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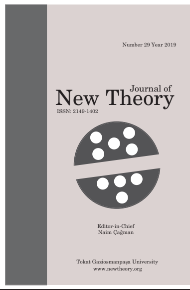

New Topologies via Weak N-Topological Open Sets and Mappings

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Abstract — One of the objectives of this paper is to introduce some weak N-topological open sets. We characterize N-topological continuous, N^*-quotient, N^*-α quotient and N^*-semi quotient mappings and derive some new topologies with suitable examples.

Keywords — *N-topology, $N\tau\alpha$-open, $N\tau$ semi- open, $N\tau$ pre-open, $N\tau\beta$-open*

1. Introduction

In 1963 Norman Levine [1] initiated the concept of semi open sets and its continuous functions. In 1965 O.Njastad [2] developed the α-open set and its properties in classical topology. Mashhour et al. [3] investigated the properties of pre open sets. Andrijevic [4] discussed the behaviour of β-open sets in classical topology. The general form of classical topology called N-topology and $N\tau$-open sets were initiated by Lellis Thivagar et al. [5]. In this paper we introduce $N\tau\alpha$-open set, $N\tau$ semi-open set, $N\tau$ pre-open set and $N\tau\beta$-open set in N-topological space. We also establish that the set of all $N\tau\alpha$-open sets forms a topology. Apart from this we investigate the properties of some N-topological continuous and quotient mappings. In this section we discuss some basic properties of N-topological spaces which are useful in sequel. Here by a space $(X, N\tau)$, we mean a N-topological space with N-topology $N\tau$ defined on X in which no separation axioms are assumed unless otherwise explicitly stated.

Definition 1.1. [5] Let X be a non empty set, τ_1, τ_2, ... , τ_N be N-arbitrary topologies defined on X and let the collection $N\tau = \{S \subseteq X : S = (\bigcup_{i=1}^{N} A_i) \cup (\bigcap_{i=1}^{N} B_i),\ A_i, B_i \in \tau_i\}$, is said to be N-topology on X if it satisfies the following axioms:

(i) $X, \emptyset \in N\tau$

(ii) $\bigcup_{i=1}^{\infty} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{\infty} \in N\tau$

(iii) $\bigcap_{i=1}^{n} S_i \in N\tau$ for all $\{S_i\}_{i=1}^{n} \in N\tau$

Then the pair $(X, N\tau)$ is called a N-topological space on X. The elements of $N\tau$ are known as $N\tau$-open set and the complement of $N\tau$-open set is called $N\tau$-closed.

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Definition 1.2. [5] Let A be a subset of N-topological space $(X, N\tau)$. Then

(i) $N\tau\text{-}int(A) = \cup\{G : G \subseteq A$ and G is $N\tau$-open$\}$

(ii) $N\tau\text{-}cl(A) = \cap\{F : A \subseteq F$ and F is $N\tau$-closed$\}$

Theorem 1.3. [5] Let $(X, N\tau)$ be a topological space on X and $A \subseteq X$. Then $x \in N\tau\text{-}cl(A)$ if and only if $G \cap A \neq \emptyset$ for every open set G containing x.

Definition 1.4. A subset A of a topological space (X, τ) is called

(i) α-open [2] if $A \subseteq int(cl(int(A)))$

(ii) semi-open [1] if $A \subseteq cl(int(A))$

(iii) pre-open [3] if $A \subseteq int(cl(A))$

(iv) β-open [4] if $A \subseteq cl(int(cl(A)))$

The complement of α-open (resp. semi-open, pre-open and β-open) set is called α-closed (resp. semi-closed, pre-closed and β-closed).

2. Weak Forms of Open Sets in N-Topological Space

In this section we investigate some classes of open sets in N-topological space and discuss the relationship between them.

Definition 2.1. A subset A of a N-topological space $(X, N\tau)$ is called

(i) $N\tau\alpha$-open set if $A \subseteq N\tau\text{-}int(N\tau\text{-}cl(N\tau\text{-}int(A)))$

(ii) $N\tau$ semi-open set if $A \subseteq N\tau\text{-}cl(N\tau\text{-}int(A))$

(iii) $N\tau$ pre-open set if $A \subseteq N\tau\text{-}int(N\tau\text{-}cl(A))$

(iv) $N\tau\beta$-open set if $A \subseteq N\tau\text{-}cl(N\tau\text{-}int(N\tau\text{-}cl(A)))$

The complement of $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open and $N\tau\beta$-open) set is called $N\tau\alpha$-closed (resp. $N\tau$ semi-closed, $N\tau$ pre-closed and $N\tau\beta$-closed). The set of all $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open and $N\tau\beta$-open) sets of $(X, N\tau)$ is denoted by $N\tau\alpha O(X)$ (resp. $N\tau SO(X)$, $N\tau PO(X)$ and $N\tau\beta O(X)$ and the set of all $N\tau\alpha$-closed (resp. $N\tau$ semi-closed, $N\tau$ pre-closed and $N\tau\beta$-closed) sets of $(X, N\tau)$ is denoted by $N\tau\alpha C(X)$ (resp. $N\tau SC(X)$, $N\tau PC(X)$ and $N\tau\beta C(X)$.

