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De-Moivre and Euler Formulae for Dual-Hyperbolic Numbers

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Abstract: In this study, we generalize the well-known formulae of de-Moivre and Euler of hyperbolic numbers to dual-hyperbolic numbers. Furthermore, we investigate the roots and powers of a dual-hyperbolic number by using these formulae. Consequently, we give some examples to illustrate the main results in this paper.

Keywords: Dual number, Hyperbolic number.

1 Introduction

The number systems of two- dimensional numbers have taken place in literature with a multi-perspective approach. The hyperbolic numbers were first introduced by J. Cockle [1] and elaborated by I.M. Yaglom [2]. At the end of the 20th century, O. Bodnar, A. Stakhov and I.S. Tkachenko revealed a hyperbolic function class with gold ratio [3]. In recent years, there have been a great number of studies referring to hyperbolic numbers [4]-[9]. One of the most important recent studies has been given by A. Harkin and J. Harkin and generalized trigonometry including complex, hyperbolic and dual numbers were studied [10]. Any hyperbolic number (or split complex number, perplex number, double number) $z = x + j\,y$ is a pair of real numbers (x, y), which consists of the real unit $+1$ and hyperbolic (unipotent) imaginary unit j satisfying $j^2 = 1, j \neq \pm 1$. Therefore, hyperbolic numbers are elements of two-dimensional real algebra

$$H = \left\{ z = x + jy \,|x, y \in R \text{ and } j^2 = 1\,(j \neq \pm 1) \right\}$$

which is generated by 1 and j. The module of a hyperbolic number z is defined by

$$|z| = \begin{cases} \mp\sqrt{x^2 - y^2} & ; \quad |x| \geq |y| \\ \mp\sqrt{y^2 - x^2} & ; \quad |x| \leq |y| \end{cases}$$

and its argument is $\varphi = \operatorname{arctanh}\left(\frac{y}{x}\right)$ and represented by $\arg(z)$. Any hyperbolic number z can be given by one of the following forms;

$$\begin{array}{l} \text{a}-)\ z = r\left(\cosh\varphi + j\sinh\varphi\right) \\ \text{b}-)\ z = r\left(\sinh\varphi + j\cosh\varphi\right). \end{array}$$

The hyperbolic number given in (a) and (b) is called the first and second type hyperbolic number, respectively, see figure 1.

On the other hand, the developments in the number theory present us new number systems including the dual numbers which are expressed by the real and dual parts similar to hyperbolic numbers. This idea was first introduced by W. K. Clifford to solve some algebraic problems [11]. Afterwards, E. Study presented different theorems with his studies on kinematics and line geometry [12].

A dual number is a pair of real numbers which consists of the real unit $+1$ and dual unit ε satisfying $\varepsilon^2 = 0$ for $\varepsilon \neq 0$. Therefore, the dual numbers are elements of two-dimensional real algebra

$$D = \left\{ z = x + \varepsilon y \mid x, y \in R, \varepsilon^2 = 0, \varepsilon \neq 0 \right\}$$

which is generated by $+1$ and ε.

Similar to the hyperbolic numbers, the module of a dual number z is defined by $|z| = |x + \varepsilon y| = |x| = r$ and its argument is $\theta = \frac{y}{x}$ and represented by $\arg(z)$. The set of all points which satisfy the equation $|z| = |x| = r > 0$ and which are on the dual plane are the lines $x = \pm r$ [2]. This circle is called the Galilean circle on a dual plane. Let S be a circle centered with O and M be a point on S. If d is the line OM, and α is the angle δ_{Od}, a Galilean circle can be seen in the following figure 2.

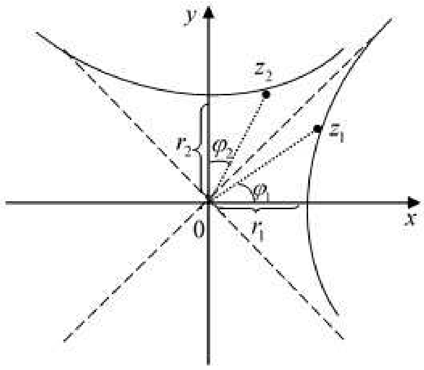

Fig. 1: Representation of hyperbolic numbers at a coordinate plane

So, one can easily see that

$$\mathrm{cosg}\alpha = \frac{|OP|}{|OM|} = 1 \quad , \quad \mathrm{sing}\alpha = \frac{|MP|}{|OM|} = \frac{\delta_{Od}}{1} = \alpha.$$

