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# EXISTENCE RESULTS FOR STEKLOV PROBLEM WITH NONLINEAR BOUNDARY CONDITION

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**Abstract:** *In this article, we study the nonlinear Steklov boundary value problem. The existence of a nontrivial weak solution is obtained on variable exponent Sobolev spaces, by means of the Mountain Pass theorem.*

**Keywords:** *p(x)-Laplace operator, variational methods, Steklov boundary value, Mountain Pass theorem.*

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## 1. Introduction

The purpose of this paper is to study the following Steklov problem involving the p(x)-Laplacian,

$$\begin{cases} -div\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)+|u|^{p(x)-2}u=f(x,u), x\in\Omega \\ a(x)|\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \upsilon}=|u|^{q(x)-2}u, x\in\partial\Omega \end{cases} \qquad (P)$$

where $\Omega\subset R^N$ $(N\geq 2)$ is a bounded with smooth boundary, $p$ is continuous functions on $\overline{\Omega}$ such that $p^- := \inf_{x\in\overline{\Omega}} p(x) > 1$, $q$ is continuous functions on $\partial\Omega$ such that $q^- := \inf_{x\in\partial\Omega} q(x) > 1$, and $p(x)\neq q(y)$ for any $x\in\overline{\Omega}, y\in\partial\Omega$, $\Delta_{p(x)}u := div\left(|\nabla u|^{p(x)-2}\nabla u\right)$ denotes the $p(x)-$ Laplace operator, $f : \Omega x R \to R$ is a Carathéodory function, $\dfrac{\partial u}{\partial \upsilon}$ is the outer unit normal derivative on $\partial\Omega$ and $a(x)$ is a function which satisfies the conditions $0 < a_1 \leq a(x) \leq a_2$ where $a_1$ and $a_2$ are positive constants.

The study of differential equations and variational problems with $p(x)-$ growth conditions is a new and interesting topic in the last few years. The interest in studying such problems was stimulated by their application in mathematical physics, more precisely in elastic mechanics **[25]**, electrorheological fluids and stationary thermo-rheological viscous flows of non-Newtonian fluids, image processing [8,12,19,20**]** and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium [3,7]. Many results have been obtained on this kind of problems, for instance, we here cite [1,4,10,13,14,16,18,21,22].

Problems of type (P) has been intensively studied by many authors [2,4,5,6,9,17]. In [24]**,** the authors investigated the existence and multiple results by using a variation of the Mountain Pass the following $p(x)-$Laplacian with nonlinear boundary conditions in bounded domain $\Omega$

$$\begin{cases} -div\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right)+b(x)|u|^{p(x)-2}u=\lambda f(x,u), x\in\Omega \\ a(x)|\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \upsilon}=c(x)|u|^{q(x)-2}u+\eta g(x,u), x\in\partial\Omega \end{cases}$$

where $f$ and $g$ functions satisfies the Ambrosetti-Rabinowitz type condition. We also mention that the authors **[11]** studied de existence result for following class of Steklov boundary value problems involving $p(x)-$Laplacian

$$\begin{cases} -\Delta_{p(x)}u+a(x)|u|^{p(x)-2}u=f(x,u), x\in\Omega \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \upsilon}=g(x,u), x\in\partial\Omega \end{cases}$$

Using the variational method, under suitable conditions $a, f$ and $g$, they obtained results on the existence of solutions.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Lebesgue and Sobolev spaces. In Section 3, using Mountain Pass theorem and the variational method we show the existence nontrivial weak solutions of problem (P).

## 2.Preliminaries

In this section, we recall in what follows some definitions and basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$[22,16,15].

Set $C_+(\overline{\Omega})=\{p; p(x)\in C_+(\overline{\Omega}), \inf\ p(x)>1, \text{for all } x\in\overline{\Omega}\}$.

For any $p(x)\in C_+(\overline{\Omega})$, we denote

$$1<p^-:=\inf_{x\in\Omega} p(x)\le p^+:=\sup_{x\in\Omega} p(x)<\infty.$$

Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega)=\left\{u \middle| u:\Omega\to R \text{ is measurable}, \text{such that } \int_\Omega |u(x)|^{p(x)}dx<\infty\ \right\}$$

We define a norm, the so-called Luxemburg norm, on this space by the formula

$$|u|_{p(x)}:=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}dx\le 1\ \right\}$$

and $\left(L^{p(x)}(\Omega),|u|_{p(x)}\right)$becomes a Banach space.

