Journal of New Results in Science (JNRS)
ISSN:1304-7981
http://dergipark.gov.tr/en/pub/jnrs
Research Article
Open Access

Volume: 8
Issue: 2
Year: 2019
Pages: 37-41

Received: 11.12.2019 ***Accepted***: 23.12.2019 ***Published***: 27.12.2019

Eigenvalues and Eigenfunctions of The Periodic Sturm-Liouville Problems with Discontinuities

Osman Yılmaz[a,*], **Mustafa Kandemir**[b], **Kadriye Aydemir**[c]

[a] *Department of Mathematics, Graduate School of Natural and Applied Sciences, Amasya, Turkey*
[b] *Department of Mathematics, Faculty of Education, Amasya University, Amasya, Turkey*
[c] *Department of Mathematics, Faculty of Science, Amasya University, Amasya, Turkey*
[*] *Corresponding author, osman.yilmaz@amasya.edu.tr*

ABSTRACT: In this study, we examine some spectral properties of a new type periodic eigenvalue problem for the differential equation $-y'' + q(x)y = \lambda y, \quad x \in [a,c) \cup (c,b]$ together with the periodic boundary conditions at the end-points $x = a, b$ given by $y(a) = y(b),\ y'(a) = y'(b)$ and with the interface conditions at the interior point of singularity $x = c$, given by $y(c+) = \alpha y(c-),\ y'(c+) = \beta y'(c-)$ where q(x) is the continuous function, α,β are real numbers and λ is complex eigenvalue parameter.

Keywords — *Periodic Sturm-Liouville problem, Transmission conditions*

1. Introduction

Eigenvalue problems associated with ordinary differential equations arise in considering physical problems,such as determining the temperature distribution of a heat conducting rod vibration problems of the wire hanging on some internal points, wave and diffusion problems and etc. by the method of separation of variables (see, for example (Woldegerima, 2011)). The typical equation that often occurs in eigenvalue problems is of the form

$$a_1(x)\frac{d^2y}{dx^2} + a_2(x)\frac{dy}{dx} + [a_3(x) + \lambda]y = 0 \tag{1.1}$$

If we introduce

$$p(x) = exp\int \frac{a_2(x)}{a_1(x)}dx, q(x) = \frac{a_3(x)}{a_1(x)}p(x), r(x) = \frac{p(x)}{a_1(x)}$$

into equation (1.1) we obtain

$$\frac{d}{dx}(p(x)\frac{dy}{dx}) + (q(x) + \lambda r(x))y = 0 \tag{1.2}$$

which is known as the Sturm-Liouville equation. In terms of the self-adjoint operator

$$Ł = \frac{d}{dx}(p\frac{d}{dx}) + q$$

the equation (1.2) can be written as

$$Ł[y] + \lambda r(x)y = 0$$

where λ is a spectral parameter independent of x, and p, q and r are real-valued functions of x. To ensure the existence of solutions, we let q and r be continuous and p be continuously differentiable in a closed finite interval $[a,b]$. Recall that the Sturm-Liouville equation (1.2) is called regular in the interval $[a,b]$ if the functions $p(x)$ and $r(x)$ are positive The Sturm-Liouville equation

$$\frac{d}{dx}(p(x)\frac{dy}{dx})+(q(x)+\lambda r(x))y=0,\ \ x\in[a,b]$$

in which $p(a)=p(b)$, together with the periodic boundary conditions $y(a)=y(b)$ and $y'(a)=y'(b)$ is called a periodic Sturm-Liouville system. Not that in different areas of natural sciences many problems arise in the form of boundary value problems involving interior singularities. In recent years, there has been growing interest in boundary- value problems with discontinuous coefficients(see, for example, (Allahverdiev et al., 2013; Kandemir and Mukhtarov, 2018; Mukhtarov and Aydemir, 2015; Panakhov and Sat, 2013; Yücel and Mukhtarov, 2018) and references cited therein).

