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Lacunary Power Series and U_m-Numbers

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Abstract

In this work, the values of certain lacunar power series with rational coefficients for U_m-number arguments were determined to be either in a particular algebraic number field or in the set of transcendental numbers under specific circumstances in the complex numbers field. The result was also applied on some of the lacunary power series with coefficients in an algebraic number field. Roth's theorem which is the essential result in Diophantine approximation to algebraic numbers was used to reach the present results.

Keywords: Algebraic number field, power series, transcendental number, U_m-number.

1. Introduction

A classification of complex transcendental numbers in terms of polynomial approximation was introduced in 1932 by Mahler [1]. Pursuant to Mahler's classification, the set of complex numbers are separated into four disjoint subsets such as A, S, T and U numbers. Depend on approximation by algebraic numbers, another classification was suggested in 1939 by Koksma. Therefore, the set of complex numbers are separated into four subsets such as A^*, S^*, T^* and U^* numbers [2]. Wirsing [3] concluded that Mahler's and Koksma's classifications are equivalent, in other words, the classes A, S, T and U are identical to the classes A^*, S^*, T^* and U^*, respectively. The class A corresponds to the set of algebraic numbers. The class U are divided into the subclasses U_m, where m is a positive integer and $m > 1$. Furthermore, $U_m \cap U_n = \emptyset$ if $m \neq n$, and $U = \cup_{m=1}^{\infty} U_m$. The theory of transcendental numbers has been studied from many different point of view such as exponents of diophantine approximation [4, 5], and the transcendence of value of some series or functions [6, 7, 8]. Notably, it was proved by Maillet [9] that the value of a rational function with rational coefficients for Liouville number arguments is a Liouville number. Inspired by this result, Mahler [10] asks the question of "Which analytic functions $f(z)$ have the property that if X is any Liouville number, then so is $f(X)$? In particular, are there entire transcendental functions with this property?". That question has interested a lot of mathematicians [11, 12, 13, 14].

In this manuscript some particular lacunary power series with rational coefficients for some transcendental number arguments were taken into account. First of all, some properties related with the coefficients of this series were given and it was shown that the convergence radius is infinite. Afterwards, using the Roth's theorem [15] it was proved in the field of complex numbers that the values of series for U_m-number arguments are associated with either a particular algebraic number field or the set of transcendental numbers. At last, these results were applied to some series with coefficients which are from a certain particular algebraic number field. Consequently, the results in [11] were generalized for the power series for Liouville number arguments to lacunary power series for U_m-number arguments.

The manuscript is organized in the three parts. In Section 2, some notations and some basic results are given. In Section 3, the main results of this manuscript are presented. In Section 3.1, some particular lacunary power series whose coefficients are rational numbers are focused on. It is proved that the values of these series for U_m-number arguments are either an element of a particular algebraic number field or a transcendental number. In Section 3.2, the values of some lacunary power series whose coefficients are in a particular algebraic number field are investigated.

2. Materials and Methods

The height $H(G)$ for a polynomial G in $\mathbb{Z}[x]$ is the maximum of the absolute values of the coefficients of

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G. The height $H(\alpha)$ and the degree $deg(\alpha)$ of the algebraic number α are defined to be the height and the degree of the minimal polynomial of α, respectively.

Let ξ be a complex number. ξ is termed as a Liouville number if for each positive integer n, there exists the rational numbers $\frac{a_n}{b_n}$ $(b_n > 1)$ such that the inequality

$$0 < \left| \xi - \frac{a_n}{b_n} \right| < b_n^{-n}$$

holds true. The set of Liouville numbers coincides with the set of U_1 numbers.

Roth [15] found an important result regarding with the approximation of irrational algebraic numbers. Suppose let α be an irrational algebraic number and ε be an arbitrarily small positive real number. Then there exist only finitely many integer solutions a and b such that the inequality

$$\left| \alpha - \frac{a}{b} \right| < b^{-(2+\varepsilon)}$$

holds.