Let $a:\partial\Omega\to R$ be measurable. Define the weighted variable exponent Lebesgue space

$$L_{a(x)}^{p(x)}(\partial\Omega)=\left\{u \middle| u:\partial\Omega\to R\, is \quad measurable \quad and \int_{\partial\Omega}|a(x)||u|^{p(x)}dx<+\infty\right\},$$

With the norm

$$|u|_{(p(x),a(x))}=\inf\left\{\kappa>0;\int_{\partial\Omega}|a(x)|\left|\frac{u}{\kappa}\right|^{p(x)}d\sigma\le 1\right\}$$

Where $d\sigma$ is the measure on the boundary. Then $L_{a(x)}^{p(x)}(\partial\Omega)$ is a Banach space. In particular, $a \in L^{\infty}(\partial\Omega), L_{a(x)}^{p(x)}(\partial\Omega) = L^{p(x)}(\partial\Omega)$.

**Proposition 2.1**[14,21] *If $p(x) \in L^{\infty}$, the conjugate space $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_{\Omega} uv dx\right| \leq \left(\frac{1}{p^{-}} + \frac{1}{p^{+}}\right)|u|_{p(x)}|v|_{p'(x)}.$$

The modular of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to R$ defined by

$$\rho_{p(x)}(u) = \int_{\Omega} |u|^{p(x)} dx, \quad \forall u \in L^{p(x)}(\Omega).$$

**Proposition 2.2 [**15,24] *If $u, u_n \in L^{p(x)}(\Omega)$ (n=1,2,...) and $p^{+} < \infty$, we have*

(i) $|u|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 (= 1, > 1)$,

(ii) $\min\left(|u|_{p(x)}^{p^{-}}, |u|_{p(x)}^{p^{+}}\right) \leq \rho_{p(x)}(u) \leq \max\left(|u|_{p(x)}^{p^{-}}, |u|_{p(x)}^{p^{+}}\right)$,

(iii) $|u_n - u|_{p(x)} \to 0 (\to \infty) \Leftrightarrow \rho_{(p(x),\Omega)}(u_n - u) \to 0 (\to \infty)$.

**Proposition 2.3** [24] *Denote $\rho_{p(x)}(u) = \int_{\partial\Omega} |u|^{p(x)} d\sigma, \ \forall u \in L^{p(x)}(\partial\Omega)$. Then*

(i) $|u|_{L^{p(x)}(\partial\Omega)} \geq 1 \Leftrightarrow |u|_{L^{p(x)}(\partial\Omega)}^{p^{-}} \leq \rho_{L^{p(x)}(\partial\Omega)}(u) \leq |u|_{L^{p(x)}(\partial\Omega)}^{p^{+}}$,

(ii) $|u|_{L^{p(x)}(\partial\Omega)} < 1 \Leftrightarrow |u|_{L^{p(x)}(\partial\Omega)}^{p^{+}} \leq \rho_{L^{p(x)}(\partial\Omega)}(u) \leq |u|_{L^{p(x)}(\partial\Omega)}^{p^{-}}$.

**Remark 2.4.** It is noted that since $L^{p(x)}(\Omega) \to L_{loc}^{1}(\Omega)$ i.e., for any compact subset $K \subset \Omega$ there exists a constant $C_K > 0$ such that $\| fX_K \|_1 \leq C_K \| f \|$. So every function in $L^{p(x)}(\Omega)$ has a distributional (weak) derivative, and variable exponent Sobolev space is well defined on $L^{p(x)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is denined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

and equipped with the norm,

$$\|u\| := \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{\nabla u(\mathrm{x})}{\lambda}\right|^{p(x)} + \int_{\Omega} \left|\frac{\mathrm{u}(\mathrm{x})}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

For $u \in W^{1,p(x)}(\Omega)$, if we define

$$\|u\|_a := \inf\left\{\lambda > 0 : \int_{\Omega} a(x)\left|\frac{\nabla u(\mathrm{x})}{\lambda}\right|^{p(x)} + \int_{\Omega} b(x)\left|\frac{\mathrm{u}(\mathrm{x})}{\lambda}\right|^{p(x)} dx \leq 1 \right\}.$$

In view of assumptions $a(x)$ of and $b(x)$ ($b(x)$ is a function which satisfies the conditions $0 < b_1 \leq b(x) \leq b_2$ where $b_1$ and $b_2$ are positive constants ), it is easy to see that $\|u\|_a$ is an equivalent norm on $W^{1,p(x)}(\Omega)$.