Moreover, the exponential representation of a dual number $z = x + \varepsilon y$ is in the form of $z = xe^{\varepsilon\alpha}$ where $\frac{y}{x}$ is dual angle and it is shown as $\arg(z) = \frac{y}{x} = \alpha$ [3]. In addition, from the definitions of Galilean cosine and sine, we realize

$$\mathrm{cosg}(\alpha) = 1 \text{ and } \mathrm{sing}(\alpha) = \frac{y}{x} = \alpha.$$

By considering the exponential rules, we write

$$\begin{aligned} \mathrm{cosg}(x+y) &= \mathrm{cosg}(x)\,\mathrm{cosg}(y) - \varepsilon^2 \mathrm{sing}(x)\,\mathrm{sing}(y), \\ \mathrm{sing}(x+y) &= \mathrm{sing}(x)\,\mathrm{cosg}(y) + \mathrm{cosg}(x)\,\mathrm{sing}(y), \\ \mathrm{cosg}^2(x) + \varepsilon^2 \mathrm{sing}^2(x) &= 1 \end{aligned}$$

[10].

E. Cho proved that de-Moivre formula for the hyperbolic numbers is admissible for quaternions [13]. Also, Yaylı and Kabadayı gave the de-Moivre formula for dual quaternions [14]. This formula was also investigated for the case of hyperbolic quaternions in [15]. In this study, we first introduce dual-hyperbolic numbers and algebraic expressions on dual hyperbolic numbers. We also generalize de-Moivre and Euler formulae given for hyperbolic and dual numbers to dual-hyperbolic numbers. Then we have found the roots and forces of the dual-hyperbolic numbers. Finally, the obtained results are supported by examples.

2 Dual-Hyperbolic numbers

A dual-hyperbolic number ω can be written in the form of hyperbolic pair (z_1, z_2) such that $+1$ is the real unit and ε is the dual unit. Thus, we denote dual-hyperbolic numbers set by

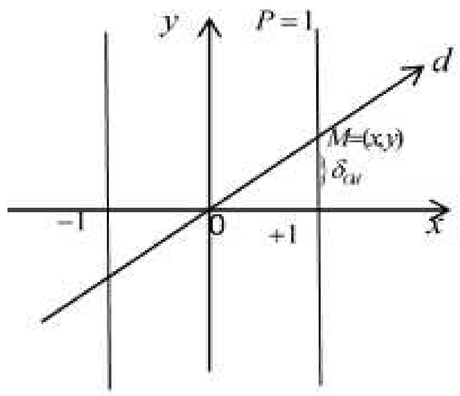

Fig. 2: Galilean unit circle

$$DH = \left\{ \omega = z_1 + \varepsilon z_2 \mid z_1, z_2 \in H \text{ and } \varepsilon^2 = 0,\ \varepsilon \neq 0 \right\}.$$

If we consider hyperbolic numbers $z_1 = x_1 + jx_2$ and $z_2 = x_3 + jx_4$, we represent a dual-hyperbolic number

$$\omega = x_1 + x_2 j + x_3 \varepsilon + x_4 \varepsilon j.$$

Here j, ε and εj are unit vectors in three-dimensional vectors space such that j is a hyperbolic unit, ε is a dual unit, and εj is a dual-hyperbolic unit [16]. So, the multiplication table of dual-hyperbolic numbers' base elements is given below.

$\times$	1	j	ε	$j\varepsilon$
1	1	j	ε	$j\varepsilon$
j	j	1	$j\varepsilon$	ε
ε	ε	$j\varepsilon$	0	0
$j\varepsilon$	$j\varepsilon$	ε	0	0

Table 1 Multiplication Table of Dual-Hyperbolic Numbers

We define addition and multiplication on dual-hyperbolic numbers as follows

$$\begin{array}{l} \omega_1 + \omega_2 = (z_1 \pm \varepsilon z_2) + (z_3 \pm \varepsilon z_4) = (z_1 \pm z_3) + \varepsilon (z_2 \pm z_4), \\ \omega_1 \times \omega_2 = (z_1 + \varepsilon z_2) \times (z_3 + \varepsilon z_4) = z_1 z_3 + \varepsilon (z_1 z_4 + z_2 z_3) \end{array}$$

where ω_1 and ω_2 are dual-hyperbolic numbers and $z_1, z_2, z_3, z_4 \in H$. On the other hand, the division of two dual-hyperbolic numbers is

$$\frac{\omega_1}{\omega_2} = \frac{z_1 + \varepsilon z_2}{z_3 + \varepsilon z_4} = \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{{z_3}^2},$$

where $\operatorname{Re}(\omega_2) \neq 0$.

