Eastern Anatolian Journal of Science
Volume I, Issue I, 2015, 29-34

Eastern Anatolian Journal of Science

Further instability result on the solutions of nonlinear vector differential equations of the fifth order

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Abstract

The purpose of this paper is to investigate instability of the trivial solution of non-linear vector differential equation of the fifth order of the form

$$X^{(5)} + AX^{(4)} + B\dddot{X} + \Psi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\ddot{X}$$

$$+G(X)\dot{X} + F(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})X = 0$$

by constructing a Lyapunov function.

Keywords: Instability, Lyapunov function, nonlinear differential equation.

1. Introduction

Instability of the trivial solution of scalar and vector differential equations of the fifth order were investigated by many authors. About the topic, we refer to the papers of (EZEILO 1978; EZEILO 1979; TIRYAKI 1987; LI and YU 1990; LI and DUAN 2000; SADEK 2003; TUNC 2004; TUNC 2005; TUNC and SELVI 2005; TUNC and KARTA 2008; KRASOVSKII 1955). In all of the papers mentioned above, authors used Krasovskii's criteria (KRASOVSKII 1955) and Lyapunov's second (or direct) method (LYAPUNOV 1966).

According to observations in the literature, firstly, for the case $n = 1$, EZEILO (1978; 1979) investigated the instability of trivial solution of the fifth order scalar non-linear differential equations, respectively,

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x}$$

$$+a_3 \ddot{x} + a_4 \dot{x} + f(x) = 0,$$

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x}$$

$$+h(\dot{x})\ddot{x} + g(x)\dot{x} + f(x) = 0,$$

$$x^{(5)} + \psi(\ddot{x})\dddot{x} + \varphi(\ddot{x}) + \theta(\dot{x}) + f(x) = 0,$$

and

$$x^{(5)} + a_1 x^{(4)} + a_2 \dddot{x} + g(\dot{x})\ddot{x}$$

$$+h(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\dot{x} + f(x) = 0$$

in which a_1, a_2, a_3, a_4, are constants and f, g, h, ψ, ϕ and θ are continuous functions depending only on the arguments shown as $f(0) = \phi(0) = \theta(0) = 0$. TIRYAKI (1987) studied the instability of trivial solution of the fifth order non-linear scalar differential equation of the form

$$x^{(5)} + a_1 x^{(4)} + k(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\dddot{x} + g(\dot{x})\ddot{x}$$

$$+h(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\dot{x} + f(x) = 0.$$

LI and YU (1990) concerned the instability of trivial solution of the fifth order non-linear scalar differential equation

Accepted date: 06.05.2015
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$$x^{(5)} + ax^{(4)} + b\dddot{x} + \psi(x, \dot{x}, \ddot{x}, \dddot{x}, x^{(4)})\ddot{x}$$

$$+g(x)\dot{x} + f(x) = 0$$

by introducing a Lyapunov function, where a and b are some positive constant.

LI and DUAN (2000) showed the instability of trivial solution of the fifth order nonlinear scalar differential equation of the for

$$\dot{x}_i = x_{i+1}\ (i = 1,2,3,4)$$

$$\dot{x}_5 = -f_5(x_4)x_5 - f_4(x_3)x_4$$

$$-f_3(x_1, x_2, x_3, x_4, x_5)x_3 - f_2(x_2) - f_1(x_1)$$

$$(f_2(0) = f_1(0) = 0)$$

and

$$\dot{x}_i = x_{i+1}(i = 1,2,3,4)$$

$$\dot{x}_5 = -a_5x_5 - f_4(x_1, x_2, x_3, x_4, x_5)x_4$$

$$-f_3(x_2)x_3 - f_2(x_1, x_2, x_3, x_4, x_5)x_2 - f_1(x_1) \tag{1.1}$$

On the other hand, SADEK (2003) examined the instability of trivial solutions of the fifth order vector differential equations described as

