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Paranorm Ideal Convergent Fibonacci Difference Sequence Spaces

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Abstract
In this paper we introduce some new sequence spaces $c_0^I(\hat{F},p)$, $c^I(\hat{F},p)$ and $\ell_\infty^I(\hat{F},p)$ for $p=(p_n)$, a sequence of positive real numbers. In addition, we study some topological and algebraic properties on these spaces. Lastly, we examine some inclusion relations on these spaces.

Keywords: Fibonacci difference matrix, I–Cauchy, I–convergence, Paranormed space
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1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ be the sets of all natural, real and complex numbers, respectively. We denote

$$\omega = \{x=(x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$$

the vector space of all real or complex sequences. Any vector subspace of ω is called a sequence space.

Definition 1.1. *Let X be a linear space. A function $g : X \to \mathbb{R}$, is called paranorm if for all $x,y \in X$,*

(i) $g(x) \geq 0$ for all $x \in X$,

(ii) $g(-x) = g(x)$,

(iii) $g(x+y) \leq g(x)+g(y), \forall x,y \in X$,

(iv) (c_n) is a sequence of scalars with $c_n \to c (n\to\infty)$ and (x_n) is a sequence of vetors with $g(x_n - x) \to 0$ as $(n\to\infty)$, then $g(x_n c_n - xc) \to 0$ as $(n\to\infty)$.

A paranorm g which $g(x) = 0$ implies that $x = \theta$ is called a total paranorm and the pair (X,g) is called a totally paranormed space. The concept of paranorm is related to the linear metric spaces given by some total paranorm [1]. The notion of paranormed sequence was studied at the initial stage by Nakano[2] and Simons [3]. Later on it was investigated by Maddox [4, 5] and others [6]. Tripathy and Hazarika [7] generalized the sequence spaces of Maddox to introduced the new idea of paranorm I–convergent sequence spaces $c_0^I(p)$, $c^I(p)$, $\ell_\infty^I(p)$ and $\ell_\infty(p)$ where $p=(p_n)$ is the sequence of strictly positive real numbers.

Initially, as a generalization of statistical convergence which was first introduced by Fast [8] and Steinhaus [9] for real and complex sequences, the notation of ideal convergence (I-convergence) was introduced and studied by Kostyrko et al.[10].

Recall that a family of sets $I \subseteq 2^{\mathbb{N}}$ is called an ideal if (i) for each $A, B \in I \Rightarrow A \cup B \in I$, (ii) for each $A \in I, B \subseteq A \Rightarrow B \in I$. An ideal I is said to be admissible if $I \neq 2^{\mathbb{N}}$ and contains every finite subset of $\mathbb{N}$ and I is said to be maximal if there cannot exist any non–trivial ideal $J \neq I$ containing I as a subset. For each ideal I there is a filter $\mathscr{F}(I)$ which corresponds to I (filter associated with ideal I), defined by $\mathscr{F}(I) = \{K \subseteq \mathbb{N} : K^c \in I\}$. The notion of I–convergence defined in [10] as the sequence $(x_n) \in \omega$ is said to be I–convergent to a number $L \in \mathbb{C}$ if, for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |x_n - L| \geq \varepsilon\}$ belongs to I. And we write $I-\lim x_n = L$. In case $L = 0$ then $(x_n) \in \omega$ is said to be I–null. Where I assumed to be admissible. Some notions for usual convergence have been extended with respect to the admissible ideal in $\mathbb{N}$, such as the notions of bounded and Cauchy sequence extended to I–bounded and I–Cauchy defined in [11], respectively, as follows: A sequence $(x_n) \in \omega$ is said to be I–Cauchy if, for every $\varepsilon > 0$, there exists a number $N = N(\varepsilon)$ such that the set $\{n \in \mathbb{N} : |x_n - x_N| \geq \varepsilon\}$ belongs to I. A sequence $(x_n) \in \omega$ is said to be I–bounded if there exists $K > 0$, such that, the set $\{n \in \mathbb{N} : |x_n| > K\}$ belongs to I. Throughout the paper, c^I, c_0^I and ℓ_∞^I represent the I–convergent, I–null and I–bounded sequence spaces, respectively. Further, details on ideal convergence see, [12, 13, 14, 15, 16, 17] and their references.

