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Some Results on Lattice (Anti-Lattice) Ordered Double Framed Soft Sets

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Abstract − In this article, we generalised the notion of the lattice (anti-lattice) ordered soft sets and introduced the notion of the lattice (anti-lattice) ordered double framed soft sets and proved some results by applying the basic operations like union, intersection, union-product and intersection-product, etc. Further, by applying the operations of restricted union and restricted intersection, we elaborated the applications of lattice ordered double framed soft sets in algebraic structure.

Keywords − *Soft set, double framed soft set, lattice ordered double framed soft set, lattice ordered Boolean-algebra*

1. Introduction

In daily life there exists certain difficulties which deal with uncertainty, vague and precise like in environmental sciences, economics and engineering etc. To face such types of difficulties, there are many theories developed like probability theory, interval mathematical theory and theory of fuzzy set. These theories are classical mathematical tools. Due to the limitations of these theories, we felt hesitation in giving a comfortable solution to solve these problems, which are known as uncertainty, vague and precise. May be dealt with using a wide range existing theory such as the theory of fuzzy (intuitionistic fuzzy) set [1, 2, 3], the theory of interval mathematics [4], theory of probability, theory of vague set [5] and theory of rough set [6]. However, due to limitations and difficulties of these theories, Molodtsov [7] pointed out these problems and solved by introducing a new theory which is known as soft set theory. Maji et al. [8] introduced the applications of soft set theory in decision-making problems. Also, Maji et al. [9] studied the theoretical work on soft set theory to polish this concept so that readers could easily understand and contributed their role to extend the scope of this theory in different fields of life. After theoretical discussion, now we discussed the contributions of those researchers whose applied this concept in different fields of algebras like Aktaş and Çağman [10] studied the notion of soft sets and soft groups and introduced the notion of soft groups. They also defined the relation between fuzzy set, rough set and soft set and discussed its properties. Ali et al. [11] initiated the concept of lattice ordered soft sets and discussed some of its properties. Lattice ordered soft sets are very helpful in particular types of decision-making problems when there is some order between the elements of the parameter set. Mahmood et al. [12] initiated the concept of lattice ordered intuitionistic fuzzy soft sets. Mahmood et al. [13] worked on lattice ordered soft near rings. Jun and Ahn [14] initiated the notion of double framed soft set. For further information, we mention the readers to the papers [15-27] regarding soft algebras and properties of soft sets. Inspiring from the above literature and especially, the concept of lattice ordered soft sets [11]. This paper courage us to extend this concept into lattice ordered double framed soft sets because in this paper mentioned that sometimes we define particular order between linguistic terms, for example, the selections of the

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brilliant student based on percentage (80% to 90%) of marks in any educational institute of PhD Mathematics class.
This paper distributed in three sections, in 2nd section, some basic concepts of soft sets, properties of soft sets, lattice (anti-lattice) ordered soft set and double framed soft sets are discussed and introduced their notations. In the 3rd Section, we initiated the concept of the lattice (anti-lattice) ordered double framed soft sets and discussed their properties by using examples and results. Also, by using the notion of the lattice (anti-lattice) ordered double framed soft set we introduced the algebraic structures of the lattice (anti-lattice) ordered double framed soft set like bounded lattice, complemented lattice and distributed lattices etc. Note that for further study, we use "S-set" instead of soft set.

2. Preliminaries

In this section, we discussed some basic notions and properties related to S-set, lattice (anti-lattice) ordered S-set and double framed S-set.

Definition 2.1. [7] Let E be a parameter set, U be a universal set and let $P(U)$ denotes the power set of U and $A \subseteq E$. Then, a set-valued function α from A to $P(U)$ is called an S-set over U and is denoted as (α, A).

Definition 2.2. [9] A S-set (α, A) is called a soft subset of (β, B), over U if

1) $A \subseteq B$.
2) $\alpha(x) \subseteq \beta(x)$ for all $x \in A$.

It is denoted as $(\alpha, A) \tilde{\subseteq} (\beta, B)$. In this case (β, B) is called a soft superset of (α, A).