Lemma 2.1. ([16]) Let K be an algebraic number field of degree g. Suppose let $\alpha_1, \dots, \alpha_k$ $(k \geq 1)$ be algebraic numbers in K, and let $G(y, x_1, \dots, x_k)$ be a polynomial with integral coefficients so that the degree of G in y is at least 1, and η be an algebraic number. If $G(\eta, \alpha_1, \dots, \alpha_k) = 0$, then the degree of η is $\leq dg$, and

$$H(\eta) \leq 3^{2dg+(l_1+\dots+l_k)g} H^g H(\alpha_1)^{l_1 g} \dots H(\alpha_k)^{l_k g}$$

where $H(\eta)$ is the height of η, $H(\alpha_i)$ is the height of α_i $(i = 1, \dots, k)$, also H, l_i $(i = 1, \dots, k)$ and d are the height of G, degree of G in x_i and degree of G in y, respectively.

Lemma 2.2. ([17]) If η is an algebraic number, then

$$|\eta| \leq H(\eta) + 1,$$

where $H(\eta)$ is the height of η.

3. Results and Discussion

In this section we assumed that the obtained results hold true for all sufficiently large n unless otherwise specified.

3.1. Lacunary Power Series with Rational Coefficients

In this subsection, we deal with the lacunary power series

$$g(x) = \sum_{k_n=0}^{\infty} c_{k_n} x^{k_n} \tag{3.1}$$

where $c_{k_n} = \frac{b_{k_n}}{d_{k_n}}$ (d_{k_n} and b_{k_n} are integers; $d_{k_n} > 1$ for sufficiently large n) is a non-zero rational number and $\{k_n\}_{n=0}^{\infty}$ is an increasing sequence of natural numbers.

Lemma 3.1. Assume the notations given above. If

$$\sigma := \liminf_{n\to\infty} \frac{\log d_{k_{n+1}}}{\log d_{k_n}} > 1 \tag{3.2}$$

then we have

$$\lim_{n\to\infty} \log d_{k_n} = +\infty \quad \text{and} \quad \lim_{n\to\infty} d_{k_n} = +\infty. \tag{3.3}$$

Proof: As a consequence of (3.2), we get that there is a sufficiently small ϵ_1 (> 0) such that $\sigma_1 = \sigma - \epsilon_1 > 1$.

Now, one arrive at

$$\log d_{k_{n+1}} > \sigma_1 \log d_{k_n}, \tag{3.4}$$

since $d_{k_n} > 1$. From (3.4) we get (3.3).

Lemma 3.2. Assume the notations given above. Let $D_{k_n} := [d_{k_0}, d_{k_1}, \dots, d_{k_n}]$ and $\epsilon_1 > 0$ be a sufficiently small number such that $\sigma_1 = \sigma - \epsilon_1 > 1$.

Then we have

$$d_{k_n} \leq D_{k_n} < C_0 \, d_{k_n}^{\left(\frac{\sigma_1}{\sigma_1 - 1}\right)}, \tag{3.5}$$

where $C_0 > 0$ is a suitable number.

Proof: From (3.4) we get (3.5).

Lemma 3.3. Assume the notations given above. If

$$\theta := \liminf_{n\to\infty} \frac{\log|b_{k_n}|}{\log d_{k_n}} < 1, \tag{3.6}$$

then the convergence radius of the series $g(x)$ in (3.1) is infinite.

Proof: Using (3.6) we arrive at

$$|b_{k_n}| < d_{k_n}^{\theta+\epsilon_2}, \tag{3.7}$$

where $\epsilon_2 > 0$ is a sufficiently small number such that $\theta + \epsilon_2 < 1$. From (3.3), (3.4) and (3.7), the convergence radius of the series $g(x)$ is infinite.

Theorem 3.1 Let $g(x)$ be the lacunary power series in (3.1) satisfying the conditions (3.2), (3.6) and

$$\lim_{n\to\infty} \frac{\log d_{k_n}}{k_n} < 1. \tag{3.8}$$

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Let $\mathfrak{H} = \mathbb{Q}(\xi)$ be an algebraic number field such that $[\mathfrak{H}:\mathbb{Q}] = m$ and α_{k_n} be algebraic number of degree m over $\mathfrak{H}$. Then let ζ be a U_m-number for which the following properties hold: ζ has an approximation with algebraic numbers α_{k_n} so that the following inequalities are true

$$|\zeta - \alpha_{k_n}| < \frac{1}{H(\alpha_{k_n})^{k_n w(k_n)}}, \quad \textbf{(3.9)}$$