$$X^{(5)} + \Psi(\ddot{X})\dddot{X} + \Phi(\ddot{X}) + \theta(\dot{X}) + F(X) = 0$$

and

$$X^{(5)} + AX^{(4)} + B\dddot{X} + \mathrm{H}(\dot{X})\ddot{X}$$

$$+G(X)\dot{X} + F(X) = 0.$$

In addition, respectively, TUNC (2004; 2005) investigated the instability of trivial solution of the fifth order vector differential equations of the form

$$X^{(5)} + AX^{(4)} + \Psi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dddot{X}$$

$$+G(\dot{X})\ddot{X} + H(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dot{X} + F(X) = 0,$$

$$X^{(5)} + AX^{(4)} + B(t)(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dddot{X}$$

$$+C(t)G(\dot{X})\ddot{X} + D(t)(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\dot{X}$$

$$+E(t)F(X) = 0$$

and

$$X^{(5)} + \Psi(\dot{X}, \ddot{X})\dddot{X} + \Phi(\dot{X}, \ddot{X}, \dddot{X}, X^{(4)})$$

$$+\theta(\dot{X}) + F(X) = 0.$$

TUNC and SEVLI (2005) showed a similar study for the instability of trivial solution of the fifth order vector differential equation

$$X^{(5)} + \Psi(\dot{X}, \ddot{X})\dddot{X} + \Phi(X, \dot{X}, \ddot{X})\ddot{X}$$

$$+\theta(\dot{X}) + F(X) = 0.$$

Furthermore, TUNC and KARTA (2008) analyzed sufficient conditions which ensure the trivial solution of vector differential equation

$$X^{(5)} + AX^{(4)} + B\dddot{X} + \Psi(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)})\ddot{X}$$

$$+G(X)\dot{X} + F(X)X = 0$$

by introducing a Lyapunov function, where A and B are constant $n \times n$ −symmetric matrices; Ψ, G and F are $n \times n$ − symmetric continuous matrix functions depending, in each case, on the arguments shown.

This article is a generalized version of the study produced by TUNC and KARTA (2008). In this study, the results constituted give additional the result to those obtained by TUNC and KARTA (2008) for equation

$$X^{(5)} + AX^{(4)} + B\dddot{X} + \Psi\big(X, \dot{X}, \ddot{X}, \dddot{X}, X^{(4)}\big)\ddot{X}$$

$$+G(X)\dot{X} + F(X)X = 0 \qquad (1.2)$$

in the real Euclidean space $\mathbb{R}^n$ (with the usual norm denoted in what follows by $\|.\|$), in which $X \in \mathbb{R}^n$; A and B are constant $n \times n$-symmetric matrices; Ψ, F and G are $n \times n -$ symmetric continuous matrix functions depending, in each case, on the arguments shown. Let $J_G(X)$ display the Jacobian matrix corresponding to the function $G(X)$ that is,

$$J_G(X) = \left(\frac{\partial g_i}{\partial x_j}\right), \text{(i,j=1,2,…,n)}$$

in which $(x_1, x_2, \ldots, x_n)$ and $(g_1, g, \ldots, g_n)$ are the components of $X$and G, respectively. In addition, it is assumed, as basic throughout the paper, that Jacobian matrix $J_G(X)$ exist and is continuous and symmetric. The symbol $\langle X, Y\rangle$ corresponding to any pair X, Y in $\mathbb{R}^n$ stands for the usual scalar product $\sum_{i=1}^{n} x_i y_i$ and the matrix $A = (a_{ij})$ is said to be positive definite if and only if the quadratic form $X^T AX$ is positive definite, where $X \in \mathbb{R}^n$ and X^T denotes the transpose of X.