Definition 2.3. [9] Let (α, A) and (β, B) be S-sets over U. Then, (α, A) and (β, B) are called soft equal if $(\alpha, A) \tilde{\subseteq} (\beta, B)$ and $(\beta, B) \tilde{\subseteq} (\alpha, A)$.

Definition 2.4. Let L be a non-empty poset. Then, L is called a lattice if for each $\{x, y\} \subseteq L$ there exist $sup\{x, y\} \in L$ and $inf\{x, y\} \in L$.

Definition 2.5. A lattice having both first and last element is called bounded lattice.

Definition 2.6. A distributive lattice with the least and the greatest element is called Boolean algebra if and only if every element has a complement in it.

Definition 2.7. A bounded distributive lattice L along with a unary operation "c" which satisfies $(x \wedge y)^c = x^c \vee y^c$ and $(x^c)^c = x$ is called De 'Morgan's algebra.

Definition 2.8. A De 'Morgan's algebra which satisfies $x \wedge x^c \leq y \vee y^c$ for all x, y is called Kleene algebra.

Definition 2.9. [11] A S-set (α, A) is said to be lattice (anti-lattice) ordered S-set if $x_1 \leq x_2$ implies $\alpha(x_1) \subseteq \alpha(x_2)$ $\big(\alpha(x_2) \subseteq \alpha(x_1)\big)$ for all $x_1, x_2 \in A$.

Definition 2.10. [14] A set $\big((\alpha, \beta), A\big)$ is said to be double framed soft set (DFS-set), where α and β both are S-sets over U and A is a subset of E (E is the set of parameters).

Definition 2.11. [14] Let $\big((\alpha, \beta), A\big)$ and $\big((\lambda, \mu), B\big)$ be double framed soft sets (DFS-sets) over U. Then, $\big((\alpha, \beta), A\big)$ is called a double framed soft subset (DFS-subset) of $\big((\lambda, \mu), B\big)$ if

1) $A \subseteq B$,
2) $\alpha(x) \subseteq \lambda(x), \beta(x) \supseteq \mu(x)$ for all $x \in A$.

We write $\big((\alpha,\beta),A\big) \tilde{\sqsubseteq} \big((\lambda,\mu),B\big)$. In this case $\big((\lambda,\mu),B\big)$ is called a DFS-superset of $\big((\alpha,\beta),A\big)$.

Definition 2.12. [14] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U. Then, their uni-int product is defined as a DFS-set $\big((H_1,H_2),D\big) = \big((\alpha,\beta),A\big) \vee \big((\lambda,\mu),B\big)$, where $D = A \times B$, $H_1 = \alpha \vee \lambda$, $H_2 = \beta \wedge \mu$ and $H_1(x_1,y_1) = \alpha(x_1) \cup \lambda(y_1)$, $H_2(x_1,y_1) = \lambda(x_1) \cap \mu(y_1)$ for all $(x_1,y_1) \in A \times B$, where $x_1 \in A$ and $y_1 \in B$. We shall call this uni-int product of DFS-set as union-product of DFS-set.

Definition 2.13. [14] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U. Then, their int-uni product is defined as a DFS-set $\big((H_1,H_2),D\big) = \big((\alpha,\beta),A\big) \wedge \big((\lambda,\mu),B\big)$, where $D = A \times B$ and $H_1(x_1,y_1) = \alpha(x_1) \cap \lambda(y_2)$, $H_2(x_1,y_1) = \lambda(x_1) \cup \mu(y_1)$ for all $(x_1,y_1) \in A \times B$, $H_1 = \alpha \wedge \lambda$, $H_2 = \beta \vee \mu$ where $x_1 \in A$ and $y_1 \in B$. We shall call this int-uni product of DFS-set as intersection-product of DFS-set.

Definition 2.14. [16] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U. Then, their extended uni-int is defined as a DFS-set $\big((H_1,H_2),A \cup B\big)$, where $H_1 = \alpha \,\tilde{\cup}\, \lambda : (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \alpha(e) & if\ e \in A\backslash B \\ \lambda(e) & if\ e \in B\backslash A \\ \alpha(e) \cup \lambda(e) & if\ e \in A \cap B \end{cases}$$

and $H_2 = \beta \,\tilde{\cap}\, \mu: (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \beta(e) & if\ e \in A\backslash B \\ \mu(e) & if\ e \in B\backslash A \\ \beta(e) \cap \mu(e) & if\ e \in A \cap B \end{cases}$$

It is denoted as $\big((\alpha,\beta),A\big) \sqcup_{\varepsilon} \big((\lambda,\mu),B\big) = \big((H_1,H_2),A \cup B\big)$. We shall call this extended uni-int of DFS-set as union of DFS-set.

