A Review on the Solutions of Difference Equations via Integer Sequences such as Fibonacci Numbers and Tribonacci Numbers

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Abstract
In this review article, we study the recent investigations on the forms of solutions of systems difference equations and difference equations in terms of well-known integer sequences such as Fibonacci numbers, Padovan numbers. We focus on the papers given some interesting relationships both between the exact solutions of difference equations and the integer sequences and between the equilibrium points of difference equations and golden ratio.

Keywords: Difference equations, Equilibrium point, Fibonacci number, Solutions, Tribonacci numbers
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1. Introduction

Difference equations and systems of difference equations are great importance in the field of mathematics as well as in other sciences. The applications of the theory of difference equations appear as discrete mathematical models of many phenomena such as in biology, economics, ecology, control theory, physics, engineering, population dynamics and so forth. Recently, there has been a growing interest in the study of finding closed-form solutions of difference equations and systems of difference equations. Some of the forms of solutions of these equations are representable via well-known integer sequences such as Fibonacci numbers, Lucas numbers, Pell numbers and Padovan numbers.

Now, we give information about integer sequences that establish a large part of our study.

- The Fibonacci sequence is defined by

$$F_n = F_{n-1} + F_{n-2}, \;\; n \geq 2 \tag{1.1}$$

with initial conditions $F_0 = 0$, $F_1 = 1$. Also, it is obtained to extend the Fibonacci sequence backward as

$$F_{-n} = (-1)^{n+1} F_n.$$

The characteristic equation of (1.1) is $x^2 - x - 1 = 0$ such that the roots

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ (golden ratio) and } \beta = \frac{1 - \sqrt{5}}{2}.$$

Also, there exists the following limit

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha,$$

where F_n is nth Fibonacci number.

- The Padovan sequence is defined by

$$P_n = P_{n-2} + P_{n-3}, \quad n \in \mathbb{N} \tag{1.2}$$

with initial conditions $P_{-2} = 0$, $P_{-1} = 0$, $P_0 = 1$.

The characteristic equation of (1.2) is $x^3 - x - 1 = 0$ such that the roots

$$
\begin{aligned}
p &= \frac{r^2 + 12}{6r} \\
q &= -\frac{r^2 + 12}{12r} - i\frac{\sqrt{3}}{2}\left(\frac{r}{6} - \frac{2}{r}\right) \\
t &= -\frac{r^2 + 12}{12r} + i\frac{\sqrt{3}}{2}\left(\frac{r}{6} - \frac{2}{r}\right)
\end{aligned}
$$

where $r = \sqrt[3]{108 + 12\sqrt{69}}$ and the unique real root is p named as plastic number. Also, there exists the following limit

$$\lim_{n\to\infty} \frac{P_{n+1}}{P_n} = p,$$

where P_n is nth Padovan number.

- Horadam sequence, a generalization of Fibonacci sequence, $(W_n(a,b;p,q))_{n\geq 0}$ or simply $(W_n)_{n\geq 0}$ is defined by

$$W_n = pW_{n-1} + qW_{n-2}, \quad W_0 = a, W_1 = b, \quad n \geq 2, \tag{1.3}$$

where a, b, p and q are arbitrary real numbers.

The characteristic equation of (1.3) is $x^2 - px - q = 0$ such that the roots

$$\lambda = \frac{p + \sqrt{p^2 + 4q}}{2} \text{ and } \mu = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

Also, there exists the following limit

$$\lim_{n\to\infty} \frac{W_{n+1}}{W_n} = \lambda,$$

where W_n is nth Horadam number.

- The generalized Padovan sequence, an extension of the padovan sequence, is defined by

$$S_n = pS_{n-2} + qS_{n-3}, \quad n \in \mathbb{N} \tag{1.4}$$

with initial conditions $S_{-2} = 0$, $S_{-1} = 0$, $S_0 = 1$, where p and q are arbitrary real numbers.

The characteristic equation of (1.4) is $x^3 - px - q = 0$ such that the roots

$$
\begin{aligned}
\phi &= \frac{R^2 + 12p}{6R} \\
\varphi &= -\frac{R^2 + 12p}{12R} + i\frac{\sqrt{3}}{2}\left(\frac{R}{6} - \frac{2p}{R}\right) \\
\psi &= -\frac{R^2 + 12p}{12R} - i\frac{\sqrt{3}}{2}\left(\frac{R}{6} - \frac{2p}{R}\right)
\end{aligned}
$$

where $R = \sqrt[3]{108q + 12\sqrt{-12p^3 + 81q^2}}$. Also, there exists the following limit

$$\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = \phi,$$

where S_n is nth generalized Padovan number.