**Proposition 2.5** [24] *Denote $\Gamma(u) = \int_{\partial\Omega} \left(a(x)|\nabla u|^{p(x)} + |u|^{p(x)}\right) d\sigma, \ \forall u \in W^{1,p(x)}(\partial\Omega)$. Then* *(i)* $\Gamma(u) \geq 1 \Rightarrow \varepsilon_1 \|u\|^{p^-} \leq \Gamma(u) \leq \varepsilon_2 \|u\|^{p^+}$

*(ii)* $\Gamma(u) \leq 1 \Rightarrow \varepsilon_3 \|u\|^{p^+} \leq \Gamma(u) \leq \varepsilon_4 \|u\|^{p^-}$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and $\varepsilon_4$ are positive constants independent of $u$.

Space $W_0^{1,p(x)}(\Omega)$ is denoted as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $|u|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm $\|u\| = |\nabla u|_{p(x)}$. Since Poincaré inequality holds [16], i.e. there exists a positive constant $C > 0$ such that

$$\|u\| \leq C |\nabla u|_{p(x)} \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

**Proposition 2.6** [16,24]

*(i)* *If* $1 < p^- \leq p^+ < \infty$ *then the spaces* $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ *and* $W_0^{1,p(x)}(\Omega)$ *are separable, reflexive and uniformly convex Banach spaces,*

*(ii)* *If* $q(x) \in C_+(\overline{\Omega})$ *and* $q(x) < p^*(x)$ *for all* $x \in \overline{\Omega}$ *then the embedding* $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$ *is compact and continuous,*

*(iii)* *If* $q(x) \in C_+(\partial\Omega)$ $q(x) < p^\partial(x)$ *and for all* $x \in \partial\Omega$ *then the trace embedding* $W^{1,p(x)}(\Omega) \to L^{q(x)}(\partial\Omega)$ *is compact and continuous.*

We define,

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)}, & \text{if } N > p(x) \\ \infty, & \text{if } N \leq p(x) \end{cases} \quad \text{and} \quad p^\partial(x) = \begin{cases} \dfrac{(N-1)p(x)}{N - p(x)}, & \text{if } N > p(x) \\ \infty, & \text{if } N \leq p(x). \end{cases}$$

**Definition 2.7** [23] *Let* $X$ *be Banach spaces and the function* $I \in C^1(X, R)$. *We say that* $I$ *satisfies Palais-Smale condition* ***(PS)*** *in* $X$ *if any sequence* $\{u_n\}$ *in* $X$ *such that* $I(u_n)$ *is bounded and* $I'(u_n) \to 0$ *in* $X^*$ *as* $n \to \infty$ *has a convergent subsequence.*

**Lemma 2.8 8 (Mountain Pass Theorem)** [23] *Let* $X$ *be a Banach space and the function* $I \in C^1(X, R)$ *satisfies Palais-Smale condition. Assume that* $I(0) = 0$ *and*

*(i)* *There exist two positive real numbers* $\eta$ *and* $r$ *such that* $I(u) \geq r$ *with* $\|u\| = r$,

*(ii)* *There exists* $u_1 \in X$ *such that* $\|u_1\| > r$, *and* $I(u) < 0$.

*Put* $G = \{\varphi \in C([0,1], X) : \varphi(0) = 0, \varphi(1) = u_1 \}$. *Set* $\beta = \inf\{\max \varphi([0,1]) : \varphi \in G \}$. *Then* $\beta \geq r$ *and* $\beta$ *is a critical value of* $I$.

Throughout this paper, the following hypotheses are assumed.