Let λ and μ be two arbitrary sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n, k \in \mathbb{N}$. By the sequence space λ_A defined by $\lambda_A := \{x = (x_k) \in \omega : Ax \in \lambda\}$, we denote the domain of the matrix A in the space λ, the sequence $Ax = \{A_n(x)\}$ for all $x \in \lambda$, the A–transform of x, is in μ defined by $A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k$, for each $n \in \mathbb{N}$. By (λ, μ), we denote the class of all matrices A such that $\lambda \subseteq \mu_A$. Many researchers have addressed this approach to constructing a new sequence space by means of the matrix domain of a particular limitation method; see, for instance, [18, 19, 20, 21, 22, 23]. Recently, by using the sequence of Fibonacci numbers $\{f_n\}_{n=0}^{\infty}$ defined by the linear recurrence equalities $f_0 = f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}, n \geq 2$, Kara [24] defined the Fibonacci band matrix $\hat{F} = (f_{nk})$ as follows:

$$\hat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n} & , (k = n-1) \\ \frac{f_n}{f_{n+1}} & , (k = n) \\ 0 & , 0 \leq k < n-1 \text{ or } k > n \end{cases}$$

for all $n, k \in \mathbb{N}$, and introduced some new difference sequence spaces by means of the matrix $\hat{F}$. Where the notion of difference sequence spaces was firstly introduced by Kizmaz[25] for more detail [26, 27, 28, 29, 30]. Afterward, Kara and Demiriz [24] introduced the paranormed sequence spaces $c_0(\hat{F}, p)$, $c(\hat{F}, p)$ and $\ell_\infty(\hat{F}, p)$ related to the matrix domain of $\hat{F}$. i.e.,

$$c_0(\hat{F}, p) = \left\{ x = (x_n) \in \omega : \lim_{n\to\infty} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^{p_n} = 0 \right\}$$

$$c(\hat{F}, p) = \left\{ x = (x_n) \in \omega : \exists L \in \mathbb{C} \ni \lim_{n\to\infty} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^{p_n} = L \right\}$$

$$\ell_\infty(\hat{F}, p) = \left\{ x = (x_n) \in \omega : \sup_{n\in\mathbb{N}} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^{p_n} < \infty \right\}.$$

Lately, by combining the definitions of Fibonacci difference matrix $\hat{F}$ and the notion of ideal convergence, Khan et al.[13] introduced the sequence spaces $c_0^I(\hat{F})$, $c^I(\hat{F})$, and $\ell_\infty^I(\hat{F})$ defined as the set of all sequences whose $\hat{F}$–transforms are in the spaces c_0^I, c^I and ℓ_∞^I, respectively, defined as follows:

$$\lambda_{\hat{F}} = \{x = (x_k) \in \omega : \hat{F}_n(x) \in \lambda\} \text{ for } \lambda = \{c_0^I, c^I, \ell_\infty^I\},$$

where the sequence $\hat{F}_n(x)$ is frequently used as the $\hat{F}$–transform of the sequence $x = (x_n)$ defined by

$$\hat{F}_n(x) = \begin{cases} \frac{f_0}{f_1} x_0 = x_0 & , n = 0 \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} & , n \geq 1 \quad \text{for all } n \in \mathbb{N}. \end{cases} \tag{1.1}$$

In this paper, by using the Fibonacci difference matrix $\hat{F}$ and same technique we introduce the paranorm ideal convergent Fibonacci difference sequence spaces $c_0^I(\hat{F}, p)$, $c^I(\hat{F}, p)$, and $\ell_\infty^I(\hat{F}, p)$ related to the matrix domain of $\hat{F}$ in the sequence spaces $c_0^I(p)$, $c^I(p)$ and $\ell_\infty^I(p)$. Further, we study some topological and algebraic properties on these spaces and examine some inclusion relations concerning these spaces.

Definition 1.2. *[13] Let $x = (x_n)$ and $z = (z_n)$ be two sequences. We say that $x_n = z_n$ for almost all n relative to I (in short a.a.n.r.I) if the set $\{n \in \mathbb{N} : x_n \neq z_n\} \in I$.*

Definition 1.3. *[31] A sequence space E is said to be symmetric, if $(x_{\pi(n)}) \in E$ whenever $(x_n) \in E$ where $\pi(n)$ is a permutation on $\mathbb{N}$.*

Definition 1.4. *[31] A sequence space E is said to be solid or normal, if $(\alpha_n x_n) \in E$ whenever $(x_n) \in E$ and for any sequence of scalars $(\alpha_n) \in \omega$ with $|\alpha_n| < 1$, for every $n \in \mathbb{N}$.*