In this study, we examine some spectral properties of a new type periodic eigenvalue problem for the differential equation

$$-y''+q(x)y=\lambda y,\ \ x\in[a,c)\cup(c,b] \tag{1.3}$$

together with the periodic boundary conditions at the end-points $x=a,b$ given by

$$y(a)=y(b),\ y'(a)=y'(b) \tag{1.4}$$

and with the interface conditions at the interior point of singularity $x=c$, given by

$$y(c+)=\alpha y(c-),\ y'(c+)=\beta y'(c-) \tag{1.5}$$

where q(x) is the continuous function, α,β are real numbers and λ is complex eigenvalue parameter.

2. Construction of the in terms of the Left-Hand and Right-Hand Eigensolutions

Let us consider the periodic Sturm-Liouville problem on two disjoint intervals together with additional interface conditions given by (1.3)-(1.5).

Theorem 2.1. *Let $\lambda\beta=1$. Then the discontinuous periodic Sturm-Liouville problem (1.3)-(1.5) is self adjoint.*

Proof. Consider the problem (1.3)-(1.5). Denote by Ly the differential operator

$$Ly=-y''+q(x)y$$

Let $y_1,y_2\in C^2[a,b]$ that satisfies the given boundary and interface conditions (1.4)-(1.5). We shall show that,

$$\int_a^{c-}[y_1Ly_2-y_2Ly_1]dx+\int_{c+}^{b}[y_1Ly_2-y_2Ly_1]dx=0$$

By definition (1.5)

$$Ly_1=-y_1''+qy_1$$

$$Ly_2=-y_2''+qy_2$$

Multiplying the first equality by y_2 and the second equality by y_1 and then subtracting yields,

$y_1Ly_2 - y_2Ly_1 = (y_1'y_2 - y_2'y_1)'$

Now integrating by parts over $[a,c)\cup(c,b]$ we obtain

$$\int_a^{c-}(y_1Ly_2 - y_2Ly_1)dx = [y_1'(c-)y_2(c-) - y_2'(c-)y_1(c-)] - [y_1'(a)y_2(a) - y_2'(a)y_1(a)]$$

and

$$\int_{c+}^{b}(y_1Ly_2 - y_2Ly_1)dx = [y_1'(b)y_2(b) - y_2'(b)y_1(b)] - [y_1'(c+)y_2(c+) - y_2'(c+)y_1(c+)]$$

Thus, we get

$$\int_a^{c-}(y_1Ly_2 - y_2Ly_1)dx + \int_{c+}^{b}(y_1Ly_2 - y_2Ly_1)dx = (y_1'y_2 - y_2'y_1)'\mid_a^{c-} + (y_1'y_2 - y_2'y_1)'\mid_{c+}^{b}$$

Since y_1 and y_2 satisfies the conditions (1.4)-(1.5) we obtain

$$(y_1'y_2 - y_2'y_1)'\mid_a^{c-} + (y_1'y_2 - y_2'y_1)'\mid_{c+}^{b} = 0$$

Consequently,

$$\int_a^{c-}[y_1Ly_2 - y_2Ly_1]dx + \int_{c+}^{b}[y_1Ly_2 - y_2Ly_1]dx = 0$$

Thus, the discontinuous Periodic Sturm-Liouville Problem (1.3)-(1.5)) is self-adjoint.

Theorem 2.2. *Let $\lambda.\beta = 1$. Then the eigenvalues of the discontinuous periodic Sturm-Liouville Problem (1.3)-(1.5) are real.*

Proof. Let (λ, y) be any eiganpair. Then we have

$$-y'' + q(x)y = \lambda y$$

$$-\overline{y}'' + q(x)\overline{y} = \overline{\lambda}\overline{y}$$

Multiplying the first equation by $\overline{y}$ and the second equation by y and then subtracting yields,

$$(\overline{y}'y - y'\overline{y})' = (\lambda - \overline{\lambda})y\overline{y}$$

