Communications in Advanced Mathematical Sciences
Vol. II, No. 4, 303-308, 2019
Research Article
e-ISSN: 2651-4001
DOI: 10.33434/cams.607382

The Univalent Function Created by the Meromorphic Functions Where Defined on the Period Lattice

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Abstract
The function $\xi(z)$ is obtained from the logarithmic derivative function $\sigma(z)$. The elliptic function $\wp(z)$ is also derived from the $\xi(z)$ function. The function $\wp(z)$ is a function of double periodic and meromorphic function on lattices region. The function $\wp(z)$ is also double function. The function $\varphi(z)$ meromorphic and univalent function was obtained by the serial expansion of the function $\wp(z)$. The function $\varphi(z)$ obtained here is shown to be a convex function.

Keywords: Convex function, Elliptic function, Latices, Meromorphic function
2010 AMS: 30C45

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Received: 20 August 2019, **Accepted:** 25 November 2019, **Available online:** 29 December 2019

1. Introduction

We begin this important paper by introducing some important functions and some important classes.

Definition 1.1. A *get the subset of complex numbers* $\mathbb{C}$. *If* A *is a group according to the collection process, then* A *in called a module defined on the ring of integers* $\mathbb{Z}$.

Definition 1.2. *If the module* A *does not have a stack point in the finite plane, then this module* A *is called a lattice. Lattices can be divided into three groups as follows.*

i. Zero dimensional lattices;

$$W_m = \{m\omega : m = 0 \in \mathbb{Z}, \omega \neq 0 \in \mathbb{C}\}$$

ii. One dimensional lattices;

$$W_m = \{m\omega_1 : m \neq 0 \in \mathbb{Z}, \omega \neq 0 \in \mathbb{C}\}$$

iii. Two dimensional lattices;

$$W_{m,n} = \{m\omega_1 + n\omega_2 : m \neq 0, n \neq 0 \in \mathbb{Z}, \omega_1 \neq 0, \omega_2 \neq 0 \in \mathbb{C}\}$$

Lemma 1.3. *The function* $\xi(z)$ *is absolute and uniform convergence [1].*

Proof.

$$\xi(z)=\frac{1}{z}+\sum_{m,n\neq(0,0)}(\frac{1}{z-W}+\frac{1}{W}+\frac{z}{W^2})$$

where

$$\begin{aligned}\sum_{m,n\neq(0,0)} &= \sum_m\sum_n\left|\frac{1}{z-W_{mn}}+\frac{1}{W_{mn}}-\frac{z}{(W_{mn})^2}\right|=\left|\frac{(W_{mn})^2+(z-W_{mn})W_{mn}+(1-W_{mn})z}{(z-W_{mn})(W_{mn})^2}\right|=\left|\frac{z}{(z-W_{mn})(W_{mn})^2}\right|\\ &= \left|\frac{z}{(1-\frac{z}{W_{mn}})(W_{mn})^2}\right|\leq\frac{|z|}{(1-\frac{|z|}{|W_{mn}|})\,|W_{mn}|^2}<\frac{2\,|z|}{|W_{mn}|^2}.\end{aligned}$$

For all m,n such that $|W|>2|z|$ the series under consideration in therefore absolutely and convergent.Thus, function $\xi(z)$ has a simple pole at point $z=W$. In that case, $\xi(z)$ is meromorphic. On the other hand it is clear that $\xi(z)$ in the odd function so $\xi(z)=-\xi(-z)$. □

Theorem 1.4. *The function $\xi(z)$ has following the power series for point $z=0$.*

$$\xi(z)=\frac{1}{z}-\frac{A_2}{3}-\frac{A_4}{5}-...=\frac{1}{z}-\sum_{k\geq2}\frac{A_{2k-2}}{2k-1}z^{2k-1}$$