- Generalized Tribonacci sequence is defined by

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, \quad n \geq 3 \tag{1.5}$$

with initial conditions $V_0 = a, V_1 = b, V_2 = c$ and r, s, t are real numbers.

The characteristic equation is $x^3 - rx^2 - sx - t = 0$, whose roots are

$$\begin{aligned}
\alpha &= \alpha(r,s,t) = \frac{r}{3} + A + B \\
\beta &= \beta(r,s,t) = \frac{r}{3} + \omega A + \omega^2 B \\
\gamma &= \gamma(r,s,t) = \frac{r}{3} + \omega^2 A + \omega B
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} + \sqrt{\Delta} \right)^{1/3}, \quad B = \left(\frac{r^3}{27} + \frac{rs}{6} + \frac{t}{2} - \sqrt{\Delta} \right)^{1/3} \\
\Delta &= \Delta(r,s,t) = \frac{r^3 t}{27} - \frac{r^2 s^2}{108} + \frac{rst}{6} - \frac{s^3}{27} + \frac{t^2}{4}, \\
\omega &= \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).
\end{aligned}$$

- When $r = 1$, $s = 1$, $t = 1$ and $a = 0$, $b = 1$, $c = 1$ in (1.5), Tribonacci sequence is defined by

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n, \tag{1.6}$$

with initial conditions $T_0 = 0$, $T_1 = 1$, $T_2 = 1$. Also, it can be extended the Tribonacci sequence backward (negative subscripts) as

$$T_{-n} = T_{-n+3} - T_{-n+2} - T_{-n+1}.$$

The characteristic equation of (1.6) is

$$x^3 - x^2 - x - 1 = 0$$

such that the roots

$$\begin{aligned}
\alpha &= \frac{1 + \sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}}}{3} \\
\beta &= \frac{1 + \omega\sqrt[3]{19 + 3\sqrt{33}} + \omega^2\sqrt[3]{19 - 3\sqrt{33}}}{3} \\
\gamma &= \frac{1 + \omega^2\sqrt[3]{19 + 3\sqrt{33}} + \omega\sqrt[3]{19 - 3\sqrt{33}}}{3}
\end{aligned}$$

where α is called Tribonacci constant and

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3)$$

is a primitive cube root of unity.

Furthermore, there exist the following limit

$$\lim_{n \to \infty} \frac{T_{n+r}}{T_n} = \alpha^r,$$

where $r \in \mathbb{Z}$ and T_n is the nth Tribonacci number.

- Lucas sequence is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, \ L_1 = 1.$$

- Pell sequence is defined by

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 0, \ P_1 = 1.$$

- Pell-Lucas sequence is defined by

$$P_n = 2P_{n-1} + P_{n-2}, \quad P_0 = 2, \ P_1 = 2.$$

- Jacobsthal sequence is defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \ J_1 = 1.$$

- Jacobsthal-Lucas sequence is defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 2, \ J_1 = 1.$$

- Perrin sequence is defined by

$$P_n = P_{n-2} + P_{n-3}, \quad P_0 = 3, \ P_1 = 0, \ P_2 = 2.$$

2. Literature review

In [1], Tollu et al. considered the following difference equations

$$x_{n+1} = \frac{1}{1 + x_n}, \quad y_{n+1} = \frac{1}{-1 + y_n}, \quad n = 0, 1, ..., \tag{2.1}$$

such that their solutions are associated with Fibonacci numbers, where initial conditions are $x_0 \in \mathbb{R} - \left\{ -\frac{F_{m+1}}{F_m} \right\}_{m=1}^{\infty}$ and $y_0 \in \mathbb{R} - \left\{ -\frac{F_{m+1}}{F_m} \right\}_{m=1}^{\infty}$ and F_m is the mth Fibonacci number.

They investigated the some relationships both between Fibonacci numbers and solutions of equations (2.1) and between the golden ratio and equilibrium points of equations (2.1). Then, they proved that: the solutions of equations (2.1) are given by

$$x_n = \frac{F_n + F_{n-1}x_0}{F_{n+1} + F_n x_0}, \quad y_n = \frac{F_{-n} + F_{-(n-1)}y_0}{F_{-(n+1)} + F_{-n}y_0},$$

where F_n is the nth Fibonacci number, and the nontrival solutions of equations (2.1) converge to $-\beta$ and β, so that β is conjugate to the golden ratio.

