L_{2n+1} - 5w_0 F_{2n+2}}{5F_{2n} - w_0 L_{2n+1}}, \\ w_{2n} &= \dfrac{5F_{2n} - w_0 L_{2n+1}}{L_{2n-1} - w_0 F_{2n}}, \\ w_{2n+1} &= \dfrac{5L_{2n+1} - 5z_0 F_{2n+2}}{5F_{2n} - z_0 L_{2n+1}}. \end{cases}$$

where $\{L_n\}_n$ is the Lucas sequence, $\{F_n\}_n$ is the Fibonacci sequence and the initial conditions z_0 and $w_0 \in \mathbb{R} - G_1$, with G_1 is the Forbidden Set of system (3.14) given by

$$G_1 = \bigcup_{n=-1}^{\infty} \{(z_0, w_0) : L_{2n-1} - z_0 F_{2n} = 0, \quad 5F_{2n} - w_0 L_{2n+1} = 0\}.$$

Let

$$\begin{cases} x_n^{(j)} = x_{(k+1)n-j}, \\ y_n^{(j)} = y_{(k+1)n-j}. \end{cases} \tag{3.16}$$

where $j \in \{0,1,\ldots k\}$.
Using notation (3.16), we can write (1.1) as

$$\begin{cases} x_{n+1}^{(j)} = \dfrac{5y_n^{(j)} - 5}{y_n^{(j)}}, \\ y_{n+1}^{(j)} = \dfrac{5x_n^{(j)} - 5}{x_n^{(j)}}. \quad n \in \mathbb{N}, \end{cases}$$

for each $j \in \{0,1,\ldots,k\}$.
So, from Theorem (3.1) we get for $n = 2,3,\ldots,$

$$\begin{cases} x_{2n}^{(j)} &= \dfrac{5F_{2n} - x_0^{(j)} L_{2n+1}}{L_{2n-1} - x_0^{(j)} F_{2n}}, \\ x_{2n+1}^{(j)} &= \dfrac{5L_{2n+1} - 5y_0^{(j)} F_{2n+2}}{5F_{2n} - y_0^{(j)} L_{2n+1}}, \\ y_{2n}^{(j)} &= \dfrac{5F_{2n} - y_0^{(j)} L_{2n+1}}{L_{2n-1} - y_0^{(j)} F_{2n}}, \\ y_{2n+1}^{(j)} &= \dfrac{5L_{2n+1} - 5x_0^{(j)} F_{2n+2}}{5F_{2n} - x_0^{(j)} L_{2n+1}}. \end{cases}$$

From all above mentioned we see that the following theorem holds.

Theorem 3.2. *Let $\{x_n, y_n\}_{n\geq -1}$ be a solution of (1.1). Then, for $n = 2,3,\ldots,$*

$$\begin{cases} x_{(k+1)2n-j} &= \dfrac{5F_{2n} - x_{-j} L_{2n+1}}{L_{2n-1} - x_{-j} F_{2n}}, \\ x_{(k+1)(2n+1)-j} &= \dfrac{5L_{2n+1} - 5y_{-j} F_{2n+2}}{5F_{2n} - y_{-j} L_{2n+1}}, \\ y_{(k+1)2n-j} &= \dfrac{5F_{2n} - y_{-j} L_{2n+1}}{L_{2n-1} - y_{-j} F_{2n}}, \\ x_{(k+1)(2n+1)-j} &= \dfrac{5L_{2n+1} - 5x_{-j} F_{2n+2}}{5F_{2n} - x_{-j} L_{2n+1}}. \end{cases}$$

where $j \in \{0,1,\ldots,k\}$, $\{L_n\}_n$ the Lucas sequence, $\{F_n\}_n$ the Fibonacci sequence and the initial conditions x_{-k}, $x_{-k+1},\ldots x_0$, y_{-k}, $y_{-k+1},\ldots,y_1$ and $y_0 \in \mathbb{R} - G_j$, with G_j is the Forbidden Set of system (1.1) given by

$$G_j = \bigcup_{n=-1}^{\infty} \{(x_{-k}, x_{-k+1}, \ldots, x_0, y_{-k}, y_{-k+1}, \ldots, y_0,) : L_{2n-1} - x_{-j} F_{2n} = 0, \quad 5F_{2n} - y_{-j} L_{2n+1} = 0, \quad j = 0,1,\ldots,k\}.$$