**(f1)** $f : \Omega x R \to R$ is Carathéodory condition such that

$$|f(x,t)| \leq c_1 + c_2 |t|^{\alpha(x)-1} \quad , \quad \forall (x,t) \in \Omega x R,$$

where $c_1, c_2 > 0$, $\alpha(x) \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$,

**(f2)** $f(x,t) = o\left(|t|^{p^+-1}\right), t \to 0$ ; for all $x \in \Omega$,

(**AR):** Ambrosetti-Rabinowitz's condition; there exist $t^* > 0$ and $\theta > p^+$ such that $0 < \theta F(x,t) \le f(x,t)t,\ |t| \ge t^*$, for all $x \in \Omega$

where $F(x,t) = \int_0^t f(x,s)ds$ and $q^- \le q(x) < p^\partial(x)$ for all $q(x) \in C(\partial\Omega)$.

**Theorem 2.9.** *Assume that conditions **(f1), (f2), (AR),** $q^- > \theta, \alpha^-, p^+$ and $\alpha^- > p^+$ are satisfied, then problem $(P)$ has at least one nontrivial weak solution.*

**3. Main Results**

Let $X$ denote the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$. We say that $u \in X/\{0\}$ is a weak solution of **(P)** if

$$\int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2}uv dx - \int_{\partial\Omega}|u|^{q(x)-2}uv d\sigma - \int_\Omega f(x,u)v dx = 0$$

for all $v \in X$ .

The energy functional corresponding to the problem $(P)$ is defined as $I = X \to R$

$$I(u) = \int_\Omega \frac{a(x)|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)} - \int_{\partial\Omega}\frac{|u|^{q(x)}}{q(x)}d\sigma - \int_\Omega F(x,u)dx$$

where $F(x,u) = \int_0^u f(x,s)ds$ and $d\sigma$ is the measure on the boundary.

**Proposition 3.1** [24] *If one denotes*

$$J(u) = \int_\Omega \frac{a(x)|\nabla u|^{p(x)} + |u|^{p(x)}}{p(x)}dx,\ \ \forall x \in X.$$

*Then $J \in C^1(X,R)$ and the derivative operator of $J$, denoted by $J'$, is*

$$\langle J'(u), v\rangle = \int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u \nabla v dx + \int_\Omega |u|^{p(x)-2}uv dx,\quad \forall u,v \in X,$$

*and one has:*

*(i)* $J' : X \to X^*$ *is a continuous, bounded, and strictly monotone operator,*

*(ii)* $J'$ *is a mapping of $(S^+)$ type, that is, if $u_n \to u$ (weak convergent) in $X$ and* $\limsup_{n\to\infty}\langle J'(u_n) - J'(u), u_n - u\rangle \le 0$, *then $u_n \to u$ (strongly convergent) in $X$,*

*(iii)* $J' : X \to X^*$ *is a homeomorphism.*

**Proposition 3.2** [5,24] *If one denotes*

$$\varphi(u) = \int_{\partial\Omega}\frac{|u|^{q(x)}}{q(x)}d\sigma, \forall u \in X,$$

*where* $q(x) \in C_+(\partial\Omega)$ and $q(x) < p*(x)$ for any $x \in \partial\Omega$, then $\varphi \in C^1(X,R)$ *and the derivative operator of $\varphi$, denoted by $\varphi'$, is*

$$\langle \varphi'(u), v\rangle = \int_{\partial\Omega} |u|^{q(x)-2} uv d\sigma, \quad \forall u, v \in X,$$

*and one has* $\varphi : X \to R$ *and* $\varphi' : X \to X^*$ *are sequentially weak- strongly continuous, bounded, namely,* $u_n \to u$ *(weakly continuous ) implies* $\varphi'(u_n) \to \varphi'(u)$ *(strongly continuous) .*

Therefore, from the assumption **(f1)**, Proposition 2.6, Proposition 3.1 and Proposition 3.2, it is easy to see that $I(u) \in C^1(X, R)$ and the critical points $I$ are weak solutions of $(P)$**.** Moreover, the derivate of $I$ is the mapping $I' : X \to X^*$

$$\langle I'(u), v\rangle = \int_\Omega \left(a(x)|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv\right)dx - \int_{\partial\Omega} |u|^{p(x)-2} uv d\sigma - \int_\Omega f(x,u) v dx$$

for any $u, v \in X$ **[6].**

**Lemma 3.3** *Suppose that **(f1), (f2), (AR)*** and $q^- > \theta$ *are satisfied, then* $I$ *satisfies the **(PS)** condition.*