Particularly if $N = 1$, then the $1\tau\alpha$-open, 1τ semi-open, 1τ pre-open and $1\tau\beta$-open set of $(X, 1\tau)$ respectively become α-open, semi-open, pre-open and β-open set of (X, τ) which are defined in definition 2.4.

Theorem 2.2. Let A be a subset of N-topological space $(X, N\tau)$. Then

(i) every $N\tau$-open set is $N\tau\alpha$-open.

(ii) every $N\tau\alpha$-open set is $N\tau$ semi-open.

(iii) every $N\tau\alpha$-open set is $N\tau$ pre-open.

(iv) every $N\tau$ semi-open set is $N\tau\beta$-open.

(v) every $N\tau$ pre-open set is $N\tau\beta$-open.

The converse of the above theorem need not be true as shown in the following examples.

Example 2.3. If we take $N = 3$, $X = \{a, b, c\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X\}$ and $\tau_3 = \{\emptyset, X, \{a, b\}\}$. Then $3\tau O(X) = \{\emptyset, X, \{a\}, \{a, b\}\}$ and $3\tau\alpha O(X) = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}\}$. Here the set $A = \{a, c\}$ is $3\tau\alpha$-open but not 3τ-open.

Example 2.4. If $N = 5$, $X = \{a, b, c, d\}$, $\tau_1 = \{\emptyset, X, \{a\}\}$, $\tau_2 = \{\emptyset, X, \{b, c\}\}$, $\tau_3 = \{\emptyset, X, \{a, b, c\}\}$, $\tau_4 = \{\emptyset, X, \{a\}, \{a, b, c\}\}$ and $\tau_5 = \{\emptyset, X, \{b, c\}, \{a, b, c\}\}$. Then, $5\tau O(X) = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ $= 5\tau\alpha O(X)$, $5\tau SO(X) = \{\emptyset, X, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$, $5\tau PO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\},$ $\{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}$ and $5\tau\beta O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\},$ $\{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$. Here the set $\{a, d\}$ is 5τ semi-open and $5\tau\beta$-open but not $5\tau\alpha$-open as well as not 5τ pre-open. Also the set $\{a, c\}$ is 5τ pre-open and $5\tau\beta$-open but not $5\tau\alpha$-open as well as 5τ semi-open.

We observe that the following theorem is analogous to the 1985 topological space result of Reilly and Vamanamurthy [6].

Theorem 2.5. Let $(X, N\tau)$ be a N-topological space. Then every $N\tau\alpha$-open set is both $N\tau$ semi-open and $N\tau$ pre-open and conversely.

Lemma 2.6. The arbitrary union of $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open, $N\tau\beta$-open) sets is $N\tau\alpha$-open (resp. $N\tau$ semi-open, $N\tau$ pre-open, $N\tau\beta$-open).

Remark 2.7. Intersection of any two $N\tau$ semi-open (resp. $N\tau$ pre-open, $N\tau\beta$-open) sets need not be a $N\tau$ semi-open (resp. $N\tau$ pre-open, $N\tau\beta$-open) set. Consider example 3.4, the sets $\{a, d\}$ and $\{b, c, d\}$ are 5τ semi-open, but $\{d\}$ is not 5τ semi-open. The sets $\{a, c, d\}$ and $\{a, b, d\}$ are 5τ pre-open, but $\{a, d\}$ is not 5τ semi-open. Also the sets $\{a, d\}$ and $\{c, d\}$ are $5\tau\beta$-open, but $\{d\}$ is not $5\tau\beta$-open.

Theorem 2.8. Let $(X, N\tau)$ be a N-topological space. Then $N\tau\alpha O(X) = \{A \subseteq X : A \cap B \in N\tau SO(X) \forall B \in N\tau SO(X)\}$.
Proof: Proof follows as similar as the Proposition 1 of [2].