Throughout this paper, we consider the following differential systems which are equivalent to the equation (1.2) which was attained as usual by setting $\dot{X} = Y, \ddot{X} = Z, \dddot{X} = W, X^{(4)} = U$ from (1.2):

$$\dot{X} = Y, \dot{Y} = Z, \dot{Z} = W, \dot{W}\dot{U} = -AU - BW$$

$$-\Psi(X, Y, Z, W, U)Z - G(X)Y -$$

$$\mathrm{F}(X, Y, Z, W, U)X. \qquad (1.3)$$

However, with respect to our observations in the literature, even though many papers have been reviewed, there are a few examples about the subject. Therefore, we give an example to indicate the importance of the topic.

2. Main result

Our main result is the following theorem.

Theorem 2.1. In addition to the basic conditions given above for coefficients A, B, Ψ, F and G of (1.2) equation, it is assumed that A and $J_G(X)$ are symmetric matrices and there are constants a, b and a positive constant k_1 such that the following conditions provide:

(i) $\lambda_i(A) > a,\ \lambda_i(B) \geq b,$

$bsgna > 0; \quad (i = 1,2, \ldots, n)$

(ii) $\lambda_i\big(F(X, Y, Z, W, U)\big)sgna$

$-\frac{1}{4|a|}\left[\lambda_i\big(\Psi(X, Y, Z, W, U)\big)\right]^2 > k_1$

$(i = 1,2, \ldots, n)$

Then trivial solution $X = 0$ of (1.2) is instability.

Now, in order to prove main result, we use the following lemma.

Lemma2.2. Let A be a real symmetric $n \times n -$matrix and

$$a^{\iota} \geq \lambda_i(A) \geq a > 0 \quad ,(i = 1,2, \ldots, n) \quad (2.1)$$

in which a^{ι}, a are constants.

Then

$$\langle X, X\rangle \geq \langle AX, X\rangle \geq a\langle X, X\rangle\,,$$

$$a^{\iota^2}\langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2\langle X, X\rangle. \quad (2.2)$$

See HORN (1994) for proof.

Proof of Theorem 2.1. As basic tool for proof of Theorem (2.1), we will use Lyapunov function $V(X, Y, Z, W, U)$ given as

$$V = V_0(X, Y, Z, W, U)sgna \qquad (2.3)$$

in which

$$V_0 = \langle Y, W\rangle + \langle Y, AZ\rangle - \langle X, U\rangle$$

$$-\langle X, AW\rangle - \langle X, BZ\rangle - \frac{1}{2}\langle Z, Z\rangle$$

$$+\frac{1}{2}\langle BY,Y\rangle-\int_0^1\langle\sigma G(\sigma X)X,X\rangle d\sigma \quad (2.4)$$

under the conditions of Theorem 2.1, it will be indicated that the Lyapunov function $V(X,Y,Z,W,U)$ satisfies the entire KRASOVSKII (1955) criteria:

(K_1) In every neighborhood of (0,0,0,0,0), there exists a point $(\xi,\eta,\varsigma,\mu,\rho)$ such as $V\ (\xi,\eta,\varsigma,\mu,\rho)>0$.

(K_2) The time derivative $\dot{V}=(\frac{d}{dt})V(X,Y,Z,W,U)$ along solution paths of system (1.3) is positive semidefinite.

(K_3) The only solution

$$V(X,Y,Z,W,U)$$

$$=(X(t),Y(t),Z(t),W(t),U(t))$$

of system (1.3) which satisfies $\dot{V}=0\ (t\geq 0)$ is the trivial solution (0,0,0,0,0). These properties guarantee that the trivial solution of (1.2) is unstable. It is clear from (2.3) and (2.4) that $V_1(0,0,0,0,0)$=0. Additionally, it is easy to see that

$$V(0,\varepsilon,sgna,0,\varepsilon,0)$$

$$=\|\varepsilon\|^2+\frac{1}{2}bsgna\|\varepsilon\|^2>0$$

for all arbitrary $\varepsilon\neq 0,\varepsilon\in\mathbb{R}^n$. If this happens, it displays (K_1) feature of Krasoskii (1955). Let