Definition 2.15. [16] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ are two DFS-sets over U. Then, their extended int-uni is defined as a DFS-set $\big((H_1,H_2),A \cup B\big)$, where $H_1 = \alpha \,\tilde{\cap}\, \lambda : (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \alpha(e) & if\ e \in A\backslash B \\ \lambda(e) & if\ e \in B\backslash A \\ \alpha(e) \cap \lambda(e) & if\ e \in A \cap B \end{cases}$$

and $H_2 = \beta \,\tilde{\cup}\, \mu: (A \cup B) \longrightarrow P(U)$ defined as

$$e \longrightarrow \begin{cases} \beta(e) & if\ e \in A\backslash B \\ \mu(e) & if\ e \in B\backslash A \\ \beta(e) \cup \mu(e) & if\ e \in A \cap B \end{cases}$$

It is denoted as $\big((\alpha,\beta),A\big) \sqcap_{\varepsilon} \big((\lambda,\mu),B\big) = \big((H_1,H_2),A \cup B\big)$. We shall call this extended int-uni of DFS-set as intersection of DFS-set.

Definition 2.16. [16] Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be DFS-sets over U such that $A \cap B \neq \emptyset$. Then, their restricted uni-int is denoted as $\big((\alpha,\beta),A\big) \sqcup \big((\lambda,\mu),B\big)$ and defined as $\big((\alpha,\beta),A\big) \sqcup \big((\lambda,\mu),B\big) = \big((H_1,H_2),D\big)$ where $D = A \cap B$ and for every $x \in D$, $H_1(x) = \alpha(x) \cup \lambda(x)$, $H_2(x) = \lambda(x) \cap \mu(x)$. We shall call this restricted uni-int of DFS-set as restricted union of DFS-set.

Definition 2.17. [16] Let $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ be DFS-set over U such that $A\cap B\neq\emptyset$. Then, their restricted int-uni is denoted as $((\alpha,\beta),A)\sqcap((\lambda,\mu),B)$ and defined as $((\alpha,\beta),A)\sqcap((\lambda,\mu),B)=((H_1,H_2),D)$, where $D=A\cap B$ and for all $x\in D$, $H_1(x)=\alpha(x)\cap\lambda(x)$, $H_2(x)=\lambda(x)\cup\mu(x)$. We shall call this restricted int-uni of DFS-set as restricted intersection of DFS-set.

Definition 2.18. [16] A DFS-set $((\alpha,\beta),A)$ over U is called relative whole DFS-set, if $\alpha{:}\,A\longrightarrow P(U)$ and $\beta{:}\,A\longrightarrow P(U)$ are defined as

$\alpha(x)=U$ and $\beta(x)=\emptyset$ for all $x\in A$.

It is denoted as $A_{(\mathfrak{U},\emptyset)}$.

Definition 2.19. [16] A DFS-set $((\alpha,\beta),A)$ over U is called relative null DFS-set, if $\alpha:A\longrightarrow P(U)$ and $\beta:A\longrightarrow P(U)$ are defined as

$\alpha(x)=\emptyset$ and $\beta(x)=U$ for all $x\in A$.

It is denoted as $A_{(\emptyset,\mathfrak{U})}$.

Definition 2.20. [16] For a DFS-set $((\alpha,\beta),A)$, the complement of $((\alpha,\beta),A)$ is defined as a DFS-set $((\alpha^c,\beta^c),A)$, where $\alpha^c:A\longrightarrow P(U)$ and $\beta^c:A\longrightarrow P(U)$ are defined as

$\alpha^c(x)=\big(\alpha(x)\big)^c$ and $\beta^c(x)=\big(\beta(x)\big)^c$ for all $x\in A$.

It is denoted as $\big((\alpha,\beta),A\big)^c\cong((\alpha^c,\beta^c),A)$.