Theorem 2.9. Let $(X, N\tau)$ be a N-topological space. Then $N\tau\alpha O(X)$ is a topology finer than $N\tau O(X)$.
Proof: Clearly $\emptyset \in N\tau\alpha O(X)$ and $\bigcup_{i\in\Lambda} A_i \in N\tau\alpha O(X)$ for every $\{A_i\}_{i\in\Lambda} \in N\tau\alpha O(X)$ by lemma 3.6. By theorem 3.8 we have $N\tau\alpha O(X)$ is a topology and clearly $N\tau O(X) \subseteq N\tau\alpha O(X)$.

Definition 2.10. Let $(X, N\tau)$ be a N-topological space. A subset A of X is said to be $N\tau$-nowhere dense set if $N\tau\text{-}int(N\tau\text{-}cl(A)) = \emptyset$.

Lemma 2.11. Let $(X, N\tau)$ be a N-topological space. A subset A of X is $N\tau\alpha$-open set, then it can be written as a difference of $N\tau$-open set and $N\tau$-nowhere dense set.

Remark 2.12. $N\tau O(X) = N\tau\alpha O(X)$ if and only if all $N\tau$-nowhere dense sets are $N\tau$-closed.

Definition 2.13. An N-topological space $(X, N\tau)$ is said to be extremely disconnected if $N\tau\text{-}cl(A)$ is $N\tau$-open for all $N\tau$-open sets A.

Lemma 2.14. $N\tau SO(X)$ is a topology if and only if $(X, N\tau)$ is extremely disconnected.

3. Weak Closure and Interior Operators in N-Topology

In this section, we introduce some weak closure and interior operators in N-topological space and investigate their properties.

Definition 3.1. Let $(X, N\tau)$ be a N-topological space and A be a subset of X.

(i) The $N\tau$-α closure of A, denoted by $N\tau\text{-}\alpha cl(A)$, and defined by

$$N\tau\text{-}\alpha cl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau\alpha\text{-closed set}\}$$

(ii) The $N\tau$-semi closure of A, denoted by $N\tau\text{-}scl(A)$, and defined by

$$N\tau\text{-}scl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau \text{ semi-closed set}\}$$

(iii) The $N\tau$-pre closure of A, denoted by $N\tau\text{-}pcl(A)$, and defined by

$$N\tau\text{-}pcl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau \text{ pre-closed set}\}$$

(iv) The $N\tau$-β closure of A, denoted by $N\tau\beta cl(A)$, and defined by

$$N\tau\text{-}\beta cl(A) = \cap\{F : A \subseteq F \text{ and } F \text{ is } N\tau\beta\text{-closed set}\}$$

Definition 3.2. Let $(X, N\tau)$ be a N-topological space and A be a subset of X.

(i) The $N\tau$-α interior of A, denoted by $N\tau\alpha int(A)$, and is defined by

$$N\tau\text{-}\alpha int(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau\alpha\text{-open set}\}$$

(ii) The $N\tau$-semi interior of A, denoted by $N\tau$-$sint(A)$, and is defined by

$$N\tau\text{-}sint(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau \text{ semi-open set}\}$$

(iii) The $N\tau$-pre interior of A, denoted by $N\tau$-$pint(A)$, and is defined by

$$N\tau\text{-}pint(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau \text{ pre-open set}\}$$

(iv) The $N\tau$-β interior of A, denoted by $N\tau$-$\beta int(A)$, and is defined by

$$N\tau\text{-}\beta int(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } N\tau\beta\text{-open set}\}$$

Theorem 3.3. Let $(X, N\tau)$ be a N-topological space on X and let $A, B \subseteq X$. Then

(i) $N\tau$-$\alpha cl(A)$ is the smallest $N\tau\alpha$-closed set which containing A.

(ii) A is $N\tau\alpha$-closed iff $N\tau$-$\alpha cl(A) = A$. In particular, $N\tau$-$\alpha cl(\emptyset) = \emptyset$ and $N\tau$-$\alpha cl(X) = X$.