$\lim_{n\to\infty} w(k_n) = \infty$ and

$$d_{k_n}^{s_1} \leq H(\alpha_{k_n})^{k_n} \leq d_{k_n}^{s_2}, \quad \textbf{(3.10)}$$

where s_1 and s_2 are real numbers such that $1 < s_1 \leq s_2$. If $\sigma(\sigma-1)(1-\theta) > 2m(\sigma + (\sigma-1)(1+\theta+s_2))$, then $g(\zeta)$ is either in $\mathfrak{H}$ or in the set of transcendental numbers.

Proof: Let us consider the polynomial

$$g_n(\mathrm{x}) = \sum_{v=0}^{n} c_{k_v} x^{k_v} \quad (n = 1, 2, \ldots).$$

Therefore we get $g_n(\alpha_{k_n}) \in K$ and then

$$\deg\left(g_n(\alpha_{k_n})\right) \leq m \quad (n = 1, 2, \ldots).$$

Taking

$$G(z, \mathrm{x}) = D_{k_n} z - \sum_{v=0}^{n} D_{k_n} c_{k_v} x^{k_v},$$

we arrive at

$$H(G) = \max_{0 \leq v \leq n} \{D_{k_n}, D_{k_n}|c_{k_v}|\} = D_{k_n} B_{k_n},$$

Where

$$B_{k_n} := \max_{0 \leq v \leq n}\{1, |b_{k_v}|\}.$$

In view of (3.3) and (3.7), we obtain

$$B_{k_n} < d_{k_n}^{\theta+\epsilon_2}. \quad \textbf{(3.11)}$$

Thus we have

$$H(G) = D_{k_n} d_{k_n}^{\theta+\epsilon_2}.$$

Since

$$G(g_n(\alpha_{k_n}), \alpha_{k_n}) = 0,$$

if we consider Lemma 2.1, then from (3.5), (3.8) and (3.10) we get

$$\begin{aligned} H\left(g_n(\alpha_{k_n})\right) &\leq 3^{2m+\mathrm{m}k_n} H(\mathrm{G})^m H(\alpha_{k_n})^{\mathrm{m}k_n} \\ &\leq d_{k_n}^{\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\theta+\epsilon_2+s_2+1\right)m} \end{aligned} \quad \textbf{(3.12)}$$

In view of (3.8)-(3.11), one can find

$$\begin{aligned} |g_n(\zeta) - g_n(\alpha_{k_n})| &= \left|\sum_{v=0}^{n} c_{k_v}\left(\zeta^{k_v} - \alpha_{k_n}^{k_v}\right)\right| \\ &\leq |\zeta - \alpha_{k_n}| \sum_{v=0}^{n} |c_{k_v}|\, k_v\, (|\zeta|+1)^{k_v-1} \\ &\leq \frac{1}{2}\, d_{k_n}^{-\{s_1 w(k_n) - \theta - \epsilon_2 - \epsilon_3\}}, \end{aligned} \quad \textbf{(3.13)}$$

where $\epsilon_3 > 0$ is a suitable constant. Now, using (3.12) and (3.13) we can infer

$$|g_n(\zeta) - g_n(\alpha_{k_n})| \leq \frac{\frac{1}{2}}{H\left(g_n(\alpha_{k_n})\right)^{\frac{s_1 w(k_n) - \theta - \epsilon_2 - \epsilon_3}{\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\theta+\epsilon_2+s_2+1\right)m}}} \quad \textbf{(3.14)}$$