$$(X,Y,Z,W,U)=$$

$$\big(X(t),Y(t),Z(t),W(t),U(t)\big)$$

be an arbitrary solution of system (1.3). Differentiating (2.3) with respect to t, along this solution, taking into account the conditions of the Theorem 2.1, we obtain

$$\dot{V}_0=\langle AZ,Z\rangle+\langle F(X,Y,Z,W,U)X,X\rangle$$

$$+\langle X,\Psi(X,Y,Z,W,U)Z\rangle. \quad (2.5)$$

It follows from (2.3) and (2.7) that

$$\dot{V}=sgna\langle Z,AZ\rangle$$

$$+\ sgna\langle X,F(X,Y,Z,W,U)X\rangle$$

$$+sgna\langle X,\Psi(X,Y,Z,W,U)Z\rangle$$

$$\geq|a|\langle Z,Z\rangle+sgna\langle X,F(X,Y,Z,W,U)X\rangle$$

$$+sgna\langle X,\Psi(X,Y,Z,W,U)Z\rangle$$

$$=|a|\left\|Z+\frac{1}{2|a|}\Psi(X,Y,Z,W,U)Xsgna\right\|^2$$

$$+[sgna\langle X,F(X,Y,Z,W,U)X\rangle$$

$$-\frac{1}{4|a|}\langle\Psi(X,Y,Z,W,U)X,\Psi(X,Y,Z,W,U)X\rangle\Big]$$

$$\geq[sgna\langle X,F(X,Y,Z,W,U)X\rangle$$

$$-\frac{1}{4|a|}\langle\Psi(X,Y,Z,W,U)X,\Psi(X,Y,Z,W,U)X\rangle\Big]$$

$$\geq k_1\|X\|^2 \quad (2.6)$$

by (i) and (ii).

Taking into account the conditions of the Theorem 2.1, we deduce from (2.8) that $\dot{V}(t)\geq 0$ for all $t\geq 0$, that is, $\dot{V}$ is positive semi-definite. This shows that (K_2) feature of KRASOVSKII (1955) is satisfied. Furthermore, $\dot{V}=0(t\geq 0)$ necessarily implies that $X=0$ for all $t\geq 0$, and also $,Z=\dot{Y}=0,W=\ddot{Y}=0,\dot{W}=\dddot{Y}=0$, for all $t\geq 0$.Thus, it follows that estimates

$X=Y=\ Z=W=U=0$ for all $t\geq 0$.

If this happens, it shows (K_3) feature of KRASOVSKII (1955) is satisfied. As a result, taking into account the conditions of Theorem 2.1, the function V ensures the entire the criteria of KRASOVSKII (1955). Thus, the fundamenta properties of the function $V(X,Y,Z,W,U)$,which were proved above imply that the zero solution of system (1.3) is unstable. The system (1.3) is equivalent to the differential equation (1.2). Therefore, the proof of Theorem (2.1) is complete.

Now, we give an example for Theorem (2.1).

Example:

As special cases of system (1.3), if we take for $n = 2$,

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}, A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} 1 & -\frac{1}{1 + x_1{}^2 + y_1{}^2} \\ -\frac{1}{1 + x_1{}^2 + y_1{}^2} & 1 \end{bmatrix}$$

$$\Psi = \begin{bmatrix} x_1{}^2 + y_1{}^2 + z_1{}^2 & 1 \\ 1 & x_1{}^2 + y_1{}^2 + z_1{}^2 \end{bmatrix},$$

$$G = \begin{bmatrix} x_1{}^2 + x_1{}^2 & x_1{}^2 \\ x_1{}^2 & x_1{}^2 + x_1{}^2 \end{bmatrix}$$