Now integrating by parts over $[a,c)\cup(c,b]$ we obtain

$$(\overline{y}'y - y'\overline{y})\mid_a^{c-} + (\overline{y}'y - y'\overline{y})\mid_{c+}^{b} = (\lambda - \overline{\lambda})\int_a^{c-} y\overline{y}dx + (\lambda - \overline{\lambda})\int_{c+}^{b} y\overline{y}dx$$

Thus, we get

$$(\overline{y}'y - y'\overline{y})\mid_a^{c-} = [\overline{y}'(c-)y(c-) - y'(c-)\overline{y}(c-)] - [\overline{y}'(a)y(a) - y'(a)\overline{y}(a)]$$

and

$$(\overline{y}'y - y'\overline{y})\mid_{c+}^{b} = [\overline{y}'(b)y(b) - y'(b)\overline{y}(b)] - [\overline{y}'(c+)y(c+) - y'(c+)\overline{y}(c+)]$$

Since (λ, y) is eiganpair and α, β are real, then $(\overline{\lambda}, \overline{y})$ must be the eigenpair of the same problem (1.3)-(1.5). Therefore, we have

$$\overline{y}(a) = \overline{y}(b),\ \overline{y}'(a) = \overline{y}'(b)$$

$\overline{y}(c+) = \overline{\alpha}\overline{y}(c-),\ \overline{y}'(c+) = \overline{\beta}\overline{y}'(c-).$

Taking in view these equalities, we have

$$(\lambda - \overline{\lambda})[\int_a^{c-} y\overline{y}dx + \int_{c+}^{b} y\overline{y}dx] = 0$$

From this it follows that

$$(\lambda - \overline{\lambda}) = 0$$

Thus, the eigenvalues of the periodic Sturm-Liouville (1.3)-(1.5) are real.

Theorem 2.3. *Let λ_n and λ_m are distinct eigenvalues of the discontinuous periodic Sturm-Liouville Boundary Value Problem (1.3)-(1.5) on $[a,c)\cup(c,b]$. Then their corresponding eigenfunctions y_n and y_m are orthogonal on $L_2[a,c)\oplus L_2(c,b]$, that is*

$$\int_a^{c-} y_n(x)y_m(x)dx + \int_{c+}^{b} y_n(x)y_m(x)dx = 0$$

Proof. By assumption

$$-y_m'' + q(x)y_m = \lambda_m y_m$$

$$-y_n'' + q(x)y_n = \lambda_n y_n.$$

Multiplying the first equation by y_n and the second equation by y_m and then subtracting yields,

$$(y_n'y_m - y_m'y_n)' = (\lambda_m - \lambda_n)y_m y_n$$

Now integrating by parts over $[a,c)\cup(c,b]$ we obtain

$$(y_n'y_m - y_m'y_n)\ |_a^{c-} + (y_n'y_m - y_m'y_n)\ |_{c+}^{b} = (\lambda_m - \lambda_n)(\int_a^{c-} y_m y_n dx) + \int_{c+}^{b} y_m y_n dx)$$

Since y_m and y_n satisfy the conditions (1.4)-(1.5) we get

$$\begin{aligned} & [y_n'(c-)y_m(c-) - y_m'(c-)y_n(c-)] - [y_n'(a)y_m(a) - y_m'(a)y_n(a)] \\ + \quad & [y_n'(b)y_m(b) - y_m'(b)y_n(b)] - [y_n'(c+)y_m(c+) - y_m'(c+)y_n(c+)] = 0 \end{aligned}$$

Consequently

$$(\lambda_m - \lambda_n)[\int_a^{c-} y_m y_n dx + \int_{c+}^{b} y_m y_n dx] = 0.$$

Using $\lambda_m \neq \lambda_n$ we find

$$\int_a^{c-} y_m y_n dx + \int_{c+}^{b} y_m y_n dx = 0$$

The proof is complete.