Next, Rabago [2] presented a theoretical explanation in deriving the closed-form solution of Eq. (2.1) which Tollu et al. studied in [1] and provided another approach in proving Sroysang's conjecture (2013).

Then, in [3], Yazlik et al. studied the following rational difference equation systems

$$x_{n+1} = \frac{x_{n-1} \pm 1}{y_n x_{n-1}}, \quad y_{n+1} = \frac{y_{n-1} \pm 1}{x_n y_{n-1}}, \quad n = 0, 1, ..., \tag{2.2}$$

such that their solutions associated with Padovan numbers. In their study, they obtained that the forms of solutions of system (2.2) are as follows

$$x_n = \begin{cases} \mp \frac{P_n x_{-1} y_0 \mp P_{n+1} x_{-1} + P_{n-1}}{P_{n-1} x_{-1} y_0 \mp P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \mp \frac{P_n y_{-1} x_0 \mp P_{n+1} y_{-1} + P_{n-1}}{P_{n-1} y_{-1} x_0 \mp P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

$$y_n = \begin{cases} \mp \frac{P_n y_{-1} x_0 \mp P_{n+1} y_{-1} + P_{n-1}}{P_{n-1} y_{-1} x_0 \mp P_n y_{-1} + P_{n-2}}, & \text{if } n \text{ is odd} \\ \mp \frac{P_n x_{-1} y_0 \mp P_{n+1} x_{-1} + P_{n-1}}{P_{n-1} x_{-1} y_0 \mp P_n x_{-1} + P_{n-2}}, & \text{if } n \text{ is even} \end{cases}$$

where P_n is the nth Padovan number. Also, they demonstrated that every solutions of the systems (2.2) converge to point (p,p) and $(-p,-p)$, where p is the plastic number.

Tollu et al. [4] considered the following four Riccati difference equations

$$x_{n+1} = \frac{1+x_n}{x_n}, \quad y_{n+1} = \frac{1-y_n}{y_n}, \quad u_{n+1} = \frac{1}{u_n+1}, \quad v_{n+1} = \frac{1}{v_n-1}, \tag{2.3}$$

in which the initial conditions are real numbers. They derived the formulae for the solutions of equations (2.3) are given by

$$
\begin{aligned}
x_n &= \frac{F_{n+1}x_0 + F_n}{F_n x_0 + F_{n-1}}, \\
y_n &= \frac{F_{-(n+1)}y_0 + F_{-n}}{F_{-n}y_0 + F_{-(n-1)}}, \\
u_n &= \frac{F_n + F_{n-1}u_0}{F_{n+1} + F_n u_0}, \\
v_n &= \frac{F_{-n} + F_{-(n-1)}v_0}{F_{-(n+1)} + F_{-n}v_0},
\end{aligned}
$$

where F_n is nth Fibonacci number, F_{-n} is nth negative Fibonacci number. In addition to, they stated the asymptotic behaviors of the solutions of these equations and introduced that every solutions of these equations converge to their positive or negative equilibrium points.

Also, they in [5] studied the systems of difference equations

$$x_{n+1} = \frac{1+p_n}{q_n}, \quad y_{n+1} = \frac{1+r_n}{s_n}, \quad n \in \mathbb{N}_0,$$

where each of the sequences p_n, q_n, r_n and s_n is some of the sequences x_n or y_n by their own. They solved fourteen systems out of sixteen possible systems. In particularly, the representation formulae of solutions of twelve systems were stated via Fibonacci numbers. Also, for ten systems, they expressed that the solutions of these systems tend to the unique point (α, α) where α is the golden ratio.