Definition 1.5. *[31] Let $K = \{n_i \in \mathbb{N} : n_1 < n_2 < \dots\} \subseteq \mathbb{N}$ and E be a sequence space. A K–step space of E is a sequence space*

$$\lambda_K^E = \{(x_{n_i}) \in \omega : (x_n) \in E\}.$$

A canonical pre–image of a sequence $(x_{n_i}) \in \lambda_K^E$ is a sequence $(y_n) \in \omega$ defined as follows:

$$y_n = \begin{cases} x_n & , \text{ if } n \in K \\ 0 & , \text{ otherwise.} \end{cases}$$

A canonical pre–image of a step space λ_K^E is a set of canonical pre–images of all elements in λ_K^E. i.e., y is in canonical pre–image of λ_K^E iff y is canonical pre–image of some element $x \in \lambda_K^E$.

Definition 1.6. *[31] A sequence space E is said to be monotone, if it contains the canonical pre–images of its step space.*

Lemma 1.7. *[31] Every solid space is monotone.*

Lemma 1.8. *[[31], Lemma 2.5] Let $K \in \mathscr{F}(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \notin I$.*

Lemma 1.9. *[Lascarides [32], Proposition 1] Let $h = \inf p_n$, $H = \sup_n p_n$. Then the following conditions are equivalent:*

(i) *$H < \infty$ and $h > 0$,*

(ii) *$c_0(p) = c_0$ or $\ell_\infty(p) = \ell_\infty$,*

(iii) *$\ell_\infty\{p\} = \ell_\infty(p)$,*

(iv) *$c_0\{p\} = c_0(p)$,*

(v) *$\ell\{p\} = \ell(p)$.*

2. Main results

In this section, we introduce the paranormed sequence spaces $c_0^I(\hat{F}, p)$, $c^I(\hat{F}, p)$ and $\ell_\infty^I(\hat{F}, p)$ related to the matrix domain of $\hat{F}$ in the sequence spaces $c_0^I(p)$, $c^I(p)$ and $\ell_\infty^I(p)$. Further, we study some inclusion theorems and study some topological and algebraic properties on these resulting. We assume throughout this section that the sequences $x = (x_n)$ and $(\hat{F}_n(x))$ are connected by relation (1.1) and $p = (p_n)$ be a sequence of positive real numbers and I is an admissible ideal of subset of $\mathbb{N}$. We define

$$c_0^I(\hat{F}, p) := \left\{x = (x_n) \in \omega : \left\{n \in \mathbb{N} : |\hat{F}_n(x)|^{p_n} \geq \varepsilon\right\} \in I\right\},$$

$$c^I(\hat{F}, p) := \left\{x = (x_n) \in \omega : \left\{n \in \mathbb{N} : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon, \text{ for some } L \in \mathbb{C}\right\} \in I\right\},$$

$$\ell_\infty^I(\hat{F}, p) := \left\{x = (x_n) \in \omega : \exists K > 0 \text{ s.t } \left\{n \in \mathbb{N} : |\hat{F}_n(x)|^{p_n} > K\right\} \in I\right\}.$$

We write

$$m_0^I(\hat{F}, p) := c_0^I(\hat{F}, p) \cap \ell_\infty(\hat{F}, p),$$

and

$$m^I(\hat{F}, p) := c^I(\hat{F}, p) \cap \ell_\infty(\hat{F}, p).$$

Theorem 2.1. *The sequence spaces $c^I(\hat{F},p)$, $c_0^I(\hat{F},p)$, $\ell_\infty^I(\hat{F},p)$, $m_0^I(\hat{F},p)$ and $m^I(\hat{F},p)$ are linear spaces.*

Proof. Let $x=(x_n)$, $y=(y_n)$ be two arbitrary elements of the space $c^I(\hat{F},p)$ and α,β be scalars. Now, since $x,y\in c^I(\hat{F},p)$, then for given $\varepsilon>0$, we have

$$\left\{n\in\mathbb{N}:|\hat{F}_n(x)-L_1|^{p_n}\geq\frac{\varepsilon}{2},\text{ for same } L_1\in\mathbb{C}\right\}\in I,$$

and

$$\left\{n\in\mathbb{N}:|\hat{F}_n(y)-L_2|^{p_n}\geq\frac{\varepsilon}{2},\text{ for same } L_2\in\mathbb{C}\right\}\in I.$$