By use of (3.4), (3.7) and (3.8) we have

$$\begin{aligned} |g(\zeta) - g_n(\zeta)| &= \sum_{v=1}^{\infty} |c_{k_{n+v}}|\, |\zeta|^{k_{n+v}} \\ &\leq \frac{2\,|\zeta|^{k_{n+1}}}{d_{k_{n+1}}^{1-\theta-\epsilon_2}} \\ &\leq \frac{\frac{1}{2}}{d_{k_n}^{(\sigma-\epsilon_1)(1-\theta-\epsilon_2-\epsilon_4)}}, \end{aligned} \quad \textbf{(3.15)}$$

where $\epsilon_4 > 0$ is a suitable constant. Hence from (3.12) and (3.15) we find

$$|g(\zeta) - g_n(\zeta)| \leq \frac{\frac{1}{2}}{H\left(g_n(\alpha_{k_n})\right)^{\frac{(\sigma-\epsilon_1)(1-\theta-\epsilon_2-\epsilon_4)}{\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\theta+\epsilon_2+s_2+1\right)m}}}. \quad \textbf{(3.16)}$$

Since $\lim_{n\to\infty} w(k_n) = \infty$ and $s_1 > 0$, the inequality

$$s_1 w(k_n) - (\theta + \epsilon_2 + \epsilon_3) > (\sigma - \epsilon_1)(1 - \theta - \epsilon_2 - \epsilon_4)$$

hold. From (3.14) and (3.16) we get

$$|g(\zeta) - g_n(\alpha_{k_n})| \leq \frac{1}{H\left(g_n(\alpha_{k_n})\right)^{\frac{(\sigma-\epsilon_1)(1-\theta-\epsilon_2-\epsilon_4)}{\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\theta+\epsilon_2+s_2+1\right)m}}}. \quad \textbf{(3.17)}$$

For the appropriate numbers ϵ_1, ϵ_2 and ϵ_4, a suitable positive number ϵ exists such that

$$\frac{(\sigma-\epsilon_1)(1-\theta-\epsilon_2-\epsilon_4)}{\left(\frac{\sigma-\varepsilon_1}{\sigma-\varepsilon_1-1}+\theta+\epsilon_2+s_2+1\right)m} > \frac{\sigma(\sigma-1)(1-\theta)}{(\sigma+(\sigma-1)(1+\theta+s_2))\,m} - \epsilon.$$

Hence it can be found a positive number ϵ satisfies the inequality

$$\frac{\sigma(\sigma-1)(1-\theta)}{\left(\sigma+(\sigma-1)(1+\theta+s_2)\right)m}-\epsilon\geq 2+\epsilon$$

Since

$$\sigma(\sigma-1)(1-\theta)>2m\left(\sigma+(\sigma-1)(1+\theta+s_2)\right).$$

Using (3.17), we get

$$\left|g(\zeta)-g_n(\alpha_{k_n})\right|\leq\frac{1}{H\left(g_n(\alpha_{k_n})\right)^{2+\varepsilon}},\quad\textbf{(3.18)}$$

where $\varepsilon>0$ is a suitable number. If the sequence $\left(g_n(\alpha_{k_n})\right)$ is constant, then $g(\zeta)$ is in $\mathfrak{H}$. Otherwise, pursuant to Roth's theorem [15], $g(\zeta)$ is a transcendental number.

3.2. Lacunary Power Series with Algebraic Coefficients

In this subsection we concentrate our study on lacunary power series with algebraic coefficients. We extend the results in the Section 3.1 considering lacunary power series with algebraic coefficients. Suppose that $\mathfrak{H}$ is an algebraic number field of degree q and

$$g(\mathrm{x})=\sum_{k_n=0}^{\infty}\frac{\tau_{k_n}}{d_{k_n}}x^{k_n}\quad\textbf{(3.19)}$$

is a lacunary power series with non-zero algebraic coefficients, where τ_{k_n} is in $\mathfrak{H}$, d_{k_n} is a rational integer such that $d_{k_n}>1$ for sufficiently large n and $\{k_n\}_{n=0}^{\infty}$ is an increasing sequence of natural numbers.

Lemma 3.4. Assume the notations given above. If

$$\sigma:=\liminf_{n\to\infty}\frac{\log d_{k_{n+1}}}{\log d_{k_n}}>1\quad\textbf{(3.20)}$$

then we have

$$\lim_{n\to\infty}\log d_{k_n}=+\infty\quad\text{and}\quad\lim_{n\to\infty}d_{k_n}=+\infty.\quad\textbf{(3.21)}$$

Proof: The proof of Lemma 3.4 can be obtained as in the proof of Lemma 3.1.