In [6], Halim concerned with the following systems of rational difference equations

$$x_{n+1} = \frac{1}{1+y_n}, \quad y_{n+1} = \frac{1}{1+x_n}, \quad n = 0,1,..., \tag{2.4}$$

and

$$x_{n+1} = \frac{1}{1-y_n}, \quad y_{n+1} = \frac{1}{1-x_n}, \quad n = 0,1,..., \tag{2.5}$$

initial conditions are arbitrary nonzero real numbers. He determined the form of solutions of system (2.4) as given below

$$
\begin{aligned}
x_{2n-1} &= \frac{F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}, & x_{2n} &= \frac{F_{2n}+F_{2n-1}x_0}{F_{2n+1}+F_{2n}x_0}, \\
y_{2n-1} &= \frac{F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}, & y_{2n} &= \frac{F_{2n}+F_{2n-1}y_0}{F_{2n+1}+F_{2n}y_0},
\end{aligned}
$$

and proved that the equilibrium point E of system (2.4) is globally asymptotically stable, where $E = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) = \left(\frac{1}{\alpha}, \frac{1}{\alpha}\right)$, where α is the golden ratio. Furthermore, he established the solutions of system (2.5) are periodic with period six and are unstable.

In [7], Bacani and Rabago studied the behavior of solutions of the following nonlinear difference equations

$$x_{n+1} = \frac{q}{p+x_n^v} \quad \text{and} \quad y_{n+1} = \frac{q}{-p+y_n^v}, \tag{2.6}$$

where $p, q \in \mathbb{R}^+$ and $v \in \mathbb{N}$. They proved that the solutions of equations (2.6) in case $v = 1$ are as follows

$$
\begin{aligned}
x_n &= \frac{qW_n + x_0 qW_{n-1}}{W_{n+1} + x_0 W_n}, \\
y_n &= \frac{qW_{-n} + y_0 qW_{-(n-1)}}{W_{-(n+1)} + y_0 W_{-n}},
\end{aligned}
$$

where W_n is the nth Horadam number.

In [8], Halim and Bayram investigated the solutions, stability character, and asymptotic behavior of the difference equation

$$x_{n+1} = \frac{\alpha}{\beta + \gamma x_{n-k}}, \ n \in \mathbb{N}_0,$$

where the initial conditions $x_{-k}, x_{-k+1}, ..., x_0$ are nonzero real numbers, such that its solutions are associated to Horadam numbers, which are generalized Fibonacci numbers. Firstly, they had the difference equation

$$x_{n+1} = \frac{q}{p + x_{n-k}}, \tag{2.7}$$

by putting $q = \frac{\alpha}{\gamma}$ and $p = \frac{\beta}{\gamma}$. Then, they proved that the forms of the solutions of difference equation (2.7) are as follows

$$x_{(k+1)n+i} = \frac{W_{n+1} + W_n x_{i-(k+1)}}{W_{n+2} + W_{n+1} x_{i-(k+1)}} q, \ i = 1, 2, ..., k+1,$$

where W_n is the nth Horadam number. Also, they obtained that the equilibrium point E of difference equation (2.7) is globally asymptotically stable, where $E = \frac{-p + \sqrt{p^2 + 4q}}{2}$.

Then, in [9] Halim considered the system of difference equations

$$x_{n+1} = \frac{1}{1 + y_{n-2}}, \ y_{n+1} = \frac{1}{1 + x_{n-2}}, \ n = 0, 1, ..., \tag{2.8}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1},$ and y_0 are real numbers. He presented the relationship between Fibonacci numbers and the solutions of system (2.8), i.e., the form of the solutions of system (2.8) are given by

$$x_{6n+i} = \frac{F_{2n+1} + F_{2n} y_{i-3}}{F_{2n+2} + F_{2n+1} y_{i-3}}, \quad i = 1, 2, 3,$$
$$y_{6n+i} = \frac{F_{2n+1} + F_{2n} x_{i-3}}{F_{2n+2} + F_{2n+1} x_{i-3}}, \quad i = 1, 2, 3,$$
$$x_{6n+i} = \frac{F_{2n+2} + F_{2n+1} x_{i-6}}{F_{2n+3} + F_{2n+2} x_{i-6}}, \quad i = 4, 5, 6,$$
$$y_{6n+i} = \frac{F_{2n+2} + F_{2n+1} y_{i-6}}{F_{2n+3} + F_{2n+2} y_{i-6}}, \quad i = 4, 5, 6,$$

where F_n is the nth Fibonacci number. Otherwise, he showed that the equilibrium point E of system (2.8) is globally asymptotically stable, where $E = \left(\frac{-1 + \sqrt{5}}{2}, \frac{-1 + \sqrt{5}}{2} \right)$.