Now, let

$$A_x=\left\{n\in\mathbb{N}:|\hat{F}_n(x)-L_1|^{p_n}<\frac{\varepsilon}{2M_1}\right\}\in\mathscr{F}(I),$$

$$A_y=\left\{n\in\mathbb{N}:|\hat{F}_n(y)-L_2|^{p_n}<\frac{\varepsilon}{2M_2}\right\}\in\mathscr{F}(I),$$

be such that $A_x^c,A_y^c\in I$, where $M_1=D\cdot\max\{1,\sup_n|\alpha|^{p_n}\}$, $M_2=D\cdot\max\{1,\sup_n|\beta|^{p_n}\}$ and $D=\max\{1,2^{H-1}\}$ and $H=\sup_n p_n\geq 0$. Then

$$\begin{aligned}\{n\in\mathbb{N}:|(\alpha\hat{F}_n(x)+\beta\hat{F}_n(y))-(\alpha L_1+\beta L_2)|^{p_n}<\varepsilon\}\supseteq\Bigg\{&\left\{n\in\mathbb{N}:|\alpha|^{p_n}|\hat{F}_n(x)-L_1|^{p_n}<\frac{\varepsilon}{2M_1}|\alpha|^{p_n}D\right\}\\ &\cap\left\{n\in\mathbb{N}:|\beta|^{pn}|\hat{F}_n(x)-L_2|^{p_n}<\frac{\varepsilon}{2M_2}|\beta|^{p_n}D\right\}\Bigg\}.\end{aligned}\tag{2.1}$$

Thus, the set on the right hand side of equation (2.1) belongs to $\mathscr{F}(I)$. By definition of filter associated with an ideal the complement of the set on the left hand side of (2.1) belongs to I. This implies that $(\alpha x+\beta y)\in c^I(\hat{F},p)$. Hence, $c^I(\hat{F},p)$ is a linear space. The proof for other spaces will follow similarly. □

Theorem 2.2. *The classes of sequences $m^I(\hat{F},p)$ and $m_0^I(\hat{F},p)$ are paranormed spaces, paranormed by $g(x_n)=\sup_n|x_n|^{\frac{p_n}{M}}$, where $M=\max\{1,\sup_n p_n\}$.*

Proof. The proof of the result is easy, so omitted. □

Theorem 2.3. *The set $m^I(\hat{F},p)$ is closed subspace of $\ell_\infty(\hat{F},p)$.*

Proof. Let $(x_n^{(m)})$ is a Cauchy sequence in $m^I(\hat{F},p)$ such that $(x^{(m)})\to x$. We show that $x\in m^I(\hat{F},p)$. Since $(x_n^{(m)})\in m^I(\hat{F},p)$, then there exists (a_m), and for every $\varepsilon>0$ such that

$$\{n\in\mathbb{N}:|\hat{F}_n^{(m)}(x)-a_m|^{p_n}\geq\varepsilon\}\in I.$$

We need to show that

(i) (a_m) converges to a.

(ii) If $A=\{n\in\mathbb{N}:|\hat{F}_n(x)-a|^{p_n}<\varepsilon\}$, then $A^c\in I$.

(i) Since $(x_n^{(m)})$ be a Cauchy sequence in $m^I(\hat{F},p)$ then for a given $\varepsilon>0$ there exists $n_0\in\mathbb{N}$ such that

$$\sup_n|\hat{F}_n^{(m)}(x)-\hat{F}_n^{(r)}(x)|^{\frac{p_n}{M}}<\frac{\varepsilon}{3},\text{ for all } m,r\geq n_0.$$

For a given $\varepsilon>0$, we have

$$B_{mr}=\{n\in\mathbb{N}:|\hat{F}_n^{(m)}(x)-\hat{F}_n^{(r)}(x)|^{p_n}<\left(\frac{\varepsilon}{3}\right)^M\},$$

$$B_r=\{n\in\mathbb{N}:|n\in\mathbb{N}:|\hat{F}_n^{(r)}(x)-a_r|^{p_n}<\left(\frac{\varepsilon}{3}\right)^M\},$$

$$B_m = \{n \in \mathbb{N} : |\hat{F}_n^{(m)}(x) - a_m|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\}.$$