(iii) $A \subseteq B \Rightarrow N\tau\text{-}\alpha cl(A) \subseteq N\tau\text{-}\alpha cl(B)$

(iv) $N\tau\text{-}\alpha cl(A \cup B) = N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$

(v) $N\tau\text{-}\alpha cl(A \cap B) \subseteq N\tau\text{-}\alpha cl(A) \cap N\tau\text{-}\alpha cl(B)$

(vi) $N\tau\text{-}\alpha cl(N\tau\text{-}\alpha cl(A)) = N\tau\text{-}\alpha cl(A)$

Proof:

(i) Since the intersection of any collection of $N\tau\alpha$-closed sets is also $N\tau\alpha$-closed, then $N\tau$-$\alpha cl(A)$ is a $N\tau\alpha$-closed set. By definition 4.1, $A \subseteq N\tau\text{-}\alpha cl(A)$. Now let B be any $N\tau\alpha$-closed set containing A. Then $N\tau\text{-}\alpha cl(A) = \cap\{F : A \subseteq F$ and F is $N\tau\alpha$-closed$\} \subseteq B$. Therefore, A is the smallest $N\tau\alpha$-closed set containing A.

(ii) Assume A is $N\tau\alpha$-closed, then A is the only smallest $N\tau\alpha$-closed set containing itself and therefore, $N\tau$-$\alpha cl(A) = A$. Conversely, assume $N\tau$-$\alpha cl(A) = A$. Then A is the smallest $N\tau\alpha$-closed set containing itself. Therefore, A is $N\tau\alpha$-closed. In particular, since $\emptyset$ and X are $N\tau\alpha$-closed sets, then $N\tau$-$\alpha cl(\emptyset) = \emptyset$ and $N\tau$-$\alpha cl(X) = X$.

(iii) Assume $A \subseteq B$, and since $B \subseteq N\tau\text{-}\alpha cl(B)$, then $A \subseteq N\tau\text{-}\alpha cl(B)$. Since $N\tau$-$\alpha cl(A)$ is the smallest $N\tau\alpha$-closed set containing A. Therefore, $N\tau\text{-}\alpha cl(A) \subseteq N\tau\text{-}\alpha cl(B)$.

(iv) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then by (iii), we have $N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B) \subseteq N\tau\text{-}\alpha cl(A \cup B)$. On the other hand, by(i), $A \cup B \subseteq N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$. Since $N\tau$-$\alpha cl(A \cup B)$ is the smallest $N\tau\alpha$-closed set containing $A \cup B$. Then $N\tau\text{-}\alpha cl(A \cup B) \subseteq N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$. Therefore, $N\tau\text{-}\alpha cl(A \cup B) = N\tau\text{-}\alpha cl(A) \cup N\tau\text{-}\alpha cl(B)$.

(v) Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, then $N\tau\text{-}\alpha cl(A \cap B) \subseteq N\tau\text{-}\alpha cl(A) \cap N\tau\text{-}\alpha cl(B)$.

(vi) Since $N\tau$-$\alpha cl(A)$ is a $N\tau\alpha$-closed set, then $N\tau\text{-}\alpha cl(N\tau\text{-}\alpha cl(A)) = N\tau\text{-}\alpha cl(A)$.

Remark 3.4. From the above theorem, we can observe that the closure operator $N\tau$-αcl satisfies the Kuratowski's closure axioms. The following theorem can be proved as the above theorem.

Theorem 3.5. Let $(X, N\tau)$ be a N-topological space on X and let $A, B \subseteq X$. Let $N\tau$-$kcl(A)$ is the intersection of all k-closed sets containing A (where k-closed set is can be any one of the following $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set). Then

(i) $N\tau$-$kcl(A)$ is the smallest k-closed set containing A.

(ii) A is k-closed iff $N\tau$-$kcl(A) = A$. In particular, $N\tau$-$kcl(\emptyset) = \emptyset$ and $N\tau$-$kcl(X) = X$.

(iii) $A \subseteq B \Rightarrow N\tau\text{-}kcl(A) \subseteq N\tau\text{-}kcl(B)$

(iv) $N\tau\text{-}kcl(A \cup B) \supseteq N\tau\text{-}kcl(A) \cup N\tau\text{-}kcl(B)$

(v) $N\tau\text{-}kcl(A \cap B) \subseteq N\tau\text{-}kcl(A) \cap N\tau\text{-}kcl(B)$

(vi) $N\tau\text{-}kcl(N\tau\text{-}kcl(A)) = N\tau\text{-}kcl(A)$.