El-Dessoky in [10] dealt with the following difference equation

$$x_{n+1} = a x_n + \frac{\alpha x_n x_{n-l}}{\beta x_n + \gamma x_{n-k}}, \ n = 0, 1, ..., \tag{2.9}$$

where the parameters α, β, γ and a and the initial conditions $x_{-t}, x_{-t+1}, \ ,x_{-1}$ and x_0 where $t = \max\{l, k\}$ are positive real numbers. He introduced the explicit formula of solutions of some special cases of Eq. (2.9) via Fibonacci numbers and also, discussed the global behavior of solutions of Eq. (2.9).

In [11], Halim and Rabago studied the systems of difference equaions

$$x_{n+1} = \frac{1}{\pm 1 \pm y_{n-k}}, \ y_{n+1} = \frac{1}{\pm 1 \pm x_{n-k}}, \ n, k \in \mathbb{N}_0,$$

where the initial conditions $x_{-k}, x_{-k+1}, ..., x_0, y_{-k}, y_{-k+1}, ..., y_0$ are nonzero real numbers.

Initially, they examined the form and behavior of solutions of system of difference equations

$$x_{n+1} = \frac{1}{1 + y_{n-k}}, \ y_{n+1} = \frac{1}{1 + x_{n-k}}. \tag{2.10}$$

Therefore, they determined that the exact solutions of system (2.10) are as follows

$$x_{2(k+1)n+i} = \frac{F_{2n+1} + F_{2n} y_{i-(k+1)}}{F_{2n+2} + F_{2n+1} y_{i-(k+1)}}, \quad i = 1, 2, ..., k+1,$$
$$y_{2(k+1)n+i} = \frac{F_{2n+1} + F_{2n} x_{i-(k+1)}}{F_{2n+2} + F_{2n+1} x_{i-(k+1)}}, \quad i = 1, 2, ..., k+1,$$
$$x_{2(k+1)n+i} = \frac{F_{2n+2} + F_{2n+1} x_{i-(2k+2)}}{F_{2n+3} + F_{2n+2} x_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2,$$
$$y_{2(k+1)n+i} = \frac{F_{2n+2} + F_{2n+1} y_{i-(2k+2)}}{F_{2n+3} + F_{2n+2} y_{i-(2k+2)}}, \quad i = k+2, ..., 2k+2,$$

and the equilibrium point of system (2.10) is globally asymptotically stable. In addition, the authors gave some results for other systems.

Then, in [12], the authors studied the rational difference equation

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma x_n x_{n-1}}, \ n \in \mathbb{N}_0, \tag{2.11}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, α, β, $\gamma \in \mathbb{R}^+$ and the initial conditions nonzero real numbers and also investigated the two-dimensional case of the this equation given by

$$x_{n+1} = \frac{\alpha x_{n-1} + \beta}{\gamma y_n x_{n-1}}, \ y_{n+1} = \frac{\alpha y_{n-1} + \beta}{\gamma x_n y_{n-1}}, \ n \in \mathbb{N}_0. \tag{2.12}$$

Firstly, they reduced the difference equation (2.11) to the difference equation

$$x_{n+1} = \frac{p x_{n-1} + q}{x_n x_{n-1}} \tag{2.13}$$

by using changes variables $p = \frac{\alpha}{\gamma}$ and $q = \frac{\beta}{\gamma}$. Then, they presented that the closed-form solution of difference equation (2.13) is given by

$$x_n = \frac{S_{n+1} x_{-1} + S_n x_0 x_{-1} + q S_{n-1}}{S_n x_{-1} + S_{n-1} x_0 x_{-1} + q S_{n-2}},$$

where S_n is the nth generalized Padovan number and the equilibrium point of Eq (2.13) is globally asymptotically stable.

Later, they reduced the system of difference equation (2.12) to the system

$$x_{n+1} = \frac{p x_{n-1} + q}{y_n x_{n-1}}, \ y_{n+1} = \frac{p y_{n-1} + q}{x_n y_{n-1}} \tag{2.14}$$

by using changes variables $p = \frac{\alpha}{\gamma}$ and $q = \frac{\beta}{\gamma}$. Then, they presented that the closed-form solutions of system (2.14) are given by

$$x_n = \begin{cases} \frac{S_{n+1} y_{-1} + S_n x_0 y_{-1} + q S_{n-1}}{S_n y_{-1} + S_{n-1} x_0 y_{-1} + q S_{n-2}}, & \text{if } n \text{ is even,} \\ \frac{S_{n+1} x_{-1} + S_n y_0 x_{-1} + q S_{n-1}}{S_n x_{-1} + S_{n-1} y_0 x_{-1} + q S_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}$$