Then $B^c_{mr},\ B^c_r,\ B^c_m \in I$. Let $B^c = B^c_{mr} \cup B^c_m \cup B^c_r$, where

$$B = \{n \in \mathbb{N} : |a_m - a_r|^{p_n} < \varepsilon\}.$$

Then $B^c \in I$. We choose $n_0 \in B^c$, then for each $m, r \geq n_0$ we have

$$\begin{aligned}\{n \in \mathbb{N} : |a_m - a_r|^{p_n} < \varepsilon\} \supseteq \Bigg[&\left\{n \in \mathbb{N} : |a_m - \hat{F}_n^{(m)}(x)|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\right\} \\ &\cap \left\{n \in \mathbb{N} : |\hat{F}_n^{(m)}(x) - \hat{F}_n^{(r)}(x)|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\right\} \\ &\cap \left\{n \in \mathbb{N} : |\hat{F}_n^{(r)}(x) - a_r|^{p_n} < \left(\frac{\varepsilon}{3}\right)^M\right\}\Bigg].\end{aligned}$$

Then (a_m) is a Cauchy sequence in $\mathbb{C}$. So, there exists a scalar $a \in \mathbb{C}$ such that $a_m \to a,$ as $m \to \infty$.
(ii) For the next step, let $0 < \delta < 1$ be given. Then, we show that if

$$A = \{n \in \mathbb{N} : |\hat{F}_n(x) - a|^{p_n} < \delta\}$$

then $A^c \in I$. Since $x^{(m)} \to x$, then there exists $q_0 \in \mathbb{N}$ such that,

$$A_1 = \{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - \hat{F}_n(x)|^{p_n} < (\frac{\delta}{3D})^M\} \tag{2.2}$$

implies $A^c_1 \in I$. The numbers q_0 can be so chosen that together with (2.2) , we have

$$A_2 = \{n \in \mathbb{N} : |a_{q_0} - a|^{p_n} < (\frac{\delta}{3D})^M\}$$

such that $A^c_2 \in I$. Since $\{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - a_{q_0}|^{p_n} \geq \delta\} \in I$, then, we have a subset A_3 of $\mathbb{N}$ such that $A^c_3 \in I$, where

$$A_3 = \left\{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - a_{q_0}|^{p_n} < (\frac{\delta}{3D})^M\right\}.$$

Let $A^c = A^c_1 \cup A^c_2 \cup A^c_3$, where $A = \{n \in \mathbb{N} : |\hat{F}_n(x) - a|^{p_n} < \delta\}$. Therefore, for each $n \in A^c$, we have

$$\begin{aligned}\{n \in \mathbb{N} : |\hat{F}_n(x)) - a|^{p_n} < \delta\} \supseteq \Bigg[&\left\{n \in \mathbb{N} : |\hat{F}_n(x) - \hat{F}_n^{(q_0)}(x)|^{p_n} < \left(\frac{\delta}{3D}\right)^M\right\} \\ &\cap \left\{n \in \mathbb{N} : |\hat{F}_n^{(q_0)}(x) - a_{q_0}|^{p_n} < \left(\frac{\delta}{3D}\right)^M\right\} \\ &\cap \left\{n \in \mathbb{N} : |a_{q_0} - a|^{p_n} < \left(\frac{\delta}{3D}\right)^M\right\}\Bigg].\end{aligned}$$

Then the result follows. □

Corollary 2.4. *The set $m^I_0(\hat{F}, p)$ is closed subspace of $\ell_\infty(\hat{F}, p)$.*

Since the inclusions $m^I(\hat{F}, p) \subset \ell_\infty(\hat{F}, p)$ and $m^I_0(\hat{F}, p) \subset \ell_\infty(\hat{F}, p)$ are strict, so in view of last theorem, we have the following result.

Theorem 2.5. *The spaces $m^I(\hat{F}, p)$ and $m^I_0(\hat{F}, p)$ are nowhere dense subsets of $\ell_\infty(\hat{F}, p)$.*

Theorem 2.6. *The spaces $c^I_0(\hat{F}, p)$ and $m^I_0(\hat{F}, p)$ are solid and monotone.*

Proof. We shall prove the result for $c_0^I(\hat{F},p)$. The other result follows similarly. Let $x=(x_k)\in c_0^I(\hat{F},p)$ and $\alpha=(\alpha_n)$ be a sequence of scalars with $|\alpha|\le 1$, for all $n\in\mathbb{N}$. Since $|\alpha|^{p_n}\le \max\{1,|\alpha_n|^{p_n}\}\le 1$, for all $n\in\mathbb{N}$, we have

$$|\hat{F}_n(\alpha x)|^{p_n}\le |\alpha\hat{F}_n(x)|^{p_n}\le |\hat{F}_n(x)|^{p_n} \text{ for all } n\in\mathbb{N}.$$