Example 3.6. Let $X = \{a, b, c, d\}$. For $N = 3$, consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b, c\}\}$ and $\tau_3 O(X) = \{X, \emptyset, \{a, b, c\}\}$. Then, we have $3\tau O(X) = \{X, \emptyset, \{a\}, \{b, c\}, \{a, b, c\}\} = N\tau\alpha O(X)$, $3\tau C(X) = \{X, \emptyset, \{d\}, \{a, d\}, \{b, c, d\}\}$. Also $3\tau SO(X) = \{\emptyset, X, \{a\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$, $3\tau PO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}\}$ and $3\tau\beta O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}$. Let $A = \{a\}$ and $B = \{b\}$. Then $3\tau\text{-}scl(A) \cup 3\tau\text{-}scl(B) = \{a\} \cup \{b, d\} = \{a, b, d\} \neq X = 3\tau\text{-}scl(A \cup B)$. Let $A = \{a\}$ and $B = \{b\}$. Then $3\tau\text{-}pcl(A) \cup 3\tau\text{-}pcl(B) = \{a\} \cup \{b\} = \{a, b\} \neq \{a, b, d\} = 3\tau\text{-}pcl(A \cup B)$. Also let $A = \{a\}$ and $B = \{b, c\}$, then $3\tau\text{-}\beta cl(A) \cup 3\tau\text{-}\beta cl(B) = \{a\} \cup \{b, c\} = \{a, b, c\} \neq X = 3\tau\text{-}\beta cl(A \cup B)$.

Theorem 3.7. Let $(X, N\tau)$ be a N-topological space on X and $A \subseteq X$. Let $N\tau$-$kcl(A)$ is the intersection of all k-closed sets containing A (where k-closed set is can be any one of the following $N\tau\alpha$-closed set, $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set). Then $x \in N\tau\text{-}kcl(A)$ if and only if $G \cap A \neq \emptyset$ for every k-open set G containing x.

Theorem 3.8. Let $(X, N\tau)$ be a N-topological space X and $A, B \subseteq X$. Then

(i) $N\tau$-$\alpha int(A)$ is the largest $N\tau\alpha$-open set contained in A.

(ii) A is $N\tau\alpha$-open set iff $N\tau$-$\alpha int(A) = A$. In particular, $N\tau$-$\alpha int(\emptyset) = \emptyset$ and $N\tau$-$\alpha int(X) = X$.

(iii) $A \subseteq B$, then $N\tau\text{-}\alpha int(A) \subseteq N\tau\text{-}\alpha int(B)$

(iv) $N\tau\text{-}\alpha int(A \cup B) \supseteq N\tau\text{-}\alpha int(A) \cup N\tau\text{-}\alpha int(B)$

(v) $N\tau\text{-}\alpha int(A \cap B) = N\tau\text{-}\alpha int(A) \cap N\tau\text{-}\alpha int(B)$

(vi) $N\tau\text{-}\alpha int(N\tau\text{-}\alpha int(A)) = N\tau\text{-}\alpha int(A)$

Proof: The proof is obvious from the fact that a set is $N\tau\alpha$-open if and only if its complement is $N\tau\alpha$-closed.

The proof of the following theorem can be proved as similar as the above theorem.

Theorem 3.9. Let $(X, N\tau)$ be a N-topological space X and $A, B \subseteq X$. Let $N\tau$-$kint(A)$ is the union of all k-open sets contained in A (where k-open set can be any one of $N\tau$ semi-open set, $N\tau$ pre-open set and $N\tau\beta$-open set). Then

(i) $N\tau$-$kint(A)$ is the largest k-open set contained in A.

(ii) A is k-open set iff $N\tau$-$kint(A) = A$. In particular, $N\tau$-$kint(\emptyset) = \emptyset$ and $N\tau$-$kint(X) = X$.

(iii) $A \subseteq B$, then $N\tau\text{-}kint(A) \subseteq N\tau\text{-}kint(B)$

(iv) $N\tau\text{-}kint(A \cup B) \supseteq N\tau\text{-}kint(A) \cup N\tau\text{-}kint(B)$

(v) $N\tau\text{-}kint(A \cap B) \subseteq N\tau\text{-}kint(A) \cap N\tau\text{-}kint(B)$

(vi) $N\tau$-$kint(N\tau$-$kint(A)) = N\tau$-$kint(A)$

Theorem 3.10. Let $(X, N\tau)$ be a N-topological space X and $A \subseteq X$. Let $N\tau$-$kint(A)$ and $N\tau$-$kcl(A)$ are the weak interior and closure operator in N-topological space. By k-closed set, we mean any one of the following $N\tau\alpha$-closed set, $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set. Then

(i) $N\tau$-$kint(X - A) = X - N\tau$-$kcl(A)$

(ii) $N\tau$-$kcl(X - A) = X - N\tau$-$kint(A)$

Remark 3.11. Let $(X, N\tau)$ be a N-topological space X and $A \subseteq X$. Let $N\tau$-$kint(A)$ and $N\tau$-$kcl(A)$ are the weak interior and closure operator in N-topological space. By k-closed set, we mean any one of the following $N\tau\alpha$-closed set, $N\tau$ semi-closed set, $N\tau$ pre-closed set and $N\tau\beta$-closed set. If we take the complement of either side of part(i) and part(ii) of previous theorems, we get