Lemma 3.5. Assume the notations given above. Let $D_{k_n}:=\left[d_{k_0},d_{k_1},\ldots,d_{k_n}\right]$. Then we have

$$d_{k_n}\leq D_{k_n}<C_0\,d_{k_n}^{\left(\frac{\sigma_1}{\sigma_1-1}\right)},\quad\textbf{(3.22)}$$

where $C_0>0$ is a suitable number and $\epsilon_1>0$ is a sufficiently small number such that $\sigma_1=\sigma-\epsilon_1>1$.
Proof: Using (3.20) we have (3.22).

Lemma 3.6. Assume the notations given above. If

$$\theta:=\liminf_{n\to\infty}\frac{\log H(\tau_{k_n})}{\log d_{k_n}}<1,\quad\textbf{(3.23)}$$

then the convergence radius of the series $g(x)$ in (3.19) is infinite.

Proof: From (3.23) we get

$$H(\tau_{k_n})<d_{k_n}^{\theta+\epsilon_2},\quad\textbf{(3.24)}$$

where $\epsilon_2>0$ is a sufficiently small number such that $\theta+\epsilon_2<1$. Using Lemma 2.2, we have

$$\left|\tau_{k_n}\right|<2{d_{k_n}}^{\theta+\epsilon_2}.\quad\textbf{(3.25)}$$

In view of (3.21) and (3.25), the convergence radius of $g(\mathrm{x})$ is infinite.

Theorem 4.1. Let $g(\mathrm{x})$ be the lacunary power series in (3.19) satisfying the conditions (3.20), (3.23) and

$$\lim_{n\to\infty}\frac{\log d_{k_n}}{k_n}=+\infty.\quad\textbf{(3.26)}$$

Let $\mathcal{L}=\mathbb{Q}(\xi)$ be an algebraic number field such that $[\mathcal{L}:\mathbb{Q}]=m$ and let α_{k_n} be algebraic number of degree m over $\mathcal{L}$. Then let ζ be a U_m-number for which the following properties hold true: ζ has an approximation with algebraic numbers α_{k_n} so that the following inequalities are true

$$\left|\zeta-\alpha_{k_n}\right|<\frac{1}{H(\alpha_{k_n})^{k_n w(k_n)}},\quad\textbf{(3.27)}$$

$\lim_{n\to\infty}w(k_n)=\infty$ and

$$d_{k_n}^{s_1}\leq H\left(\alpha_{k_n}\right)^{k_n}\leq d_{k_n}^{s_2}\quad\textbf{(3.28)}$$

for sufficiently large n, where s_1 and s_2 are two real numbers such that $1<s_1\leq s_2$. If $\sigma(\sigma-1)(1-\theta)>2t\left(\sigma(1+\theta)+(1+s)(\sigma-1)\right)$, then $g(\zeta)$ is either in $\mathcal{M}$ or a transcendental number, where $\mathfrak{H}\subseteq\mathcal{M},\mathcal{L}\subseteq\mathcal{M}$ and $[\mathcal{M}:\mathbb{Q}]=t$.

Proof. Let us start by considering the polynomials

$$g_n(\mathrm{x})=\sum_{v=0}^{n}\frac{\tau_{k_n}}{d_{k_n}}x^{k_n}\qquad(n=1,2,\ldots).$$

Since $\gamma_n=g_n(\alpha_{k_n})$ is an algebraic number in $\mathcal{M}$ we have

$$\deg(\gamma_n)\leq t\qquad(n=1,2,\ldots).$$

By using of Lemma 2.1 with the polynomial

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$$G\,(y\,,\,x_0\,,\ \ldots\,,\,x_n,x_{n+1}) = D_{k_n}y - \sum_{v=0}^{n} \frac{D_{k_n}}{d_{k_v}}\, x_v\, x_{n+1}^{k_v}$$

from (3.22), (3.24), (3.26) and (3.28) one arrive at

$$H\,(\gamma_n) \le d_{k_n}^{\left(\left(\frac{\sigma_1}{\sigma_1-1}+\epsilon_3\right)(1+\theta+\epsilon_2)+s_2+1\right)t}, \qquad \textbf{(3.29)}$$