From this we have

$$\{n\in\mathbb{N}:|\hat{F}_n(\alpha x)|^{p_n}\ge\varepsilon\}\subseteq\{n\in\mathbb{N}:|\hat{F}_n(x)|^{p_n}\ge\varepsilon\}\in I$$

which implies

$$\{n\in\mathbb{N}:|\hat{F}_n(\alpha x)|^{p_n}\ge\varepsilon\}\in I.$$

Therefore, $(\alpha x_n)\in c_0^I(\hat{F},p)$. Hence, the space $c_0^I(\hat{F},p)$ is solid, and hence, by Lemma 1.7 the space $c_0^I(\hat{F},p)$ is monotone. □

Theorem 2.7. *The spaces $c^I(\hat{F},p)$, $m^I(\hat{F},p)$ are neither monotone nor solid in general.*

Proof. Here we give a counter example for establishment of this result.

Example 2.8. *Let $I=I_f=\{A\subseteq\mathbb{N}: A \text{ is finite }\}$. Let $p_n=1$ if n is even and $p_n=2$ if n is odd. Consider the K–step spaces $E_K(\hat{F},p)$ of $E(\hat{F},p)$ defined as follows: Let $x=(x_n)\in E(\hat{F},p)$ and $y=(y_n)\in E_K(\hat{F},p)$ be such that*

$$\hat{F}_n(y)=\begin{cases}\hat{F}_n(x) & \text{, if } n \text{ is even}\\ 0 & \text{, otherwise.}\end{cases}$$

Consider the sequence $x=(x_n)\in\omega$ such that $\hat{F}_n(x)=\frac{1}{n}$, for all $n\in\mathbb{N}$. Then $(x_n)\in E(\hat{F},p)$, but its K^{th}–step space pre–image does not belong to $E(\hat{F},p)$, where $E=c^I$ and m^I. Thus $c^I(\hat{F},p)$ and $m^I(\hat{F},p)$ are not monotone and hence by Lemma1.7 the spaces $c^I(\hat{F},p)$ and $m^I(\hat{F},p)$ are not solid.

□

Theorem 2.9. *Let (p_n) and (q_n) be two sequences of positive real numbers. Then $m_0^I(\hat{F},q)\subseteq m_0^I(\hat{F},p)$, if and only if $\lim_{n\in A} inf\dfrac{p_n}{q_n}>0$, where $A\subseteq\mathbb{N}$ such that $A\in\mathscr{F}(I)$.*

Proof. Let $\lim_{n\in A} inf\dfrac{p_n}{q_n}>0$ and $(x_n)\in m_0^I(\hat{F},q)$. Then there exists $\beta>0$ such that $p_n>\beta q_n$, for all sufficiently large $n\in A$. Since $(x_n)\in m_0^I(\hat{F},q)$, for a given $\varepsilon>0$, we have

$$B=\{n\in\mathbb{N}:|\hat{F}_n(x)|^{q_n}\ge\varepsilon\}\in I. \tag{2.3}$$

Let $G=A^c\cup B$. Then $G\in I$. Then for all sufficiently large $n\in G$,

$$\{n\in\mathbb{N}:|\hat{F}_n(x)|^{p_n}\ge\varepsilon\}\subseteq\{n\in\mathbb{N}:|\hat{F}_n(x)|^{\beta q_n}\ge\varepsilon\}\in I.$$

Therefore, $(x_n)\in m_0^I(\hat{F},p)$. The converse part of the result follows obviously. □

Corollary 2.10. *Let (p_n) and (q_n) be two sequences of positive real numbers. Then $m_0^I(\hat{F},p)=m_0^I(\hat{F},q)$ and only if $\lim_{n\in A}\inf p_nq_n>0$, where $A\subseteq\mathbb{N}$ such that $A\in\mathscr{F}(I)$.*

Theorem 2.11. *If I neither maximal nor $I=I_f$, then the space $H(\hat{F},p)$ are not symmetric, where $H=c_0^I, c^I, m_0^I$, and m^I.*

Proof. We prove the result with the help of the following example.