(i) $N\tau$-$kcl(A) = X - N\tau$-$kint(X - A)$

(ii) $N\tau$-$kint(A) = X - N\tau$-$kcl(X - A)$

4. Some Weak Continuous Functions in N-topology

In this section, we introduce some weak form of continuous functions in N-topological space and investigate the relationship between them. By the spaces X and Y, we means the N-topological spaces $(X, N\tau)$ and $(Y, N\sigma)$ respectively.

Definition 4.1. Let X and Y be two N-Topological spaces. A function $f : X \to Y$ is said to be N^*-α continuous (resp. N^*-semi continuous, N^*-pre continuous, N^*-β continuous) on X if the inverse image of every $N\sigma$-open set in Y is a $N\tau\alpha$-open set (resp. $N\tau$ semi-open, $N\tau$ pre-open, $N\tau\beta$-open) in X.

Theorem 4.2. A function $f : X \to Y$ is N^*-α continuous (resp. N^*-semi continuous, N^*-pre continuous, N^*-β continuous) on X if and only if the inverse image of every $N\sigma$-closed set in Y is a $N\tau\alpha$-closed set (resp. $N\tau$ semi-closed, $N\tau$ pre-closed, $N\tau\beta$-closed) in X.

Theorem 4.3. A function $f : X \to Y$ is N^*-continuous on X, then it is N^*-α continuous function on X.
Proof: Assume $f : X \to Y$ be a N^*-continuous function on X and let $A \subseteq Y$ be a $N\sigma$-open set. Then $f^{-1}(A) \subseteq X$ is $N\tau$-open set in X. Since every $N\tau$-open set is $N\tau\alpha$-open set, then f is N^*-α continuous on X.

The converse of the above theorem need not be true as shown in the following example.

Example 4.4. For $N = 2$, let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset\}$, $\tau_2 O(X) = \{X, \emptyset, \{a\}\}$ and $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$ and $\sigma_2 O(Y) = \{Y, \emptyset, \{x, y\}\}$. Then $2\tau O(X) = \{X, \emptyset, \{a\}\}$ and $2\sigma O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$ and $f(c) = z$. Therefore, f is 2^*-α continuous function on X but not 2^*-continuous.

Theorem 4.5. A function $f : X \to Y$ is N^*-α continuous on X if and only if it is N^*-semi continuous and N^*-pre continuous.
Proof: The proof follows from the theorem 3.5.

Theorem 4.6. A function $f : X \to Y$ is N^*-semi continuous on X, then it is N^*-β continuous.

Theorem 4.7. A function $f : X \to Y$ is N^*-pre continuous on X, then it is N^*-β continuous.

The converse of the above theorems need not be true as shown in the following example.

Example 4.8. If $N = 2$, $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b, c\}\}$ and also $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$, $\sigma_2 O(Y) = \{Y, \emptyset, \{x, y\}\}$. Then $2\tau O(X) = \{X, \emptyset, \{a\}, \{b, c\}\}$, $2\sigma O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = z$ and $f(c) = y$. Then f is 2^*-pre continuous and 2^*-β continuous function on X but it is not 2^*-semi continuous and not 2^*-α continuous function. Also if $N = 3$, $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b\}, \{a, b\}\}$, $\tau_3 O(X) = \{X, \emptyset, \{a, b\}\}$ and also $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$, $\sigma_2 O(Y) = \{Y, \emptyset, \{y, z\}\}$, $\sigma_3 O(Y) = \{Y, \emptyset\}$. Then $3\tau O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $3\sigma O(Y) = \{Y, \emptyset, \{x\}, \{y, z\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$ and $f(c) = z$. Then f is 3^*-semi continuous and 3^*-β continuous on X but it is not 3^*-pre continuous and not 3^*-α continuous.

5. Quotient Mappings in N-Topology

In this section, we introduce and establish the properties of some new types of quotient mappings in N-topological spaces.

Definition 5.1. Let X and Y be N-topological spaces, then a surjective map $f : X \to Y$ is said to be

(i) N^*-quotient map if f is N^*-continuous and for each subset G of Y, $f^{-1}(G)$ is $N\tau$-open (or $N\tau$-closed) in X implies G is $N\sigma$-open (or $N\sigma$-closed) in Y.