$$y_n = \begin{cases} \frac{S_{n+1} x_{-1} + S_n y_0 x_{-1} + q S_{n-1}}{S_n x_{-1} + S_{n-1} y_0 x_{-1} + q S_{n-2}}, & \text{if } n \text{ is even,} \\ \frac{S_{n+1} y_{-1} + S_n x_0 y_{-1} + q S_{n-1}}{S_n y_{-1} + S_{n-1} x_0 y_{-1} + q S_{n-2}}, & \text{if } n \text{ is odd,} \end{cases}$$

and the equilibrium point of the system (2.14) is global attractor.

Then, in [13], Stevic et al. the following nonlinear second-order difference equation

$$x_{n+1} = a + \frac{b}{x_n} + \frac{c}{x_n x_{n-1}}, \ n \in \mathbb{N}_0, \tag{2.15}$$

in which parameters a, b, c and the initial values x_{-1} and x_0 are complex numbers such that $c \neq 0$. Next, they used the following change of variables

$$x_n = \frac{y_n}{y_{n-1}},$$

and obtained the following third-order linear difference equation with constant coefficients

$$y_{n+1} = a y_n + b y_{n-1} + c y_{n-2}.$$

After, they introduced that the representation formula of every solution of Eq. (2.15) is

$$x_n = \frac{(s_{n+1} - a s_n) x_{-1} + s_n x_0 x_{-1} + c s_{n-1}}{(s_n - a s_{n-1}) x_{-1} + s_{n-1} x_0 x_{-1} + c s_{n-2}},$$

where s_n is the nth generalized Padovan number. Note that, Eq. (2.11) is a special case of Eq. (2.15) such that $a = 0$.

Alotaibi et al. in [14] considered the following systems of difference equations

$$x_{n+1} = \frac{y_n y_{n-2}}{x_{n-1} + y_{n-2}}, \quad y_{n+1} = \frac{x_n x_{n-2}}{\pm y_{n-1} \pm x_{n-2}}, \quad n = 0, 1, ..., \tag{2.16}$$

where the initial conditions $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0$ are arbitrary positive real numbers. They analyzed the solutions of the systems (2.16) such that their solutions are associated with Fibonacci numbers.

In [15], El-Dessoky et al. examined the following difference equation

$$y_{n+1} = \alpha y_n + \frac{\beta y_n y_{n-3}}{A y_{n-4} + B y_{n-3}}, \quad n = 0, 1, ..., \tag{2.17}$$

where α, β, A, and B are real numbers and the initial values $y_{-4}, y_{-3}, y_{-2}, y_{-1}$ and y_0 are positive real numbers. They presented the solutions of Eq. (2.17) in terms of Fibonacci numbers according to some special cases of the parameters α, β, A, and B.

Then, in [16], Matsunaga and Suzuki studied the following system of rational difference equations

$$x_{n+1} = \frac{a y_n + b}{c y_n + d}, \quad y_{n+1} = \frac{a x_n + b}{c x_n + d}, \quad n = 0, 1, ..., \tag{2.18}$$

where the parameters a, b, c, d and the initial values x_0, y_0 are real numbers. They obtained that the explicit solutions of system (2.18) are as follows

$$
\begin{aligned}
x_{2n-1} &= \frac{(ay_0 + b) G_{2n-1} + (bc - ad) y_0 G_{2n-2}}{G_{2n} + (cy_0 - a) G_{2n-1}}, \quad x_{2n} = \frac{(ax_0 + b) G_{2n} + (bc - ad) x_0 G_{2n-1}}{G_{2n+1} + (cx_0 - a) G_{2n}}, \\
y_{2n-1} &= \frac{(ax_0 + b) G_{2n-1} + (bc - ad) x_0 G_{2n-2}}{G_{2n} + (cx_0 - a) G_{2n-1}}, \quad y_{2n} = \frac{(ay_0 + b) G_{2n} + (bc - ad) y_0 G_{2n-1}}{G_{2n+1} + (cy_0 - a) G_{2n}},
\end{aligned}
$$

where G_n is a generalized Fibonacci sequence defined by

$$G_{n+2} = (a + d) G_{n+1} + (bc - ad) G_n,$$

with $G_0 = 0$ and $G_1 = 1$. Moreover, they presented that every solution of system (2.18) converges to its equilibrium points.