Then, respectively, we have

$$\lambda_1(A) = 1, \quad \lambda_2(A) = 5,$$

$$\lambda_1(B) = 1, \quad \lambda_2(B) = 3,$$

$$\lambda_1(\Psi) = \frac{x_1{}^2 + y_1{}^2}{\mathbf{1} + x_1{}^2 + y_1{}^2},$$

$$\lambda_2(\Psi) = \frac{2 + x_1{}^2 + y_1{}^2}{1 + x_1{}^2 + y_1{}^2}$$

$$\lambda_1(F) = x_1{}^2 + y_1{}^2 + z_1{}^2 + 3,$$

$$\lambda_1(F) = x_1{}^2 + y_1{}^2 + z_1{}^2 + 5$$

$$\lambda_1(G) = x_1{}^2, \quad \lambda_2(G) = 2x_1{}^2 + x_2{}^2$$

Hence, following inequalities are obtained:

(i) $\lambda_1(A) > 1,\ \lambda_2(B) \geq 1,\ bsgna > 0$

and

$$\lambda_1(F) - \frac{1}{4}\lambda_1(\Psi)^2$$

$$= x_1{}^2 + y_1{}^2 + z_1{}^2 + 3 - \frac{1}{4}\frac{(x_1{}^2 + y_1{}^2)^2}{(1 + x_1{}^2 + y_1{}^2)^2}$$

$$= \frac{4(x_1{}^2 + y_1{}^2 + z_1{}^2 + 3)(1 + x_1{}^2 + y_1{}^2)^2}{4(1 + x_1{}^2 + y_1{}^2)^2}$$

$$- \frac{(x_1{}^2 + y_1{}^2)^2}{4(1 + x_1{}^2 + y_1{}^2)^2} > 0$$

$$\lambda_2(F) - \frac{1}{4}\lambda_2(\Psi)^2$$

$$= x_1{}^2 + y_1{}^2 + z_1{}^2 + 5 - \frac{1}{4}\frac{(2 + x_1{}^2 + y_1{}^2)^2}{(1 + x_1{}^2 + y_1{}^2)^2}$$

$$= \frac{4(x_1{}^2 + y_1{}^2 + z_1{}^2 + 5)(1 + x_1{}^2 + y_1{}^2)^2}{4(1 + x_1{}^2 + y_1{}^2)^2}$$

$$- \frac{(2 + x_1{}^2 + y_1{}^2)^2}{4(1 + x_1{}^2 + y_1{}^2)^2} > 0$$

$$\lambda_2(G) - \frac{1}{4}\lambda_2(\theta)^2$$

$$= x_1{}^2 + y_1{}^2 + 1 - \frac{1}{4}\frac{\left(2x_1{}^2 + 2y_1{}^2\right)^2}{(1 + x_1{}^2 + y_1{}^2)^2}$$

$$= \frac{4(x_1{}^2 + y_1{}^2 + 1)(1 + x_1{}^2 + y_1{}^2)^2}{4(1 + x_1{}^2 + y_1{}^2)^2}$$

$$- \frac{\left(2x_1{}^2 + 2y_1{}^2\right)^2}{4(1 + x_1{}^2 + y_1{}^2)^2} > 0$$

$$\lambda_2(G) - \frac{1}{4}\lambda_2(\theta)^2$$

$$= x_1{}^2 + y_1{}^2 + 3$$

$$- \frac{1}{4}\frac{\left(4 + 2x_1{}^2 + 2y_1{}^2\right)^2}{(1 + x_1{}^2 + y_1{}^2)^2}$$

$$= \frac{4({x_1}^2 + {y_1}^2 + 3)(1 + {x_1}^2 + {y_1}^2)^2}{4(1 + {x_1}^2 + {y_1}^2)^2}$$

$$- \frac{\left(4 + 2{x_1}^2 + 2{y_1}^2\right)^2}{4(1 + {x_1}^2 + {y_1}^2)^2} > 0$$

Clearly, these last expressions imply that

$$\lambda_i\big(F(X,Y,Z,W,U)\big)sgna$$
$$- \frac{1}{4|a|}\left[\lambda_i\big(\Psi(X,Y,Z,W,U)\big)\right]^2 > k_1$$
$$(i = 1,2)$$

Thus, it is shown that all the assumptions of theorem (2.1) are provided.

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