(ii) N^*-α quotient map if f is N^*-α continuous and for each subset G of Y, $f^{-1}(G)$ is $N\tau$-open (or $N\tau$-closed) in X implies G is $N\sigma\alpha$-open (or $N\sigma\alpha$-closed) in Y.

(iii) N^*-semi quotient map if f is N^*-semi continuous and for each subset G of Y, $f^{-1}(G)$ is $N\tau$-open (or $N\tau$-closed) in X implies G is $N\sigma$ semi-open (or $N\sigma$ semi-closed) in Y.

Proposition 5.2. Let X, Y be two N-topological spaces and $f : X \to Y$ be a surjective map. Then

(i) every N^*-quotient map is N^*-α quotient.

(ii) every N^*-quotient map is N^*-semi quotient.

(iii) every N^*-α quotient map is N^*-semi quotient.

Proof: The proof is straightforward from the definition.

The following examples show that the converse of the above proposition need not be true.

Example 5.3. For $N = 2$, let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\}$, $\tau_2 O(X) = \{X, \emptyset\}$ and $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}\}$ and $\sigma_2 O(Y) = \{Y, \emptyset, \{x, y\}\}$. Then $2\tau O(X) = \{X, \emptyset, \{a\}\}$ and $2\sigma O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}\}$. Define $f : X \to Y$ by $f(a) = x$, $f(b) = y$ and $f(c) = z$. Therefore, f is 2^*-α quotient and 2^*-semi quotient map but not 2^*-quotient.

Example 5.4. For $N = 3$, let $X = \{a, b, c\}$ and $Y = \{x, y, z\}$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}, \{a, b\}\}$, $\tau_2 O(X) = \{X, \emptyset, \{b\}, \{a, b\}\}$, $\tau_3 O(X) = \{X, \emptyset, \{b\}\}$ and $\sigma_1 O(Y) = \{Y, \emptyset, \{x\}, \{x, z\}\}$, $\sigma_2 O(Y) = \{Y, \emptyset, \{y\}, \{x, y\}\}$ and $\sigma_3 O(Y) = \{Y, \emptyset, \{x\}, \{x, y\}, \{x, z\}\}$. Then $3\tau O(X) = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $3\sigma O(Y) = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}\}$. Define $f : X \to Y$ by $f(a) = y$, $f(b) = x$ and $f(c) = z$. Therefore, f is 3^*-semi quotient map but not 3^*-α quotient and not 3^*-quotient.

Definition 5.5. Let X and Y be two N-topological spaces, then a map $f : X \to Y$ is said to be

(i) N^*-open (or N^*-closed) if for every $N\tau$-open ($N\tau$-closed) set G of X, $f(G)$ is $N\sigma$-open (or $N\sigma$-closed) in Y.

(ii) N^*-α open (or N^*-α closed) if for every $N\tau$-open ($N\tau$-closed) set G of X, $f(G)$ is $N\sigma\alpha$-open (or $N\sigma\alpha$-closed) in Y.

(iii) N^*-semi open (or N^*-semi closed) if for every $N\tau$-open ($N\tau$-closed) set G of X, $f(G)$ is $N\sigma$ semi-open (or $N\sigma$ semi-closed) in Y.

Theorem 5.6. (i) Every surjective N^*-continuous map $f : X \to Y$ which is either N^*-open or N^*-closed is N^*-quotient map.

(ii) Every surjective N^*-α continuous map $f : X \to Y$ which is either N^*-α open or N^*-α closed is N^*-α quotient map.

(iii) Every surjective N^*-semi continuous map $f : X \to Y$ which is either N^*-semi open or N^*-semi closed is N^*-semi quotient map.

Proof: The proof is trivial from the definition.

Lemma 5.7. Let X be a N-topological space, Y be a set and $f : X \to Y$ be a surjective map. Then define $N\tau_f = \{G \subseteq Y : f^{-1}(G) \in N\tau O(X)\}$ is a topology on Y relative to which f is a N^*-quotient map. It is called N^*-quotient topology on Y induced by f.

Proof: The proof follows from the facts that $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(Y) = X$, $f^{-1}(\cup_{i=1}^{\infty} G_i) = \cup_{i=1}^{\infty} f^{-1}(G_i)$ and $f^{-1}(\cap_{i=1}^{n} G_i) = \cap_{i=1}^{n} f^{-1}(G_i)$.

The following lemmas can be proved similarly as the above lemma.