Example 2.12. *Let $I=I_c=\{A\subseteq\mathbb{N}:\sum_{n\in A}n^{-1}<\infty\}$, (see [33]). Let*

$$A=\{n: n=s^2 \text{ or } t^3, \text{ for } s,t\in\mathbb{N}\}=\{n\in\mathbb{N}: n=s^2, \text{ for } n\in\mathbb{N}\}\cup\{n\in\mathbb{N}: n=t^3, t\in\mathbb{N}\},$$

then

$$\sum_{n\in A}n^{-1}<\infty.$$

Let

$$p_n = \begin{cases} 1 & \text{, if } n \text{ is even ,} \\ 2 & \text{, if } n \text{ is odd .} \end{cases}$$

Consider the sequence $x = (x_n)$ such that

$$\hat{F}_n(x) = \begin{cases} n^{-1} & \text{, if } n = t^3, t \in \mathbb{N}, \\ 0 & \text{, otherwise.} \end{cases}$$

Consider the rearrangement $\hat{F}_n(y)$ of $\hat{F}_n(x)$ defined by

$$\hat{F}_n(y) = (\hat{F}_1(x), \hat{F}_3(x), \hat{F}_3(x), \hat{F}_8(x), \hat{F}_4(x), \hat{F}_5(x), \hat{F}_{27}(x), \hat{F}_6(x), \hat{F}_7(x), \hat{F}_{64}(x), \hat{F}_8(x), \hat{F}_9(x), \dots).$$

Then $(y_n) \notin H(\hat{F}, p)$, but $(x_n) \in H(\hat{F}, p)$, where $H = c_0^I, c^I, m_0^I$, and m^I.

□

Theorem 2.13. *The spaces $m_0^I(\hat{F}, p)$ and $m^I(\hat{F}, p)$ are not separable.*

Proof. Let $A = \{m_1 < m_2 < \dots\}$ be an infinite subset of $\mathbb{N}$ such that $A \in I$. Let

$$p_n = \begin{cases} 1, & \text{if } n \in A; \\ 2, & \text{otherwise.} \end{cases}$$

Let $P = \{(\hat{F}_n(x)) : \hat{F}_n(x) = 0 \text{ or } 1, \text{if } n \in A; \hat{F}_n(x) = 0, \text{otherwise}\}$. Since A is infinite, so P is uncountable. Consider the class of open balls $B_1 = \{B(\hat{F}_n(z), \frac{1}{2}) : \hat{F}_n(z) \in P\}$. Let C_1 be an open cover of $m_0^I(\hat{F}, P)$ or $m^I(\hat{F}, p)$ containing B_1. Since B_1 is uncountable, so C_1 cannot be reduced to a countable subcover for $m_0^I(\hat{F}, p)$ as well as $m^I(\hat{F}, p)$. Thus, $m_0^I(\hat{F}, p)$ and $m^I(\hat{F}, p)$ are not separable. □

Theorem 2.14. *Let $H = \sup_n p_n < \infty$ and I be a maximal admissible ideal. Then the following are equivalent:*

(a) $(x_n) \in c^I(\hat{F}, p)$,

(b) *There exists $(y_n) \in c(\hat{F}, p)$ such that $x_n = y_n$, for a.a.n.r.I,*

(c) *There exists $(y_n) \in c(\hat{F}, p)$ and $(z_n) \in c_0^I(\hat{F}, p)$ such that $x_n = y_n + z_n$ for all $n \in \mathbb{N}$ and $\{n \in \mathbb{N} : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon\} \in I$.*

(d) *There exists a subset $K = \{n_i : i \in \mathbb{N}, n_1 < n_2 < n_3 < \dots\}$ of $\mathbb{N}$, such that $K \in \mathscr{F}(I)$ and* $\lim_{n \to \infty} |\hat{F}_{n_i}(x) - L|^{p_{n_i}} = 0$.