Proposition 2.21. (De Morgan's Laws)

Let (α,A) and (β,B) be LOS-sets (ALOS-sets) over U. Then,

(1) $\big((\alpha,A)\sqcup_{\mathcal{E}}(\beta,B)\big)^c=(\alpha,A)^c\sqcap_{\mathcal{E}}(\beta,B)^c$, if $A=B$.

(2) $\big((\alpha,A)\sqcap_{\mathcal{E}}(\beta,B)\big)^c=(\alpha,A)^c\sqcup_{\mathcal{E}}(\beta,B)^c$, if $A=B$.

(3) $\big((\alpha,A)\vee(\beta,B)\big)^c=(\alpha,A)^c\wedge(\beta,B)^c$.

(4) $\big((\alpha,A)\wedge(\beta,B)\big)^c=(\alpha,A)^c\vee(\beta,B)^c$.

Proposition 2.22 If (α,A), (β,B) and (γ,C) be any LOS-sets (ALOS-sets) over U. Then, followings are LOS-sets (ALOS-sets),

(1) $(\alpha,A)\vee\big((\beta,B)\sqcup_{\mathcal{E}}(\gamma,C)\big)$

(2) $(\alpha,A)\vee\big((\beta,B)\sqcap_{\mathcal{E}}(\gamma,C)\big)$

(3) $(\alpha,A)\wedge\big((\beta,B)\sqcup_{\mathcal{E}}(\gamma,C)\big)$

(4) $(\alpha,A)\wedge\big((\beta,B)\sqcap_{\mathcal{E}}(\gamma,C)\big)$

(5) $(\alpha,A)\vee\big((\beta,B)\sqcap(\gamma,C)\big)$

(6) $(\alpha,A)\wedge\big((\beta,B)\sqcup(\gamma,C)\big)$

If $\sqcup_{\mathcal{E}}$ and $\sqcap_{\mathcal{E}}$ are LOS-set (ALOS-set).

Theorem 2.23. $\big(DFSS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is an MV-algebra.

Proof. (1) $\big(DFSS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is a commutative monoid.

(2) $\Big(A_{(\alpha_1,\beta_1)^c}\Big)^c=A_{(\alpha_1,\beta_1)}$

The other conditions satisfied trivially. Hence, $\big(DFSS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is MV-algebra.

Theorem 2.24. $\big(DFSS(U)_A,\ \sqcap,{}^c,A_{(\mathfrak{U},\emptyset)}\big)$ is an MV-algebra.

Proof. It follows from the above theorem.

3. Lattice (Anti-Lattice) Ordered Double Framed Soft Sets

In this section, our primary purpose is to define lattice (anti-lattice) ordered double framed S-set and discuss their properties and results with the help of examples. Note that we write LODFS-set and ALODFS-set for lattice ordered double framed soft set, and anti-lattice ordered double framed soft set respectively unless otherwise specified.

Definition 3.1. A DFS-set $\big((\alpha,\beta),A\big)$ is called LODFS-set (ALODFS-set) if $x_1 \leq x_2$ implies $\alpha(x_1) \subseteq \alpha(x_2)$ and $\beta(x_1) \supseteq \beta(x_2)$ $\big(\alpha(x_1) \supseteq \alpha(x_2)$ and $\beta(x_1) \subseteq \beta(x_2)\big)$ for all $x_1, x_2 \in A$.

Example 3.2. Let a company prepare a different design of cars in different colours like, $A = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ be the set of parameters which represent different types of colours of cars, where $e_7 = white, e_6 = black, e_5 = Grey, e_4 = red, e_3 = blue, e_2 = green, e_1 = yellow$ and $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}$ be the set of new designs of cars in different colours. To sell these cars, the company define lattice order between the parameters which depend upon the demand of people under the supervision of two experts say α and β. The order between the elements of A is shown in Fig. 1. $\alpha, \beta: A \longrightarrow P(U)$ are two set-valued mappings representing high-cost and low-cost of cars. Therefore, DFS-set $\big((\alpha,\beta),A\big)$ showing high-cost and low-cost for design in colours may be considered as