Lemma 5.8. Let X be a N-topological space, Y be a set and $f : X \to Y$ be a surjective map. Then define $N\tau\alpha_f = \{G \subseteq Y : f^{-1}(G) \in N\tau\alpha O(X)\}$ is a topology on Y relative to which f is a N^*-α quotient map. It is called N^*-α quotient topology on Y induced by f.

Lemma 5.9. Let X be a N-topological space, Y be a set and $f : X \to Y$ be a surjective map. Then define $N\tau S_f = \{G \subseteq Y : f^{-1}(G) \in N\tau SO(X)\}$ is a generalized topology on Y relative to which f is a N^*-semi quotient map but it need not be a topology. It is called N^*-semi quotient generalized topology on Y induced by f. If X is an extremally disconnected N-topological space, the intersection of two $N\tau$ semi-open sets in X is $N\tau$ semi-open and hence $N\tau S_f$ becomes a topology on Y.

Example 5.10. For $N = 2$, let $X = \{a, b, c\} = Y$. Consider $\tau_1 O(X) = \{X, \emptyset, \{a\}\} = \sigma_1 O(Y)$ and $\tau_2 O(X) = \{X, \emptyset\} = \sigma_2 O(Y)$. Then $2\tau O(X) = \{X, \emptyset, \{a\}\} = 2\sigma O(Y)$ and $2\tau\alpha O(X) = 2\tau SO(X) = 2\sigma\alpha O(Y) = 2\sigma SO(Y) = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Define $f : X \to Y$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. Clearly f is 2^*-quotient, 2^*-α quotient and 2^*-semi quotient map. Therefore, $2\tau_f = \{Y, \emptyset, \{a\}\}$ and $2\tau\alpha_f = 2\tau S_f = \{Y, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$.

Example 5.11. In example 6.4, f is 3^*-semi quotient map and therefore, $3\tau S_f = \{Y, \emptyset, \{x\}, \{y\}, \{x, y\}, \{x, z\}, \{y, z\}\}$ is not a topology on Y.

Theorem 5.12. Let X, Y, Z be N-topological spaces, $f : X \to Y$ be a N^*-quotient map and $h : X \to Z$ be a map that is constant on each set $f^{-1}(\{y\})$, for $y \in Y$. Then h induces a map $g : Y \to Z$ such that $g \circ f = h$. Then the induced map g is N^*-continuous if and only if h is N^*-continuous; g is N^*-quotient map if and only if h is N^*-quotient map.

Proof: Since h is constant on each set $f^{-1}(\{y\})$, for each $y \in Y$, the set $h(f^{-1}(\{y\}))$ is a one-point set in Z. Let us take this point as $g(y)$, then the map $g : Y \to Z$ such that for each $x \in X$, $g(f(x)) = h(x)$. If g is N^*-continuous, then $h = g \circ f$ is N^*-continuous. Conversely, assume h is N^*-continuous, for each $N\eta$-open set G of Z, $h^{-1}(G) = f^{-1}(g^{-1}(G))$ is $N\tau$-open in X. Since f is N^*-quotient, $g^{-1}(G)$ is $N\sigma$-open in Y and hence g is N^*-continuous.

If g is N^*-quotient map, then h is the composite of two N^*-quotient map and so is a N^*-quotient map. Conversely, assume h is a N^*-quotient map and since h is surjective, then g is surjective. Let $g^{-1}(G)$ be a $N\sigma$-open set in Y and since f is N^*-continuous, then the set $f^{-1}(g^{-1}(G)) = h^{-1}(G)$ is $N\tau$-open in X. Since h is a N^*-quotient map, G is $N\eta$-open in Z.

The following theorems can be proved similarly as the above theorem.

Theorem 5.13. Let X, Y, Z be N-topological spaces, $f : X \to Y$ be a N^*-α quotient map and $h : X \to Z$ be a N^*-continuous map that is constant on each set $f^{-1}(\{y\})$, for $y \in Y$. Then h induces a N^*-α continuous map $g : Y \to Z$ such that $g \circ f = h$.

Theorem 5.14. Let X, Y, Z be N-topological spaces, $f : X \to Y$ be a N^*-semi quotient map and $h : X \to Z$ be a N^*-continuous map that is constant on each set $f^{-1}(\{y\})$, for $y \in Y$. Then h induces a N^*-semi continuous map $g : Y \to Z$ such that $g \circ f = h$.

6. Conclusion

In this paper we established some weak form of open sets and its respective continuous and quotient mappings in our N-topological spaces. These concepts can be extended to other applicable research areas of topology such as Nano topology, Fuzzy topology, Supra topology and so on.

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