Proof. Let

$$\xi(z)=\frac{1}{z}+\sum_{m,n\neq(0,0)}\left(\frac{1}{z-W}+\frac{1}{W}+\frac{z}{W^2}\right)$$

$$\xi(z)=\frac{1}{z}+\sum_{m,n\neq(0,0)}\left(\frac{1}{-W(1-\frac{z}{W})}+\frac{1}{W}+\frac{z}{W^2}\right)$$

then

$$\begin{aligned}\xi(z) &= \frac{1}{z}+\sum_{m,n\neq(0,0)}\left[-\frac{1}{W}(1+\frac{z}{W}+(\frac{z}{W})^2+...+\frac{1}{W}+\frac{z}{W^2}\right]\\ &= \frac{1}{z}+\sum_{m,n\neq(0,0)}\frac{1}{-\Delta_{mn}}\left[1+\frac{z}{\Delta_{mn}}+\left(\frac{z}{\Delta_{mn}}\right)^2+...+\frac{1}{\Delta_{mn}}+\left(\frac{z}{(\Delta_{mn})^2}\right)\right]\\ &= \frac{1}{z}-\sum_{m,n\neq(0,0)}\frac{1}{-\Delta_{mn}}\left[\frac{z^2}{(\Delta_{mn})^3}+\frac{z^3}{(\Delta_{mn})^4}+\frac{z^4}{(\Delta_{mn})^5}+...\right]\\ &= \frac{1}{z}+\sum_{m,n\neq(0,0)}\frac{1}{-W}\left[\frac{z^2}{W^3}+\frac{z^3}{W^4}+\frac{z^4}{W^5}+...\right]\\ &= \frac{1}{z}-\sum_{m,n\neq(0,0)}\sum_{k=2}\frac{1}{W^{k+1}}z^k=\frac{1}{z}-\sum_{k=2}A_{k+1}.z^k\\ &= \frac{1}{z}-\sum_{k\geq2}(z^2+z^3+z^4+...).A_{k+1}\end{aligned}$$

where $A_{k+1}=\sum_{m,n\neq(0,0)}$.

Coefficients of toms z^{2k} in evidently zero for k=1,2,3, since the functions $\xi(z)$ is an odd function, ie equality is as follows

$$\xi(z)=\frac{1}{z}-\frac{A_2}{3}-\frac{A_4}{5}-...=\frac{1}{z}-\sum_{k\geq2}\frac{A_{2k-2}}{2k-1}z^{2k-1}.$$

□

Definition 1.5. *Weierstrass's function $\wp(z)$ is defined by the double series as*

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n\neq(0,0)} \left[\frac{1}{(z-w)^2} + \frac{1}{W^2} \right]$$

$-\frac{d}{dz}\xi(z) = \wp(z)$ *equality can be seen here. That is to say $\wp(z)$ is double function [1].*

The function $\wp(z)$ is meromorphic function in the complex plan ($|z| < 1$) with second order poles at the lattices points $z = W$. It is in double periodic with periods ω_1 and ω_2 . This mean that $\wp(z)$ satisfies. Considering the following equality $\wp(z) = \frac{1}{z^2} + \sum_{k\geq 2} A_{2k-2}.z^{2k-2}$ for $\frac{1}{z} - \sum_{k\geq 2} \frac{A_{2k-2}}{2k-1} z^{2k-1}$ where $-\frac{d}{dz}\xi(z) = \wp(z)$. The funtions $\wp(z)$ is a meromorphic and elliptic funtion which has $z = W$ second order pole points.

Theorem 1.6. *The series $\wp(z)$ is absolutely and uniformly convergent for every $z = W$.*

Proof.

$$\left| \frac{1}{(z-W)^2} - \frac{1}{W^2} \right| = \left| \frac{W^2 - (z-W)^2}{(z-W)^2.W^2} \right| = \left| \frac{(2W-z).z}{(z-W)^2.W^2} \right| \leq \frac{|z|.\left(2|W| + \frac{|W|}{2}\right)}{\frac{1}{4}W^2W^2} = \frac{10|z|}{|W|^3}$$

where $|z| < \frac{1}{2}|W|$. Thus,

$$\sum_{m,n\neq(0,0)} \left| \frac{1}{(z-W)^2} - \frac{1}{W^2} \right| = \sum_{m,n\neq(0,0)} \frac{10|z|}{W^2}.$$