$$\{\alpha(e_1) = \{u_1\}, \alpha(e_2) = \{u_1, u_2\}, \alpha(e_3) = \{u_1, u_3\}, \alpha(e_4) = \{u_1, u_2, u_3, u_4\}, \alpha(e_5) = \{u_1, u_3, u_5\},$$
$$\alpha(e_6) = \{u_1, u_2, u_3, u_4, u_5, u_6\}, \alpha(e_7) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}, \beta(e_7) = \{u_1, u_8\},$$
$$\beta(e_6) = \{u_1, u_3, u_8\}, \beta(e_5) = \{u_1, u_3, u_8\}, \beta(e_4) = \{u_1, u_2, u_3, u_4, u_5, u_8\}, \beta(e_3) = \{u_1, u_3, u_5, u_8\},$$
$$\beta(e_2) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_8\}, \beta(e_1) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8\}\big\}$$

It is more appropriate to characterise DFS-set in the form of a table, for computer application.

The tabular form of DFS-set $\big((\alpha,\beta),A\big)$ is defined in Table 1. If a car having a different colour in a set U has high cost or low cast we write 1, otherwise we write 0.
From Table 1, we can easily see that

$\alpha(e_1) \subseteq \alpha(e_2) \subseteq \alpha(e_4) \subseteq \alpha(e_6) \subseteq \alpha(e_7)$, $\alpha(e_1) \subseteq \alpha(e_3) \subseteq \alpha(e_5) \subseteq \alpha(e_6) \subseteq \alpha(e_7)$, $\alpha(e_1) \subseteq \alpha(e_3) \subseteq \alpha(e_4) \subseteq \alpha(e_6) \subseteq \alpha(e_7)$ and $\beta(e_1) \supseteq \beta(e_2) \supseteq \beta(e_4) \supseteq \beta(e_6) \supseteq \beta(e_7)$, $\beta(e_1) \supseteq \beta(e_3) \supseteq \beta(e_4) \supseteq \beta(e_6) \supseteq \beta(e_7)$, $\beta(e_1) \supseteq \beta(e_3) \supseteq \beta(e_5) \supseteq \beta(e_6) \supseteq \beta(e_7)$.

Example 3.3. Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ (universe set) be the set of seven buildings and $B = \{e_1, e_2, e_3, e_4\}$ be a set of parameters, where

e_1; one-floor building.
e_2; two-floor building.
e_3; three-floor building.
e_4; four-floor building.

There is an order between the elements of B. This order can be nominated as $e_1 \leq e_2 \leq e_3 \leq e_4$. Now the DFS-set $\big((\lambda,\mu),B\big)$ defined as $\big\{\lambda(e_1) = \{u_1, u_3\}, \lambda(e_2) = \{u_1, u_3, u_5\}, \lambda(e_3) = \{u_1, u_2, u_3, u_4, u_5\}, \lambda(e_4) = \{u_1, u_2, u_3, u_4, u_5, u_6\}, \mu(e_4) = \{u_1, u_5\}, \mu(e_3) = \{u_1, u_2, u_5\}, \mu(e_2) = \{u_1, u_2, u_3, u_4, u_5\}, \mu(e_1) = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}\big\}$.

Then, $\lambda(e_4) \supseteq \lambda(e_3) \supseteq \lambda(e_2) \supseteq \lambda(e_1)$ and $\mu(e_4) \subseteq \mu(e_3) \subseteq \mu(e_2) \subseteq \mu(e_1)$. Thus, $\big((\lambda,\mu),B\big)$ is ALODFS-set. The tabular form of ALODFS-set $\big((\lambda,\mu),B\big)$ is defined in Table 2.