***Proof.*** Assume that $\{u_n\} \subset X$ is a sequence which satisfies the properties:

$$I(u_n) \to C \text{ and } I'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty, \tag{3.1}$$

where $X^*$ is dual space of X and $C$ is a positive constant. We prove that $\{u_n\}$ possesses a convergent subsequence. First, we show that $\{u_n\}$ is bounded in X. We assume by contradiction $\|u_n\| \to \infty$ as $n \to \infty$. Using ***(AR*** ) $q^- > \theta$, (3.1), Proposition 2.2, Proposition 2.3 and considering $\|u_n\| > 1$, for n large enough, we can write

$$\begin{aligned}
C + \|u_n\| &\geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\
&= \int_\Omega \frac{1}{p(x)}\left(a(x)|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx - \int_{\partial\Omega} \frac{1}{q(x)}|u_n|^{q(x)} d\sigma - \int_\Omega F(x,u_n)dx \\
&- \frac{1}{\theta}\left(\int_\Omega \left(a(x)|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)dx - \int_{\partial\Omega} |u_n|^{q(x)} d\sigma - \int_\Omega f(x,u_n) u_u dx\right) \\
&\geq \left(\frac{\varepsilon_1}{p^+} - \frac{1}{\theta}\right)\|u_n\|^{p^-}
\end{aligned}$$

where $c_3 > 0$ is constant. Since $\theta > p^+$ we obtain that $\{u_n\}$ is bounded in X. Next, we show the strong converges to $u_n$ $X$ in. Since it $\{u_n\}$ is bounded in X, there exists $u$ in $X$ such that, up to a subsequence, $u_n$ converges weakly to $u$ in $X$ . Taking into account (3.1), we have $\langle I'(u_u), u_n - u\rangle \to 0$. So, we have

$$\begin{aligned}
&\langle I'(u_n), u_n - u\rangle \\
&= \int_\Omega \left(a(x)|\nabla u_n|^{p(x)-2} \nabla u_n (\nabla u_n - \nabla u) + |u_n|^{p(x)-2} u_n (u_n - u)\right)dx \\
&- \int_{\partial\Omega} |u_n|^{q(x)-2} u_n (u_n - u) d\sigma - \int_\Omega f(x,u_n)(u_n - u)dx.
\end{aligned}$$

Using **(f1)** and Proposition 2.1, it follows

$$\left|\int_\Omega f(x,u_n)(u_n-u)dx\right| \le \left|\int_\Omega \left(c_1+c_2|u_n|^{\alpha(x)-1}\right)(u_n-u)dx\right|$$
$$\le c_1\int_\Omega |u_n-u|dx + c_4\left\||u_n|^{\alpha(x)-1}\right\|_{\alpha'(x)}|u_n-u|_{\alpha(x)}$$

where $c_4>0$ is constant. Because $\alpha(x)<p^*(x)$ (Proposition 2.6 (ii)), there exists $u$ such that $u_n$ converges weakly to $u$ in $X$. Thanks to the compact embedding $X\to L^{\alpha(x)}(\Omega)$, we get

$$u_n \to u(strongly)\text{ in } L^{\alpha(x)}$$
$$u_n \to u\ a.e. x\in\Omega$$

So,

$$\int_\Omega f(x,u_n)(u_n-u)dx\to 0.$$

Similarly, by Proposition 2.1, Proposition 2.6 (iii) and Proposition 3.2, we have

$$\int_{\partial\Omega}|u_n|^{q(x)-2}u_n(u_n-u)d\sigma\to 0.$$

Thus,

$$\int_\Omega \left(a(x)|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)+|u_n|^{p(x)-2}u_n(u_n-u)\right)dx\to 0.$$

Finally, from Proposition 3.1**,** we deduce that $u_n$ converges strongly to $u$ $X$ in. Therefore, $I$ it satisfies the (PS) condition.

**Lemma 3.4** *Assume that conditions **(f1), (f2)**,* $q^->p^+,\alpha^-$ and $\alpha^->p^+$ *are fulfilled. Then, there exist two positive real numbers* $\rho$ *and* $r$ *such that* $I(u)\ge r$ *with* $\|u\|=\rho$.