where $\epsilon_3 > 0$ is a sufficiently small number and

$$G\,(\gamma_n\,,\tau_{k_0}\,,\ \ldots\,,\,\tau_{k_n},\tau_{k_{n+1}}) = 0.$$

In view of (3.20), (3.25), (3.26) and (3.29) we get

$$\begin{aligned}|g(\zeta) - g_n(\zeta)| &\le \frac{2\,|\zeta|^{k_{n+1}}}{d_{k_{n+1}}^{1-\theta-\epsilon_2}} \Bigg\{1 + \left(\frac{d_{k_{n+1}}}{d_{k_{n+2}}}\right)^{1-\theta-\epsilon_2} |\zeta|^{k_{n+2}-k_{n+1}} + \cdots\Bigg\} \\ &\le \frac{4\,|\zeta|^{k_{n+1}}}{d_{k_{n+1}}^{1-\theta-\epsilon_2}} \\ &\le \frac{\frac{1}{2}}{d_{k_n}^{(\sigma-\epsilon_1)(1-\theta-\epsilon_2-\epsilon_4)}} \\ &\le \frac{\frac{1}{2}}{H\,(\gamma_n)^{\frac{\sigma_1(1-\theta-\epsilon_2-\epsilon_4)}{\left(\left(\frac{\sigma_1}{\sigma_1-1}+\epsilon_3\right)(1+\theta+\epsilon_2)+s_2+1\right)t}}}, \end{aligned} \qquad \textbf{(3.30)}$$

where $\epsilon_4 > 0$ is a sufficiently small number. As a consequence of (3.21), (3.25)-(3.29) it follows that

$$\begin{aligned}|g_n(\zeta) - \gamma_n| &\le |\zeta - \alpha_{k_n}| \sum_{v=0}^{n} |\tau_{k_n}|\, k_v\, (|\zeta|+1)^{k_v-1} \\ &\le \frac{\frac{1}{2}}{d_{k_n}^{s_1 w(k_n)-(3+\theta+\epsilon_2)}} \\ &\le \frac{\frac{1}{2}}{H\,(\gamma_n)^{\frac{s_1 w(k_n)-(3+\theta+\epsilon_2)}{\left(\left(\frac{\sigma_1}{\sigma_1-1}+\epsilon_3\right)(1+\theta+\epsilon_2)+s_2+1\right)t}}}. \end{aligned} \qquad \textbf{(3.31)}$$

From (3.27), (3.30) and (3.31) we get

$$|g_n(\zeta) - \gamma_n| \le \frac{1}{H\,(\gamma_n)^{\frac{\sigma_1(1-\theta-\epsilon_2-\epsilon_4)}{\left(\left(\frac{\sigma_1}{\sigma_1-1}+\epsilon_3\right)(1+\theta+\epsilon_2)+s_2+1\right)t}}}. \qquad \textbf{(3.32)}$$

Since $\sigma(\sigma-1)(1-\theta) > 2t\big(\sigma(1+\theta)+(1+s_2)(\sigma-1)\big)$, there is a suitable number $\epsilon > 0$ such that

$$\begin{aligned}\frac{\sigma_1(1-\theta-\epsilon_2-\epsilon_4)}{\left(\left(\frac{\sigma_1}{\sigma_1-1}+\epsilon_3\right)(1+\theta+\epsilon_2)+s_2+1\right)t} &> \frac{\sigma(\sigma-1)(1-\theta)}{\big(\sigma(1+\theta)+(1+s_2)(\sigma-1)\big)\,t} - \epsilon \\ &\ge 2+\epsilon \end{aligned} \qquad \textbf{(3.33)}$$

for the suitable numbers $\epsilon_1, \epsilon_2, \epsilon_3$ and ϵ_4. Now, it can be shown that

$$|g(\zeta) - \gamma_n| \le \frac{1}{H\,(\gamma_n)^{2+\varepsilon}}, \qquad \textbf{(3.34)}$$

where ϵ is a suitable positive number. If the sequence γ_n is constant, then $g(\zeta)$ is in $\mathcal{M}$. Otherwise, as stated in Roth's theorem [15], $g(\zeta)$ is a transcendental number.

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Ethics

There are no ethical issues after the publication of this manuscript.

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