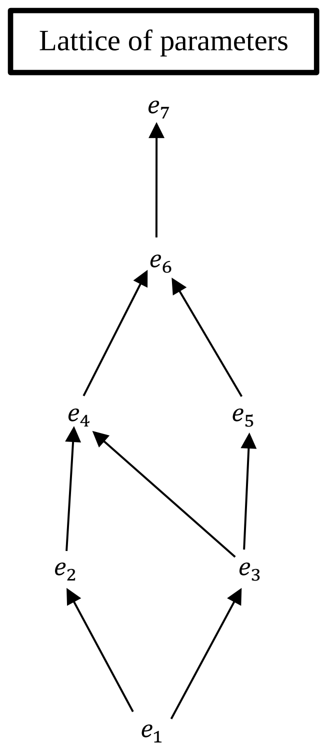

Fig. 1. Lattice of parameters

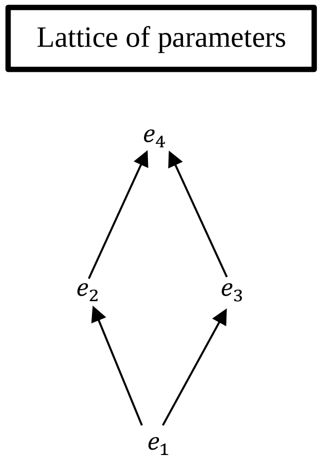

Fig. 2. Lattice of parameters

Table 1 LODFS-set $((\alpha,\beta),A)$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
e_1	$(1,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$
e_2	$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
e_3	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(0,1)$	$(0,0)$	$(0,0)$	$(0,1)$
e_4	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,0)$	$(0,0)$	$(0,0)$	$(0,1)$
e_5	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,0)$	$(0,0)$	$(0,0)$	$(0,1)$
e_6	$(1,1)$	$(1,0)$	$(1,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,0)$	$(0,1)$
e_7	$(1,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(0,1)$

Table 2 ALODFS-set $((\lambda,\mu),B)$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
e_1	$(1,1)$	$(0,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$
e_2	$(1,1)$	$(0,1)$	$(1,1)$	$(0,1)$	$(1,1)$	$(0,0)$	$(0,0)$
e_3	$(1,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(1,1)$	$(0,0)$	$(0,0)$
e_4	$(1,1)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,1)$	$(1,0)$	$(0,0)$

Note that, we can easily understand LODFS-set and ALODFS-set from Table 1. and 2.

Proposition 3.4. Restricted union of two LODFS-set (ALODFS-set) $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ is a LODFS-set (ALODFS-set).

Proof. Let $((\alpha,\beta),A)$ and $((\lambda,\mu),B)$ are two LODFS-set. Then, their restricted union is defined as such that $((\alpha,\beta),A) \sqcup ((\lambda,\mu),B) = ((H_1,H_2),C)$ where $H_1 = \alpha \mathbin{\tilde{\cup}} \lambda$, $H_2 = \beta \mathbin{\tilde{\cap}} \mu$ and $C = A \cap B$. If $A \cap B = \emptyset$, then the result is trivial. Now assume that $A \cap B \neq \emptyset$, since $A, B \subseteq E$, then both A and B inherit partial order from E. So, if $x_1 \leq_A x_2$ for all $x_1, x_2 \in A$, then $\alpha(x_1) \subseteq \alpha(x_2)$ and $\beta(x_1) \supseteq \beta(x_2)$. Similarly, if $y_1 \leq_B y_2$ for all $y_1, y_2 \in B$, then $\lambda(y_1) \subseteq \lambda(y_2)$ and $\mu(y_1) \supseteq \mu(y_2)$. Therefore, for any $z_1, z_2 \in C$ such that $\alpha(z_1) \subseteq \alpha(z_2)$, $\beta(z_1) \supseteq \beta(z_2)$ and $\lambda(z_1) \subseteq \lambda(z_2), \mu(z_1) \supseteq \mu(z_2)$. Then, $\alpha(z_1) \cup \lambda(z_1) \subseteq \alpha(z_2) \cup \lambda(z_2)$ and $\beta(z_1) \cap \mu(z_1) \supseteq \beta(z_2) \cap \mu(z_2)$ implies that $H_1(z_1) \subseteq$

$H_1(z_2)$ and $H_2(z_1) \supseteq H_2(z_2)$ for $(z_1, z_2) \in \leq_C$. Thus, we conclude that the restricted union of two DFS-set is also double framed soft set.

Proposition 3.5. The restricted intersection of two LODFS-set (ALODFS-set) $\big((\alpha, \beta), A\big)$ and $\big((\lambda, \mu), B\big)$ is a LODFS-set (ALODFS-set).

Proof. The proof is like to Proposition 3.4, by using the definition of the restricted intersection**.**
The following example illustrates that in general, the union and intersection of LODFS-set (ALODFS-set) may not be a LODFS-set (ALODFS-set).
From now to onward, we use a table to understand LODFS-set and ALODFS-set.