***Proof:*** For $\|u\|<1$, by Proposition 2.5, we have

$$I(u)\ge\frac{\varepsilon_3}{p^+}\|u\|^{p^+}-\frac{1}{q^-}\int_{\partial\Omega}|u|^{q(x)}d\sigma-\int_\Omega F(x,t)dx$$

where $c_7$ is constant. Since we have the continuous embeddings $X\to L^{q(x)}(\partial\Omega)$ $X\to L^{p^+}(\Omega)$, and $X\to L^{\alpha(x)}(\Omega)$ (Proposition 2.6), there exist $c_5$, $c_6$ and $c_7$ positive constants such that for all $u\in X$

$$|u|_{q(x),\partial\Omega}\le c_5\|u\|,\quad |u|^{p^+}\le c_6\|u\| \text{ and } |u|_{\alpha(x),\Omega}\le c_8\|u\|. \qquad (3.2)$$

Choose $\varepsilon>0$ small enough such that $\left(\varepsilon c_6^{p^+}+c_\varepsilon c_7^{\alpha^-}\right)<\frac{\varepsilon_4}{2p^+}$. Using **(f1)** and **(f2),** we have

$$|F(x,t)|\le\varepsilon|t|^{p^+}+c_\varepsilon|t|^{\alpha(x)},\ \text{for all}(x,t)\in\Omega\times\mathrm{R}. \qquad (3.3)$$

Thus, using (3.2) and (3.3) for $\|u\|<1$, we get

$$I(u)\ge\frac{\varepsilon_4}{p^+}\|u\|^{p^+}-\frac{c_5}{q^-}\|u\|^{q^-}-\varepsilon c_6^{p^+}\|u\|^{p^+}-c_\varepsilon c_7^{\alpha^-}\|u\|^{\alpha^-}$$
$$\ge\frac{\varepsilon_4}{p^+}\|u\|^{p^+}-\frac{c_5}{q^-}\|u\|^{q^-}-\varepsilon c_6^{p^+}\|u\|^{p^+}-c_\varepsilon c_7^{\alpha^-}\|u\|^{p^+}.$$
$$\ge\frac{\varepsilon_4}{2p^+}\|u\|^{p^+}-\frac{c_5}{q^-}\|u\|^{q^-}$$

Choose $c_5 \le \frac{q^- \varepsilon_4}{2p^+}$. It follows that there exist $r > 0$ s and $\rho > 0$ such that $I(u) \ge r$ with $\|u\| = \rho$. The proof of Lemma 3.4 is completed.

**Lemma 3.5** *If **(f1)**, **(f2)** and **(AR)** hold, there exists* $\phi \in X$ *such that* $\|\phi\| > \eta$ *and* $I(t\phi) < 0$ *for* $t > 0$.

***Proof.*** Thanks to **(AR),** we obtain $|F(x,t)| \ge c_8 |t|^\theta$ for all $(x,t) \in \Omega \times R$. Moreover, when $t > 1$ is large enough, from Proposition 2.2, we obtain that

$$I(t\phi) = \int_\Omega \frac{a(x)|\nabla t\phi|^{p(x)} + |t\phi|^{p(x)}}{p(x)} dx - \int_{\partial\Omega} \frac{|t\phi|^{q(x)}}{q(x)} d\sigma - \int_\Omega F(x, t\phi) dx$$
$$\le \frac{t^{p^+}}{p^-} \int_\Omega \left( a(x)|\nabla \phi|^{p(x)} + |\phi|^{p(x)} \right) dx - \frac{t^{q^-}}{q^+} \int_{\partial\Omega} |\phi|^{q(x)} d\sigma - c_8 t^\theta \int_\Omega |\phi| dx$$

Since $\theta > p^+$ we conclude that $I(t\phi) \to -\infty$ $t \to \infty$ as. The proof is completed.

***Proof of Theorem 2.9***. From Lemma 3.3, Lemma 3.4, Lemma 3.5 and $I(0) = 0$, $I$ satisfies all statements of Lemma 2.8. Therefore, $I$ has at least one nontrivial critical point, i.e., problem ***(P)*** has a nontrivial weak solution. The proof is completed.

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