In [17], Öcalan and Duman considered the following nonlinear recursive difference equation

$$x_{n+1} = \frac{x_{n-1}}{x_n}, \quad n = 0, 1, ..., \tag{2.19}$$

with any nonzero initial values x_{-1} and x_0. Then, they extended their all results to solutions of the following nonlinear recursive equations

$$x_{n+1} = \left(\frac{x_{n-1}}{x_n} \right)^p, \quad p > 0 \text{ and } n = 0, 1, ..., \tag{2.20}$$

with any nonzero initial values x_{-1} and x_0. Later, they obtained that the exact solution of Eq. (2.19) is

$$
x_n = \begin{cases}
\dfrac{x_{-1}^{f_{n-1}}}{x_0^{f_n}} & \text{if } n = 1, 3, 5, ..., \\[3mm]
\dfrac{x_0^{f_n}}{x_{-1}^{f_{n-1}}} & \text{if } n = 2, 4, 6, ...,
\end{cases}
$$

where f_n is the nth Fibonacci number. Under the special case of initial values, they determined that there exist non-oscillatory positive solutions of Eq. (2.19), which converge monotonically to the equilibrium point 1.

Furthermore, they given that the exact solution of Eq. (2.20) is

$$
x_n = \begin{cases}
\dfrac{x_{-1}^{g_{n-1}(p)}}{x_0^{f_n(p)}} & \text{if } n = 1, 3, 5, ..., \\[3mm]
\dfrac{x_0^{f_n(p)}}{x_{-1}^{f_{n-1}(p)}} & \text{if } n = 2, 4, 6, ...,
\end{cases}
$$

where $f_n(p)$ and $g_n(p)$ are the nth Fibonacci-type number. And also, under the special case of initial values, they demonstrated that there exist non-oscillatory positive solutions of Eq. (2.20), which converge monotonically to the equilibrium point 1 and the Eq. (2.20) has unbounded solutions.

Next, Akrour et al. [18] studied the following system of difference equations

$$x_{n+1} = \frac{ay_nx_{n-1} + bx_{n-1} + c}{y_nx_{n-1}}, \ y_{n+1} = \frac{ax_ny_{n-1} + by_{n-1} + c}{x_ny_{n-1}}, \ n = 0, 1, ...,$$

where the parameters a, b, c are arbitrary real numbers with $c \neq 0$ and the initial values x_{-1}, x_0, y_{-1} and y_0 are arbitrary nonzero
real numbers. They examined that the explicit solutions of system (2.10) are given by

$$x_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})x_{-1} + J_{2n+1}x_{-1}y_0},$$

$$x_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})y_{-1} + J_{2n+3}x_0y_{-1}}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}},$$

$$y_{2n+1} = \frac{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})y_{-1} + J_{2n+2}x_0y_{-1}}{cJ_{2n} + (J_{2n+2} - aJ_{2n+1})y_{-1} + J_{2n+1}x_0y_{-1}},$$

$$y_{2n+2} = \frac{cJ_{2n+2} + (J_{2n+4} - aJ_{2n+3})x_{-1} + J_{2n+3}x_{-1}y_0}{cJ_{2n+1} + (J_{2n+3} - aJ_{2n+2})x_{-1} + J_{2n+2}x_{-1}y_0},$$

where J_n is defined by the recurrent relation

$$J_{n+3} = aJ_{n+2} + bJ_{n+1} + cJ_n, \ n \in \mathbb{N},$$

such that $J_0 = 0, J_1 = 1, J_2 = a$.

Then, Okumuş and Soykan in [19] considered the following four difference equations

$$x_{n+1} = \frac{1}{x_n(x_{n-1} - 1) - 1}, \ n = 0, 1, ...,$$

$$x_{n+1} = \frac{1}{x_n(x_{n-1} + 1) + 1}, \ n = 0, 1, ..., \tag{2.21}$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} - 1) + 1}, \ n = 0, 1, ...,$$

$$x_{n+1} = \frac{-1}{x_n(x_{n-1} + 1) - 1}, \ n = 0, 1, ...,$$

and determined the solutions of these difference equations are associated to Tribonacci numbers. For example, the solutions of
Eq.(2.21) are

$$x_n = \frac{T_{n-1}x_{-1}x_0 + (T_{n+1} - T_n)x_0 + T_n}{T_nx_{-1}x_0 + (T_{n-1} + T_n)x_0 + T_{n+1}},$$

where the initial conditions $x_{-1}, x_0 \in \mathbb{R} - F_2$, with F_2 is the forbidden set of Eq.(2.21) given by

$$F_2 = \cup_{n=-1}^{\infty} \{(x_{-1}, x_0) : T_nx_{-1}x_0 + (T_{n-1} + T_n)x_0 + T_{n+1} = 0\},$$

and for the others see [19].