The function $\wp(z)$ is meromorphic region $|z| < 1$ whether the function $\wp(z)$ is not analytical region $|z| < 1$. If we get consecutive derivatives from the equation as

$$\wp(z) = \frac{1}{z^2} + \sum_{k\geq 2} A_{2k-2}.z^{2k-2}$$

$$\wp'(z) = -\frac{1.2}{z^3} + \sum_{k\geq 2} (2k-2).A_{2k-2}.z^{2k-3}$$

$$\wp''(z) = \frac{1.2.3}{z^4} + \sum_{k\geq 2} (2k-2).(2k-3).A_{2k-2}.z^{2k-4} \vdots$$

$$\wp^n(z) = (-1)^n \frac{(n+1)!}{z^{n+2}} + \sum_{k\geq 2} (2k-2).(2k-3)...(2k-(n+1)).A_{2k-2}.z^{2k-(n+1)}.$$

In that case

$$\wp^{2n-1}(z) = -\frac{(2n)!}{z^{2n+1}} + \sum_{k\geq 2} (2k-2).(2k-3)...(2k-2n).A_{2k-2}.z^{(2k-2n)}$$

$$\wp^{2n-1}(z) = -\frac{(2n)!}{z^{2n+1}} + \sum_{k\geq 2} (2k-2).(2k-3)...(2k-2n).A_{2k-2}.z^{(2k-2n)}$$

$$\wp^{2n-2}(z) = \frac{(n-1)!}{z^{2n+1}} + \sum_{k\geq 2} (2k-2).(2k-3)...(2k-(2n-1)).A_{2k-2}.z^{(2k-(2n-1))}$$

□

Theorem 1.7. *If α_i and β_i $(i = 1,2,\ldots\ldots..,r)$ be the zeros and poles respectively of an elliptic function $f(z)$ in a cell, then*

$$\sum_{i=1}^{r}\alpha_i \equiv \sum_{i=1}^{r}\beta_i \qquad (mod.2\omega_1, 2\omega_2)$$

where every zero or pole is counted as many times as the multiplicity indicates.

Proof. We have

$$\begin{aligned}
\sum_{i=1}^{r}\alpha_i - \sum_{i=1}^{r}\beta_i &= \frac{1}{2\pi i}\int_P \frac{zf'(z)}{f(z)}dz \qquad \text{(P is any suitably chosen contour)}\\
&= \frac{1}{2\pi i}\left[\int_{z_0}^{z_0+2\omega_1}\frac{zf'(z)}{f(z)}dz + \int_{z_0+2\omega_1}^{z_0+2\omega_1+2\omega_2}\frac{zf'(z)}{f(z)}dz + \int_{z_0+2\omega_1+2\omega_2}^{z_0+2\omega_2}\frac{zf'(z)}{f(z)}dz + \int_{z_0+2\omega_2}^{z_0}\frac{zf'(z)}{f(z)}dz\right]\\
&= \frac{1}{2\pi i}\left[\int_{z_0}^{z_0+2\omega_1}(z-(z+2\omega_2))\frac{f'(z)}{f(z)}dz + \int_{z_0}^{z_0+2\omega_2}(z+2\omega_1-z)\frac{f'(z)}{f(z)}dz\right]\\
&= \frac{1}{2\pi i}\left[2\omega_1\int_{z_0}^{z_0+2\omega_2}\frac{f'(z)}{f(z)}dz - 2\omega_2\int_{z_0}^{z_0+2\omega_1}\frac{f'(z)}{f(z)}dz\right]\\
&= \frac{1}{2\pi i}\left\{2\omega_1\Big[log f(z)\Big]_{z_0}^{z_0+2\omega_2} - 2\omega_2\Big[log f(z)\Big]_{z_0}^{z_0+2\omega_1}\right\} = \frac{1}{2\pi i}(4\pi i m\omega_1 - 4\pi i n\omega_2) = (m2\omega_1 + 2n\omega_2) \qquad (n=-n).
\end{aligned}$$