Example 3.6. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a lattice ordered set which is defined in fig. 4. Let $A = \{e_1, e_2, e_4, e_5\}$ and $B = \{e_1, e_2, e_3, e_4\}$. Consider two LODFS-set $\big((\alpha, \beta), A\big)$ and $\big((\lambda, \mu), B\big)$ over an initial universal set $U = \{u_1, u_2, u_3, u_4, u_5\}$ are defined as shown in Table 3 and 4 respectively such that $\alpha(e_1) \subseteq \alpha(e_2) \subseteq \alpha(e_4) \subseteq \alpha(e_5)$ and $\beta(e_1) \supseteq \beta(e_2) \supseteq \beta(e_4) \supseteq \beta(e_5)$.

Table 3 LODFS-set $\big((\alpha, \beta), A\big)$

	u_1	u_2	u_3	u_4	u_5
e_1	$(1, 1)$	$(0, 1)$	$(0, 1)$	$(0, 0)$	$(0, 1)$
e_2	$(1, 1)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(0, 0)$
e_4	$(1, 1)$	$(0, 0)$	$(0, 1)$	$(1, 0)$	$(1, 0)$
e_5	$(1, 1)$	$(1, 0)$	$(0, 0)$	$(1, 0)$	$(1, 0)$

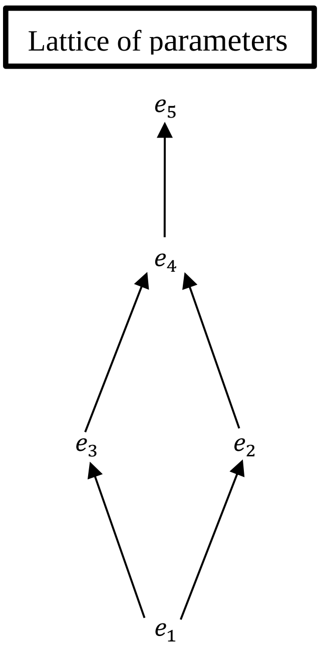

Fig. 3 Lattice of parameters

Table 4. LODFS-set $\big((\lambda, \mu), B\big)$

	u_1	u_2	u_3	u_4	u_5
e_1	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$	$(0, 1)$
e_2	$(0, 0)$	$(0, 1)$	$(0, 1)$	$(1, 0)$	$(1, 0)$
e_3	$(0, 0)$	$(1, 1)$	$(1, 0)$	$(1, 1)$	$(0, 0)$
e_4	$(0, 0)$	$(1, 1)$	$(1, 0)$	$(1, 0)$	$(1, 0)$

Now by definition of union, we have $\big((\alpha,\beta),A\big) \sqcup_{\mathcal{E}} \big((\lambda,\mu),B\big) = \big((H_1,H_2\ ),C\big)$, where $H_1 = \alpha \,\tilde{\cup}\, \lambda$, $H_2 = \beta \,\tilde{\cap}\, \mu$ and $C = A \cup B$, so we have the following table for the union.

Table 5. The union of the LODFS-sets

	u_1	u_2	u_3	u_4	u_5
e_1	$(1,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,1)$
e_2	$(1,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$
e_3	$(0,0)$	$(1,1)$	$(1,0)$	$(1,1)$	$(0,0)$
e_4	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$	$(1,0)$
e_5	$(1,1)$	$(1,0)$	$(0,0)$	$(1,0)$	$(1,0)$

From Table 5, we note that the union of $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ is not LODFS-set because $H_1(e_4) \nsubseteq H_1(e_5)$, $H_2(e_1) \nsupseteq H_2(e_3)$ and $H_2(e_4) \nsupseteq H_2(e_5)$ so $\big((H_1,H_2\ ),C\big)$ is not a LODFS-set.

Now by definition of intersection, we have $\big((\alpha,\beta),A\big) \sqcap_{\mathcal{E}} \big((\lambda,\mu),B\big) = \big((H_3,H_4),D\big)$ where $H_3 = \alpha \,\tilde{\cap}\, \lambda$, $H_4 = \beta \,\tilde{\cup}\, \mu$ and $C = A \cup B$, so we have the following table for the intersection.