3.3. Global stability of positive solutions

In this section we study the global stability character of the solutions of system (1.1). It is easy to show that (1.1) has a unique real positive equilibrium point given by

$$E=(\bar{x},\bar{y})=\left(\sqrt{5}\alpha,\sqrt{5}\alpha\right),$$

where α is the golden number.
Let $I=J=(0,+\infty)$ and consider the functions

$$f:\ I^{k+1}\times J^{k+1}\longrightarrow I,\ g:\ I^{k+1}\times J^{k+1}\longrightarrow J$$

defined by

$$f(u_0,u_1,\ldots,u_k,v_0,v_1,\ldots,v_k)=\frac{5v_k-5}{v_k},$$
$$g(u_0,u_1,\ldots,u_k,v_0,v_1,\ldots,v_k)=\frac{5u_k-5}{u_k}.$$

Theorem 3.3. *The equilibrium point E is locally asymptotically stable.*

Proof. The the linearized system about the equilibrium point

$$\overline{W}=\left(\sqrt{5}\alpha,\ldots,\sqrt{5}\alpha,\sqrt{5}\alpha,\ldots,\sqrt{5}\alpha\right)\in I^{k+1}\times J^{k+1}$$

is given by

$$X_{n+1}=AX_n,\quad X_n=(x_n,x_{n-1},\ldots,x_{n-k},y_n,y_{n-1},\ldots,y_{n-k})^T,$$

and

$$A=\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & \frac{1}{\alpha^2}\\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0\\ 0 & 0 & \ldots & \frac{1}{\alpha^2} & 0 & 0 & \ldots & 0\\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}.$$

So, after some elementary calculations, we get

$$P(\lambda)=\det(A-\lambda I_{2k+2})=\lambda^{2k+2}-\frac{1}{\alpha^4}.$$

Now, consider the two functions defined by

$$a(\lambda)=\lambda^{2k+2},\quad b(\lambda)=\frac{1}{\alpha^4}.$$

We have

$$|b(\lambda)|<|a(\lambda)|,\forall\lambda:|\lambda|=1.$$

Thus, by Rouche's Theorem, all zeros of $P(\lambda)=a(\lambda)-b(\lambda)=0$ lie in $|\lambda|<1$. So, by Theorem (2.4), we get that E is locally asymptotically stable. □

Theorem 3.4. *The equilibrium point E is globally asymptotically stable.*

Proof. Let $\{x_n,y_n\}_{n\geq -k}$ be a solution of (1.1). By Theorem (3.3) we need only to prove that E is global attractor, that is

$$\lim_{n\to\infty}(x_n,y_n)=E.$$

To do this, we prove that for $j=0,1,\ldots,k$ we have

$$\lim_{n\to+\infty}x_{(k+1)2n-j}=\lim_{n\to+\infty}x_{(k+1)(2n+1)-j}=\lim_{n\to+\infty}y_{(k+1)2n-j}=\lim_{n\to+\infty}y_{(k+1)(2n+1)-j}=\sqrt{5}\alpha.$$

For $j=0,1,\ldots,k$, it follows from Theorem (3.2) that

$$\lim_{n\to+\infty}x_{(k+1)2n-j}=\lim_{n\to+\infty}\frac{5F_{2n}-x_{-j}L_{2n+1}}{L_{2n-1}-x_{-j}F_{2n}}$$
$$=\lim_{n\to+\infty}\frac{5-x_{-j}\frac{L_{2n+1}}{F_{2n}}}{\frac{L_{2n-1}}{F_{2n}}-x_{-j}}.$$

Using

$$\lim_{n\to+\infty}\frac{F_{n+1}}{L_n}=\frac{\alpha}{\sqrt{5}},\quad \lim_{n\to+\infty}\frac{L_{n+1}}{F_n}=\sqrt{5}\alpha$$

we get

$$\lim_{n\to+\infty}x_{(k+1)2n-j}=\alpha\sqrt{5}.$$

Similarly we get

$$\lim_{n\to+\infty}x_{(k+1)(2n+1)-j}=\lim_{n\to+\infty}y_{(k+1)2n-j}=\lim_{n\to+\infty}y_{(k+1)(2n+1)-j}=\sqrt{5}\alpha.$$

□

3.4. Numerical confirmation

This subsection is included to verify and confirm the results we obtained in this work. As an illustration of our results, we consider the following numerical examples.