Thus, dual-hyperbolic numbers yield a commutative ring whose characteristic is 0. If we consider both algebraic and geometric properties of dual-hyperbolic numbers, we define five possible conjugations of dual-hyperbolic numbers. These are

$$\begin{array}{ll} \omega^{\dagger_1} = \bar{z}_1 + \varepsilon \bar{z}_2, & (\text{hyperbolic conjugation}), \\ \omega^{\dagger_2} = z_1 - \varepsilon z_2, & (\text{dual conjugation}), \\ \omega^{\dagger_3} = \bar{z}_1 - \varepsilon \bar{z}_2, & (\text{coupled conjugation}), \\ \omega^{\dagger_4} = \overline{z_1} \left(1 - \varepsilon \frac{z_2}{z_1}\right) & (\omega \in DH - A), \quad (\text{dual} - \text{hyperbolic conjugation}), \\ \omega^{\dagger_5} = z_2 - \varepsilon z_1, & (\text{anti} - \text{dual conjugation}), \end{array}$$

where "-" denotes the standard hyperbolic conjugation and the zero divisors of DH is defined by the set A [17].

In regards to these definitions, we give the following proposition for modules of dual-hyperbolic numbers.

Proposition 1. *Let $\omega = z_1 + \varepsilon z_2$ be a dual-hyperbolic number. Then we write*

$$\begin{array}{l} |\omega|_{\dagger_1}^2 = \omega \times \omega^{\dagger_1} = |z_1|^2 + 2\varepsilon \operatorname{Re}(z_1 \bar{z}_2) \in D \\ |\omega|_{\dagger_2}^2 = \omega \times \omega^{\dagger_2} = z_1^2 \in H \\ |\omega|_{\dagger_3}^2 = \omega \times \omega^{\dagger_3} = |z_1|^2 - 2j\varepsilon \operatorname{Im}(z_1 \bar{z}_2) \in DH \\ |\omega|_{\dagger_4}^2 = \omega \times \omega^{\dagger_4} = |z_1|^2 \in R \ (\omega \in DH - A) \\ |\omega|_{\dagger_5}^2 = \omega \times \omega^{\dagger_5} = z_1 z_2 + \varepsilon \left(z_2^2 - z_1^2\right) \in DH \end{array}$$

[17].

3 De-Moivre and Euler formulae for Dual-Hyperbolic number

The exponential representation of a dual-hyperbolic number is $\omega = z_1\, e^{\frac{z_2}{z_1}\varepsilon}$, where $\omega = z_1 + \varepsilon z_2 \in DH$ is a dual-hyperbolic number and $(z_1 \neq 0)$. The dual-hyperbolic angle $\frac{z_2}{z_1}$ is called the argument of dual-hyperbolic number and it is denoted by $\arg \omega = \frac{z_2}{z_1} = \varphi$ [17].

Theorem 1. *Let $\omega = z_1 + \varepsilon z_2 \in DH - A$ be a dual-hyperbolic number and φ be the principal argument of ω. Every dual-hyperbolic number can be written in the form of*

$$\begin{aligned} w &= z_1 e^{\varepsilon \varphi} \\ &= z_1 (\operatorname{cosg}(\varphi) + \varepsilon \operatorname{sing}(\varphi)) = \left\{ \begin{array}{l} r (\cosh \varphi + j \sinh \varphi) (\operatorname{cosg}(\varphi) + \varepsilon \operatorname{sing}(\varphi)),\ |x_1| > |y_1| \\ r (\sinh \varphi + j \cosh \varphi) (\operatorname{cosg}(\varphi) + \varepsilon \operatorname{sing}(\varphi)),\ |y_1| > |x_1| \end{array} \right. \end{aligned}$$

such that $\operatorname{cosg}(\varphi) = 1$ *and* $\operatorname{sing}(\varphi) = \varphi$.