Proof. **(a) implies (b).** Let $x = (x_n) \in c^I(\hat{F}, p)$, then for any $\varepsilon > 0$, there exists a number $L \in \mathbb{C}$ such that

$$\{n \in \mathbb{N} : |\hat{F}_n(x) - L|^{p_n} \geq \varepsilon\} \in I.$$

Let (m_t) be an increasing sequence with $m_t \in \mathbb{N}$ such that

$$\{n \leq m_t : |\hat{F}_n(x) - L|^{p_n} \geq t^{-1}\} \in I.$$

Define a sequence $y = (y_n)$ as $y_n = x_n$ for all $n \leq m_1$. For $m_t < n < m_{t+1}$, for $t \in \mathbb{N}$,

$$y_n = \begin{cases} x_n, & \text{if } |\hat{F}_n(x) - L|^{p_n} < t^{-1} \\ L, & \text{otherwise.} \end{cases}$$

Then $(y_n) \in c(\hat{F}, p)$ and from the following inclusion

$$\{n \leq m_t : x_n \neq y_n\} \subseteq \{n \in \mathbb{N} : |\hat{F}_n(x) - L| \geq \varepsilon\} \in I$$

we get $x_n = y_n$ for *a.a.n.r.I*.

(b) implies (c). For $x=(x_n)\in c^I(\hat{F},p)$ there exists $y=(y_n)\in c(\hat{F},p)$ such that $x_n=y_n$ for $a.a.n.r.I$. Let $K=\{n\in\mathbb{N}:x_n\neq y_n\}$, then $K\in I$. Define a sequence $z=(z_n)$ as follows:

$$z_n=\begin{cases}x_n-y_n, & \text{if } n\in K\\ 0, & \text{otherwise.}\end{cases}$$

Then $(z_n)\in c_0^I(\hat{F},p)$ and so $(y_n)\in c(\hat{F},p)$.

(c) implies (d). Suppose (c) holds. Let $\varepsilon>0$ be given. Let $P=\{n\in\mathbb{N}:|\hat{F}_n(x)|^{p_n}\geq\varepsilon\}\in I$, and

$$K=P^c=\{(n_i\in\mathbb{N}:i\in\mathbb{N},n_1<n_2<n_3<\ldots\}\in\mathscr{F}(I).$$

Then we have

$$\lim_{i\to\infty}|\hat{F}_{n_i}(x)-L|^{p_{n_i}}=0.$$

(d) implies (a). Let $\varepsilon>0$ be given and suppose that (c) holds. Then for any $\varepsilon>0$, and by Lemma 1.9 we have

$$\{n\in\mathbb{N}:|\hat{F}_n(x)-L|^{p_n}\geq\varepsilon\}\subseteq K^c\cup\{n\in K:|\hat{F}_n(x)-L|^{p_n}\geq\varepsilon\}.$$

Thus $(x_n)\in c^I(\hat{F},p)$.

□

Theorem 2.15. *The sequence spaces:*

(i) *$c^I(\hat{F},p)$ and $\ell_\infty(\hat{F},p)$ overlap but neither one contains the other,*

(ii) *$c_0^I(\hat{F},p)$ and $\ell_\infty(\hat{F},p)$ overlap but neither one contains the other.*

Proof. **(i)** We prove that $c^I(\hat{F},p)$ and $\ell_\infty(\hat{F},p)$ are not disjoint. Consider the sequence $x=(x_n)\in\omega$ such that $\hat{F}_n(x)=\frac{1}{n}$ for $n\in\mathbb{N}$. Then, $x\in c^I(\hat{F},p)$ but $x\in\ell_\infty(\hat{F},p)$. Next, define the sequence $x=(x_n)\in\omega$ such that

$$\hat{F}_n(x)=\begin{cases}\sqrt{n}, & \text{if } n \text{ is square}\\ 0, & \text{otherwise.}\end{cases}$$

Thus, $x\in c^I(\hat{F},p)$ but $x\notin\ell_\infty(\hat{F},p)$. Next, choose the sequence $x=(x_n)\in\omega$ such that

$$\hat{F}_n(x)=\begin{cases}n, & \text{if is even}\\ 0, & \text{otherwise.}\end{cases}$$

Then $(x)\in\ell_\infty(\hat{F},p)$ but $x\notin c^I(\hat{F},p)$.

(ii) The proof is similar to proof of part one.

□

3. Conclusion

In this paper, we defined some new paranorm ideal convergent Fibonacci difference sequence spaces $c_0^I(\hat{F},p)$, $c^I(\hat{F},p)$ and $\ell_\infty^I(\hat{F},p)$ as the sets of all sequences are in the space $c_0^I(p)$, $c^I(p)$ and $\ell_\infty^I(p)$ respectively. Furthermore, we studied some topological and algebraic properties of these spaces such as solidity, monotonicity and overlap. Also, we provided an example to show that these, new sequence spaces are not symmetric and show that the sets $m_0^I(\hat{F},p)$ and $m^I(\hat{F},p)$ are not separable.

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