Hence we conclude

$$\sum_{i=1}^{r}\alpha_i \equiv \sum_{i=1}^{r}\beta_i \qquad (mod.2\omega_1, 2\omega_2)[1].$$

□

Theorem 1.8. *The sum, difference, product and the quotient of any two co-periodic elliptic functions are also elliptic function of the same period.*

Proof. Since $f_i(z+2\omega) = f_i(z)$, where $2\omega = 2\omega_1$ and $2\omega_2$ $(i = 1,2)$ therefore

$$f_1(z+2\omega) \pm f_2(z+2\omega) = f_1(z) \pm f_2(z)$$

$$f_1(z+2\omega).f_2(z+2\omega) = f_1(z).f_2(z)$$

$$f_1(z+2\omega)/f_2(z+2\omega) = f_1(z)/f_2(z).$$

Again since the set of all meromorphic functions forms a field and $f_1(z) \pm f_2(z)$, $f_1(z).f_2(z)$ and $f_1(z)/f_2(z)$ are meromorphic and periodic with periods $2\omega_1$ and $2\omega_2$. So they are elliptic functions with the same periods [1]. □

Theorem 1.9. *Let $f(z)$ be regular and univalent in the closed disk $D : |z| \le R$. Then $f(z)$ maps D onto a convex domain if and only if*

$$Re\left[1 + \frac{zf'(z)}{f(z)}\right] \ge 0, \qquad \text{for } z \text{ on} \qquad D : |z| \le R.$$

Suppose further that $f(0) = 0$. Then $f(z)$ maps D onto a region that is starlike with respect to $w = 0$ if and only if

$$Re\left[\frac{zf'(z)}{f(z)}\right] \ge 0, \qquad \text{for } z \text{ on} \qquad D : |z| \le R.$$

We must assume that $f(z)$ is univalent (or replace this with some order condition) or we fall into error. Indeed, suppose that $f(z)=z^2$. Then the inequality becomes for starlike $2\geq 0$ and also for convex domain becomes $2\geq 0$. $f(z)=z^2$ is not really a convex or starlike domain. The concepts of convexity and starlikeness can be extended to multi-sheeted regions, and indeed these extensions have been thoroughly explored, but for the present we consider only plane regions. We observe that if $f(z)$ is univalent in D, then $f'(z)\neq 0$ in and hence the expression on the left is a harmonic function in D and takes its minimum on the boundary D . Thus, if $f(z)$ maps D onto a closed convex curve, then for each $r<R$, $f(z)$ maps D onto a convex curve, and hence maps D onto a convex domain. The same type of reasoning can be applied because if $f(z)$ is in S, then the singularity at $z=0$ is a removable singularity [2].

Theorem 1.10. *The function $\wp(z)$ and the function $\xi(z)$ have the following equality*

$$\frac{\wp^{(2n-1)}(z_1)}{\wp^{(2n-2)}(z_1)-\wp^{(2n-2)}(z_2)}=2\xi(z_2-z_1)-2n(\xi(z_1)-\xi(z_2)).$$

Lemma 1.11. *The sum, difference, product and quotient of any co-periodic elliptic functions are also elliptic function of the same period.*

Lemma 1.12. *If the elliptic function $f(z)$ has simple pole at and only at the points $\beta_1,\beta_2,\beta_3,...,\beta_n$ in cell with residues $A_1,A_2,A_3,...,A_n$, then*

$$\wp(z)=A_0+\sum_{r=1}^{s}(z-r).A_r,$$

where A_0 is a constant. It is in the fact that a constant A_0 in zero. In that case, the function

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}$$

is an elliptical function with poles at $z_2,-z_2$. 0 with residues 1, 1, -2n respectively. If the last equation is written in place of z, then the following equation is found