Proof: The exponential representation of a dual-hyperbolic number $\omega = z_1 + \varepsilon z_2 \in DH - A$ is $\omega = z_1\, e^{\frac{z_2}{z_1}\varepsilon}$, where dual-hyperbolic number $\frac{z_2}{z_1}$ is the principal argument φ. Thus, if we write ω in the form of

$$\omega = z_1 e^{\varepsilon\varphi} = z_1 \left(1 + \varepsilon\varphi + \frac{(\varepsilon\varphi)^2}{2!} + \frac{(\varepsilon\varphi)^3}{3!} + ...\right)$$

from properties of the dual unit, we see that

$$\omega = z_1 e^{\varepsilon\varphi} = z_1\,(1 + \varepsilon\varphi) = z_1\,(\mathrm{cosg}(\varphi) + \varepsilon \mathrm{sing}(\varphi))\,.$$

Eventually, by considering each case of $|x_1| > |y_1|$ or $|y_1| > |x_1|$ if we substitute the hyperbolic number $z_1 = x_1 + j\, y_1 \in H$ into the last equation we get

$$\omega = \begin{cases} r\,(\cosh\varphi + j\sinh\varphi)\,(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))\,, & |x_1| > |y_1|\,, \\ r\,(\sinh\varphi + j\cosh\varphi)\,(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))\,, & |y_1| > |x_1|\,. \end{cases}$$

□

Theorem 2. *Let $\omega = z_1 + \varepsilon z_2 \in DH - A$ be a dual-hyperbolic number and* $\arg\omega = \frac{z_2}{z_1} = \varphi$. *Then* $\frac{1}{e^{\varepsilon\varphi}} = e^{\varepsilon(-\varphi)}$.

Proof: If we use the Euler formula for $\frac{1}{e^{\varepsilon\varphi}}$, we have

$$\begin{aligned} \tfrac{1}{e^{\varepsilon\varphi}} &= \frac{1}{\left(1+\varepsilon\varphi+\frac{(\varepsilon\varphi)^2}{2!}+\frac{(\varepsilon\varphi)^3}{3!}+....................\right)} \\ &= \frac{1}{\cos g(\varphi)+\varepsilon\sin g(\varphi)}. \end{aligned}$$

If we multiply both the numerator and the denominator of the last fraction by $\mathrm{cosg}(\varphi) - \varepsilon\mathrm{sing}(\varphi)$, we get

$$\begin{aligned} \tfrac{1}{e^{\varepsilon\varphi}} &= \frac{1}{\mathrm{cosg}(\varphi)+\varepsilon\mathrm{sing}(\varphi)}\frac{(\mathrm{cosg}(\varphi)-\varepsilon\mathrm{sing}(\varphi))}{(\mathrm{cosg}(\varphi)-\varepsilon\mathrm{sing}(\varphi))} \\ &= \frac{\mathrm{cosg}(\varphi)-\varepsilon\mathrm{sing}(\varphi)}{\mathrm{cosg}^2(\varphi)}. \end{aligned}$$

If we consider equality $\mathrm{cosg}^2(\varphi) = 1$, we have

$$\frac{1}{e^{\varepsilon\varphi}} = \mathrm{cosg}(\varphi) - \varepsilon\mathrm{sing}(\varphi).$$

This gives us the relation

$$\frac{1}{e^{\varepsilon\varphi}} = \mathrm{cosg}(\varphi) - \varepsilon\mathrm{sing}(\varphi) = \mathrm{cosg}(-\varphi) + \varepsilon\mathrm{sing}(-\varphi).$$

As a consequence, we get $\frac{1}{e^{\varepsilon\varphi}} = e^{\varepsilon(-\varphi)}$. □

Theorem 3. *Let $\omega = z_1 + \varepsilon z_2 \in DH - A$ be a dual-hyperbolic number and $\omega = z_1 e^{\varepsilon\varphi} = z_1\,(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))$ be its polar representation. Then, the equation*

$$\omega^n = \left(z_1 e^{\varepsilon\varphi}\right)^n = (z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))^n = {z_1}^n\,(\mathrm{cosg}(n\varphi) + \varepsilon\mathrm{sing}(n\varphi))$$

yields for all non-negative integers.