Example 3.5. *Let $k=0$ in system (1.1), then we obtain the system*

$$x_{n+1}=\frac{5y_n-5}{y_n},\qquad y_{n+1}=\frac{5x_n-5}{x_n},\qquad n\in\mathbb{N}_0. \tag{3.17}$$

Assume $x_0=0.7$ and $y_0=1.5$. (See Fig (3.1))

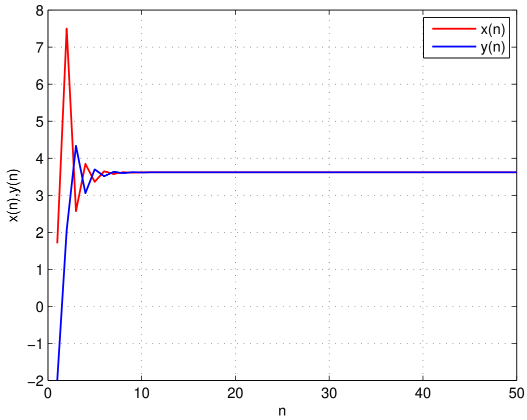

Figure 3.1: This figure shows that the solution of the system (3.17) is globally asymptotically stable

Example 3.6. *Let $k=2$ in system (1.1), then we obtain the system*

$$x_{n+1}=\frac{5y_{n-2}-5}{y_{n-2}},\qquad y_{n+1}=\frac{5x_{n-2}-5}{x_{n-2}},\qquad n\in\mathbb{N}_0. \tag{3.18}$$

Assume $x_{-2}=0.5,\ x_{-1}=0.7,\ x_0=1.6,\ y_{-2}=0.6,\ y_{-1}=-50$ and $y_0=1.7$. (See Fig(3.2))

Example 3.7. *Let $k=3$ in system (1.1), then we obtain the system*

$$x_{n+1}=\frac{5y_{n-3}-5}{y_{n-3}},\qquad y_{n+1}=\frac{5x_{n-3}-5}{x_{n-3}},\qquad n\in\mathbb{N}_0. \tag{3.19}$$

Assume $x_{-3}=0.8,\ x_{-2}=0.7,\ x_{-1}=0.6,\ x_0=0.9,\ y_{-3}=1.1,\ y_{-2}=1.8,\ y_{-1}=1.3$ and $y_0=1.6$. (See Fig(3.3))

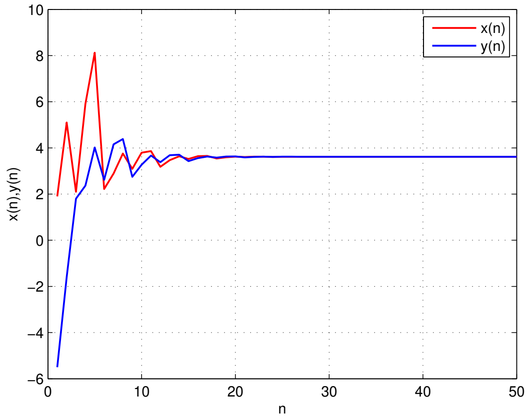

Figure 3.2: This figure shows that the solution of the system (3.18) is globally asymptotically stable

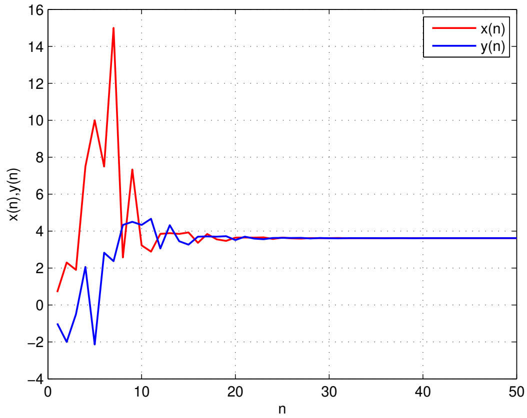

Figure 3.3: This figure shows that the solution of the system (3.19) is globally asymptotically stable

4. Conclusion

In this work, we have successfully established a theoretical explanation related to the closed-form solution of the system of two higher-order difference equations

$$x_{n+1} = \frac{5y_{n-k} - 5}{y_{n-k}}, \qquad y_{n+1} = \frac{5x_{n-k} - 5}{x_{n-k}}, \qquad n,k \in \mathbb{N}_0.$$

Also, we obtained stability results for the positive solutions of this system. Particularly, we have shown that the positive solutions of this system tends to a computable finite number, and is in fact expressible in terms of the well-known golden number.
This work we leave to the interested readers.