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=A_0+\xi(z-z_2)+\xi(z-z_2)-2n\xi(z).$$

If in the above equation z is written instead of $(-z)$ then $\wp$ is an even function and $\xi(z)$ is an odd function

$$-\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=A_0-\xi(z+z_2)-\xi(z-z_2)+2n\xi(z).$$

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=-A_0+\xi(z+z_2)+\xi(z-z_2)-2n\xi(z)$$

equations are obtained. If $A_0=0$ and z_1 are written instead of z then the following equation is continue

$$\frac{\wp^{(2n-1)}(z)}{\wp^{(2n-2)}(z)-\wp^{(2n-2)}(z_2)}=\xi(z_1+z_2)+\xi(z_1-z_2)-2n\xi(z_1).$$

The function $\varphi(z)$ defined as follows

$$\varphi(z)=\wp(z)+\frac{z^3-1}{z^2}=z+\sum_{k\geq 2}A_{2k-2}.z^{2k-2}=z+A_2z^2+A_4z^4+...$$

The function $\varphi(z)$ is an analytical function for every $z\in|z|<1$. Also because of its $\varphi(0)=0$ and $\varphi'(0)=1$,this function is class A.

2. Main Theorem

Theorem 2.1. *The function $\varphi(z)$ is an univalent function.*

Proof. If $\varphi(z_1) - \varphi(z_2) = 0$, then

$$\varphi(z_1) - \varphi(z_2) = z_1 + \sum_{k\geq 2} A_{2k-2}.z_1^{2k-2} - z_2 - \sum_{k\geq 2} A_{2k-2}.z_2^{2k-2} = 0$$

$$(z_1 - z_2)\left(1 + \sum_{k\geq 2} A_{2k-2}(z_1^{2k-3} - z_1^{2k-4}z_2 + \ldots + z_2^{2k-3})\right) = 0$$

$$1 + \sum_{k\geq 2} A_{2k-2}(z_1^{2k-3} - z_1^{2k-4}z_2 + \ldots + z_2^{2k-3}) \neq 0$$

$z_1 - z_2 = 0$ be must because $1 + \sum_{k\geq 2} A_{2k-2}(z_1^{2k-3} - z_1^{2k-4}z_2 + \ldots + z_2^{2k-3}) \neq 0$ for every $z \in |z| < 1$.

Thus, the function $\varphi(z)$ is in class S. The subclass of S consisting of the convex functions is defined by K , and S^* denotes the subclass of starlike functions. Thus $K \subset S^* \subset S$ [3].

We can do this proof in another way as follows: $|z| < 1$ is clear that there is convex region.

Note that $\varphi(z_1) - \varphi(z_2) = \int_{z_1}^{z_2} \varphi'(\eta)d\eta$.

If

$\eta = tz_2 + (1-t)z_1,\ 0 \leq t0 \leq 1$, then $z_1 - \varphi(z_2) = \int_0^1 \varphi'(tz_2 + (1-t)z_1)d\eta$.

Because,

$\eta = (tz_2 + (1-t)z_1) \in |z| < 1$ and $Re\varphi'(z) = Re\varphi'(tz_2 + (1-t)z_1) > 0$.

Thus

$\varphi'(\eta) = \varphi'(tz_2 + (1-t)z_1) \neq 0$. Therefore, if $z_1 - z_2 \neq 0$, then $\varphi(z_1) - \varphi(z_2) \neq 0$. This means that $\varphi(z)$ is univalent in $|z| < 1$. On the other hand,

$$Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) = Re\left(\frac{1 + 4A_2z + 14A_4z^3 + 36A_6z^5 + \ldots}{1 + 2A_2z + 4A_4z^3 + 6A_6z^5 + 8A_8z^7 + \ldots}\right) = Re(1 + 2A_2z - 4A_2A_2z^2 + (10A_4 + 8A_2A_2A_2)z^3 + \ldots) > 0$$

since for every $z \in |z| < 1$.

□

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