Also, in [21], they examined the following systems of difference equations

$$x_{n+1} = \frac{\pm 1}{y_n(x_{n-1} \pm 1) + 1}, \ y_{n+1} = \frac{\pm 1}{x_n(y_{n-1} \pm 1) + 1}, \ n = 0, 1, ...,$$

and proved the exact solutions of these systems of difference equations via Tribonacci numbers. E.g. the form of solutions
$\{x_n, y_n\}_{n=-1}^{\infty}$ of one of these systems is given by

$$x_{2n-1} = \frac{T_{2n-2}x_{-1}y_0 + (T_{2n} - T_{2n-1})y_0 + T_{2n-1}}{T_{2n-1}x_{-1}y_0 + (T_{2n-2} + T_{2n-1})y_0 + T_{2n}},$$

$$x_{2n} = \frac{T_{2n-1}y_{-1}x_0 + (T_{2n+1} - T_{2n})x_0 + T_{2n}}{T_{2n}y_{-1}x_0 + (T_{2n-1} + T_{2n})x_0 + T_{2n+1}},$$

$$y_{2n-1} = \frac{T_{2n-2}y_{-1}x_0 + (T_{2n} - T_{2n-1})x_0 + T_{2n-1}}{T_{2n-1}y_{-1}x_0 + (T_{2n-2} + T_{2n-1})x_0 + T_{2n}},$$

$$y_{2n} = \frac{T_{2n-1}x_{-1}y_0 + (T_{2n+1} - T_{2n})y_0 + T_{2n}}{T_{2n}x_{-1}y_0 + (T_{2n-1} + T_{2n})y_0 + T_{2n+1}},$$

where the initial conditions $x_{-1}, y_{-1}, x_0, y_0 \in \mathbb{R} - F_1$, with F_1 is the forbidden set of system given by

$$F_1 = \cup_{n=-1}^{\infty} \{(x_{-1}, y_{-1}, x_0, y_0) : A_n = 0 \text{ or } B_n = 0 \text{ or } C_n = 0 \text{ or } D_n = 0\}$$

where

$$
\begin{aligned}
A_n &= T_{2n-1}x_{-1}y_0 + (T_{2n-2} + T_{2n-1})y_0 + T_{2n}, \\
B_n &= T_{2n}y_{-1}x_0 + (T_{2n-1} + T_{2n})x_0 + T_{2n+1}, \\
C_n &= T_{2n-1}y_{-1}x_0 + (T_{2n-2} + T_{2n-1})x_0 + T_{2n}, \\
D_n &= T_{2n}x_{-1}y_0 + (T_{2n-1} + T_{2n})y_0 + T_{2n+1}.
\end{aligned}
$$

Next, they in [22] studied the following difference equation

$$x_{n+1} = \frac{\gamma}{x_n(x_{n-1} + \alpha) + \beta}, \quad n = 0, 1, \ldots, \tag{2.22}$$

where the parameters α, β and γ are nonnegative real numbers with $\gamma \neq 0$ and the initial values x_{-1} and x_0 are arbitrary nonzero real numbers. They examined that the exact solutions of Eq.(2.22) is given by

$$x_n = \frac{tV_{n-1}x_{-1}x_0 + (V_{n+1} - rV_n)x_0 + V_n}{tV_nx_{-1}x_0 + (V_{n+2} - rV_{n+1})x_0 + V_{n+1}},$$

where V_n is defined by the recurrent relation

$$V_{n+3} = rV_{n+2} + sV_{n+1} + tV_n, \quad n \in \mathbb{N},$$

such that $V_0 = 0$, $V_1 = 1$, $V_2 = r$.

Besides these studies, for related studies on solving difference equations and systems of difference equations and investigating the asymptotic behavior of their solutions, see [20, 23-38].

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