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References

[1] E. M. Elsayed, *On a system of two nonlinear difference equations of order two*, Proc. Jangeon Math. Soc., **18**(1)(2015), 353-368.
[2] E. M. Elsayed and T. F. Ibrahim, *Periodicity and solutions for some systems of nonlinear rational difference equations*, Hacet. J. Math. Stat., **44**(1)(2015), 1361-1390.
[3] E. M. Elsayed, *Solution for systems of difference equations of rational form of order two*, Comp. Appl. Math., **33**(1)(2014), 751-765.
[4] E. Camouzis and G. Ladas, *Dynamics of third-order rational difference equations with open problems and conjectures*, Vol5, CRC Press, (2007).
[5] M. Gumus, *The global asymptotic stability of a system of difference equations*, J. Difference Equ. Appl., **24**(6)(2018), 976-991.
[6] M. Gumus and R. Abo-Zeid, *On the solutions of a $(2k+2)$th order difference equation*, Dyn. Contin. Discrete Impuls. Syst., Ser. B, Appl. Algorithms, **25**(2)(2018), 129-143.
[7] Y. Halim and J. F. T. Rabago, *On the solutions of a second-order difference equation in terms of generalized Padovan sequences*, Math. Slovaca, **68**(3)(2018), 625-638.
[8] Y. Halim and J. F. T. Rabago,*On some solvable systems of difference equations with solutions associated to Fibonacci numbers*, Electron J. Math. Analysis Appl, **5**(1)(2017), 166-178.
[9] Y. Halim, *A system of difference equations with solutions associated to Fibonacci numbers*, Int. J. Difference Equ., **11**(1)(2016), 65-77.

[10] Y. Halim and M. Bayram, *On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences*, Math. Methods Appl. Sci., **39**(1)(2016), 2974-2982.
[11] Y. Halim, N. Touafek and E. M. Elsayed, *Closed form solution of some systems of rational difference equations in terms of Fibonacci numbers*, Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal., **21**(6)(2014), 473-486.
[12] Y. Halim, *Global character of systems of rational difference equations*, Electron. J. Math. Analysis Appl., **3**(1)(2015), 204-214.
[13] Y. Halim, *A system of difference equations with solutions associated to Fibonacci numbers*, Int. J Difference Equ., **11**(2016), 65–77.
[14] V. L. Kocic and G. Ladas, *Global behavior of nonlinear difference equations of higher order with applications*, Kluwer Academic Publishers, Dordrecht, (1993).
[15] T. Koshy, *Fibonacci and Lucas numbers with applcations*, Departement of mathematics, Framingham State University, (2017).
[16] M. R. S. Kulenovic and G. Ladas, *Dynamics of second order rational difference equations: With open problems and conjectures*, Chapman and Hall/CRC, (2001).
[17] H. Matsunaga and R. Suzuki, *Classification of global behavior of a system of rational difference equations*, Appl. Math. Lett., **85**(1)(2018), 57–63.
[18] O. Ocalan and O. Duman, *on solutions of the recursive equations* $x_{n+1} = x_{n-1}^p / x_n^p (p > 0)$ *via Fibonacci-type sequences*, Electron. J. Math. Analysis Appl., **7**(1)(2019), 102-115.
[19] S. Stevic, *Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences*, Electron. J. Qual. Theory Differ. Equ., **67**(1)(2014), 15 pages.
[20] S. Stevic, *More on a rational recurrence relation*, Appl. Math. E-Notes, **4**(1)(2004), 80-85.
[21] S. Stevic, *Representation of solutions of a solvable nonlinear difference equation of second order*, Electron. J. Qual. Theory Differ. Equ., **95**(1)(2018), 18 pages.
[22] D. T. Tollu, Y. Yazlik and N. Taskara, *On the solutions of two special types of Riccati difference equation via Fibonacci umbers*, **174**(1)(2013), 7 pages.
[23] D. T. Tollu, Y. Yazlik and N. Taskara, *The solutions of four Riccati difference equations associated with Fibonacci numbers*, Balkan J. Math., **2**(1)(2014), 163-172.
[24] D. T. Tollu, Y. Yazlik and N. Taskara, *On fourteen solvable systems of difference equations*, Appl. Math. & Comp., **233**(1)(2014), 310-319.
[25] N. Touafek, *On some fractional systems of difference equations*, Iran. J. Math. Sci. Inform., **9**(2)(2014), 303-305.
[26] N. Touafek and Y. Halim, *On max type difference equations: expressions of solutions*, Int. J. Appl. Nonlinear Sci., **11**(4)(2011), 396-402.
[27] N. Touafek and E. M. Elsayed, *On the periodicity of some systems of nonlinear difference equations*, Bull. Math. Soc. Sci. Math. Roum., Nouv. Ser., **55**(1)(2012), 217-224.
[28] N. Touafek and E. M. Elsayed, *On the solutions of systems of rational difference equations*, Math. Comput. Modelling, **55**(1)(2012), 1987-1997.
[29] S. Vajda, *Fibonacci and Lucas numbers and the golden section : Theory and applications*, Department of Mathematics, University of Sussex, Ellis Horwood Limited, (1989).
[30] Y. Yazlik, D. T. Tollu and N. Taskara, *On the solutions of difference equation systems with Padovan numbers*, Appl Math., **12**(1)(2013), 15-20.
[31] Y. Yazlik, D. T. Tollu and N. Taskara, *behaviour of solutions for a system of two higher-order difference equations* , J. Sci. Arts,**45**(4)(2018), 813-826.