Proof: First, let's prove that de-Moivre formula is correct for $n \in N$. For this, under consideration the Galilean trigonometric identities, for $n = 2$ the dual-hyperbolic number $\omega = z_1 e^{\varepsilon\varphi} \in DH - A$ becomes

$$\begin{aligned} (z_1 e^{\varepsilon\varphi})^2 &= z_1\,(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))\, z_1\,(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)) \\ &= {z_1}^2 \left(\mathrm{cosg}^2(\varphi) + \varepsilon\,(\mathrm{cosg}(\varphi)\mathrm{sing}(\varphi) + \mathrm{sing}(\varphi)\mathrm{cosg}(\varphi))\right) \\ &= {z_1}^2\,(\mathrm{cosg}(2\varphi) + \varepsilon\mathrm{sing}(2\varphi))\,. \end{aligned}$$

Suppose that the equality is true for $n = k$, that is,

$$(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))^k = {z_1}^k\,(\mathrm{cosg}(k\varphi) + \varepsilon\mathrm{sing}(k\varphi))\,.$$

Then for the case $n = k + 1$, we find

$$\begin{aligned} (z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))^{k+1} &= z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi))^k\,(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)) \\ &= {z_1}^k\,(\mathrm{cosg}(k\varphi) + \varepsilon\mathrm{sing}(k\varphi))\, z_1\,(\mathrm{cosg}(k\varphi) + \varepsilon\mathrm{sing}(k\varphi)) \\ &= {z_1}^k\,(\mathrm{cosg}(k\varphi)\mathrm{cosg}(\varphi) + \varepsilon\,(\mathrm{cosg}(k\varphi)\mathrm{sing}(\varphi) + \mathrm{sing}(k\varphi)\mathrm{cosg}(\varphi))) \\ &= {z_1}^{k+1}\,(\mathrm{cosg}((k+1)\varphi) + \varepsilon\mathrm{sing}((k+1)\varphi))\,. \end{aligned}$$

Here $z_1^k = r^k\left(\cosh\left(k\varphi\right) + j\sinh\left(k\varphi\right)\right)$ for $|x_1| > |y_1|$ and $r = |z_1| = \mp\sqrt{{x_1}^2 - {y_1}^2}$. Moreover, $z_1^k = r^k\left(\sinh\left(k\varphi\right) + j\cosh\left(k\varphi\right)\right)$ for $|y_1| > |x_1|$ and $r = |z_1| = \mp\sqrt{{y_1}^2 - {x_1}^2}$. On the other hand, for $\omega = z_1 e^{\varepsilon\varphi} \in DH - A$ and $n \in N$ we can write

$$\begin{aligned} w^{-n} &= {z_1}^{-n}\left(\mathrm{cosg}(n\varphi) - \varepsilon\mathrm{sing}(n\varphi)\right) \\ &= {z_1}^{-n}\left(\mathrm{cosg}(-n\varphi) + \varepsilon\mathrm{sing}(-n\varphi)\right). \end{aligned}$$

Thus, for all $n \in Z$ we obtain

$$\omega^n = \left(z_1 e^{\varepsilon\varphi}\right)^n = \left(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)^n = {z_1}^n\left(\mathrm{cosg}(n\varphi) + \varepsilon\mathrm{sing}(n\varphi)\right).$$

□

Theorem 4. *The n-th degree root of ω is*

$$\sqrt[n]{\omega} = \sqrt[n]{z}\left(\mathrm{cosg}\left(\frac{\varphi}{n}\right) + \varepsilon\mathrm{sing}\left(\frac{\varphi}{n}\right)\right)$$

where $\omega = z_1 + \varepsilon z_2 \in DH - A$ is a dual-hyperbolic number.

Proof: Polar representation of $\omega = z_1 + \varepsilon z_2 \in DH - A$ is $\omega = z_1\left(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)$. From Theorem 3, we know that

$$\omega^n = \left(z_1 e^{\varepsilon\varphi}\right)^n = \left(z_1(\mathrm{cosg}(\varphi) + \varepsilon\mathrm{sing}(\varphi)\right)^n = {z_1}^n\left(\mathrm{cosg}(n\varphi) + \varepsilon\mathrm{sing}(n\varphi)\right).$$

So, we get

$$\begin{aligned} \sqrt[n]{\omega} = \omega^{\frac{1}{n}} &= z_1^{\frac{1}{n}}\left(\mathrm{cosg}\left(\tfrac{1}{n}\varphi\right) + \varepsilon\mathrm{sing}\left(\tfrac{1}{n}\varphi\right)\right) \\ &= \sqrt[n]{z_1}\left(\mathrm{cosg}\left(\tfrac{\varphi}{n}\right) + \varepsilon\mathrm{sing}\left(\tfrac{\varphi}{n}\right)\right). \end{aligned}$$

This completes the proof. □

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