Theorem 2.4. *Let $y = y_1(x)$ be the solution of the equation $L_1 y := y'' + q_1(x)y = 0$ satisfying transmission conditions at the point of interaction $x = c$, given by*

$$y(c+) = \alpha y(c-),\ y'(c+) = \beta y'(c-), \tag{2.1}$$

let $y = y_2(x)$ be the solution of the equation $L_2y := -y'' + q_2(x)y = 0$ satisfying the same transmission conditions and let $\lambda\beta = 1$. If $q_2(x) < q_1(x)$ on $[a,c) \cup (c,b]$, then between any to consecutive zeros of $y_1(x)$ there is at least one zero of $y_2(x)$.

Proof. By assumption

$$y_1'' + q_1(x)y_1 = 0$$

$$y_2'' + q_2(x)y_2 = 0$$

Multiplying the first equation by y_2 and the second equation by y_1 and then subtracting yields,

$$(y_1'y_2 - y_2'y_1)' = (q_2(x) - q_1(x))y_1y_2$$

Let x_1 and x_2 with $x_1 < x_2$ be consecutive zeroes of y_1. Now integrating by ports over (x_1, x_2) we obtain

$$(y_1'y_2 - y_2'y_1)' \mid_{x_1}^{x_2} = \int_{x_1}^{x_2} (q_2(x) - q_1(x))y_1y_2dx$$

since $y_1(x_1) = y_1(x_2) = 0$,

$$y_1'(x_2)y_2(x_2) - y_1'(x_1)y_2(x_1) = \int_{x_1}^{x_2} (q_2(x) - q_1(x))y_1y_2dx \qquad (2.2)$$

Suppose,it possible, that y_2 does not have a zero on (x_1, x_2). Consider the case $a \leq x_1 < x_2 < c$. Without loss of generality we can suppose that $y_1(x) > 0$ and $y_2(x) > 0$ over (x_1, x_2). These conditions ensure that the integral on the right of (2.2) is positive. However, on the left, we have $y_1'(x_1) > 0$ and $y_1'(x_2) < 0$. These conditions ensure that the left side of (2.2) is negative, which presents us with contradiction: right-hand side > 0 and left-hand side < 0.Thus $y_2(x)$ is vanish at least once between the zeroes of $y_1(x)$. Since the conditions describing $y_1(x)$ are given, we conclude that $y_2(x)$ must change sign between $x = x_1$ and $x = x_2$.

The cases $c < x_1 < x_2 \leq b$ and $a \leq x_1 < c < x_2 \leq b$ are similar.

Acknowledgement

This work was supported by the research fund of Amasya University under grant no: FMB-BAP 19-0391.

References

Allahverdiev, P. B., Bairamov, E., Ugurlu E., 2013. Eigenparameter Dependent Sturm-Liouville Problems in Boundary Conditions with Transmission Conditions. Journal of Mathematical Analysis and Applications, 401(1): 388-396.

Kandemir, M., Mukhtarov, O. Sh., 2018. Solvability of Fourth Order Sturm-Liouville Problems with Abstract Linear Functionals in Boundary-Transmission Conditions. Mathematical Methods in the Applied Sciences, 41(10): 3643-3652.

Mukhtarov, O. Sh., Aydemir, K., 2015. Eigenfunction Expansion for Sturm-Liouville Problems with Transmission Conditions at One Interior Point. Acta Mathematica Scientia, 5(3): 639-649.

Panakhov, E. S., Sat, M., 2013. Reconstruction of Potential Function for Sturm-Liouville Operator with Coulomb Potential Boundary Value Problems, 2013(49): 1-9.

Woldegerima, W. A., 2011. The Sturm-Liouville Boundary Value Problems and Their Applications, LAP Lambert Academic Publishing, Germany.

Yücel, M., Mukhtarov, O. Sh., 2018. A New Treatment of the Decomposition Method for Nonclassical Boundary Value Problems. Journal of Advanced Physics, 7(2): 161-166.