Table. 6 The intersection of the LODFS-sets

	u_1	u_2	u_3	u_4	u_5
e_1	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$	$(0,1)$
e_2	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(0,0)$
e_3	$(0,0)$	$(1,1)$	$(1,0)$	$(1,1)$	$(0,0)$
e_4	$(0,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$
e_5	$(1,1)$	$(1,0)$	$(0,0)$	$(1,0)$	$(1,0)$

From Table. 6, we note that intersection of two LODFS-set $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ is not LODFS-set because $H_3(e_3) \nsubseteq H_3(e_4)$ and $H_4(e_3) \nsupseteq H_4(e_5)$ implies $\big((H_3,H_4),D\big)$ is not a LODFS-set.

Notice that from the above example, in general union and intersection of two LODFS-set may not a LODFS-set. Similarly, in general union and intersection of two ALODFS-set may not be an ALODFS-set. However, we can define the following.

Proposition 3.7. The intersection of two LODFS-set (ALODFS-set) $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ is LODFS-set (ALODFS-set) if $\big((\alpha,\beta),A\big) \subseteq \big((\lambda,\mu),B\big)$ or $\big((\lambda,\mu),B\big) \subseteq \big((\alpha,\beta),A\big)$.

Proof. Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be LODFS-set, then by definition of intersection we have $\big((\alpha,\beta),A\big) \sqcap_{\mathcal{E}} \big((\lambda,\mu),B\big) = (H,C)$, where $C = A \cup B$, $H = (H_1,H_2)$ and $H_1 = \alpha \,\tilde{\cap}\, \lambda$, $H_2 = \beta \,\tilde{\cup}\, \mu$. Now without any loss of generality, we say that $\big((\alpha,\beta),A\big) \subseteq \big((\lambda,\mu),B\big)$. Since $A \subseteq B$, then $A \cup B = B$ implies $B = C$. As $B = C$ so $H(z) = (H_1,H_2)(z)$ for all $z \in C$ implies that (H,C) is LODFS-set. Hence the intersection of two LODFS-set is also LODFS-set if one of them is contained into other.

Similarly, we can prove for ALODFS-set.

Proposition 3.8. Union of two LODFS-set (ALODFS-set) $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ is LODFS-set (ALODFS-set) if $\big((\alpha,\beta),A\big) \subseteq \big((\lambda,\mu),B\big)$ or $\big((\lambda,\mu),B\big) \subseteq \big((\alpha,\beta),A\big)$.

Proof. Like the above Proposition using the definition of the union of LODFS-set (ALODFS-set). The complement of LODFS-set $\big((\alpha,\beta),A\big)$ is denoted as $\big((\alpha,\beta),A\big)^c$ and defined as $\big((\alpha,\beta),A\big)^c =$

$((\alpha,\beta)^c, A) = \big((\alpha^c, \beta^c), A\big)$, where $(\alpha,\beta)^c = (\alpha^c, \beta^c)$ and $\alpha^c, \beta^c : A \longrightarrow P(U)$ are defined as $\alpha^c(a) = U\backslash\alpha(a)$ and $\beta^c(a) = U\backslash\beta(a)$ for all $a \in A$ is called ALODFS-set.

Similarly, the complement of ALODFS-set is LODFS-set.

Proposition 3.9. (De Morgan Laws)

Let $\big((\alpha,\beta), A\big)$ and $\big((\lambda,\mu), B\big)$ be LODFS-sets (ALODFS-sets) over U. Then,

1) $\Big(\big((\alpha,\beta),A\big) \sqcup_\varepsilon \big((\lambda,\mu),B\big)\Big)^c = \big((\alpha,\beta),A\big)^c \sqcap_\varepsilon \big((\lambda,\mu),B\big)^c$, if $A = B$.
2) $\Big(\big((\alpha,\beta),A\big) \sqcap_\varepsilon \big((\lambda,\mu),B\big)\Big)^c = \big((\alpha,\beta),A\big)^c \sqcup_\varepsilon \big((\lambda,\mu),B\big)^c$, if $A = B$.
3) $\Big(\big((\alpha,\beta),A\big) \sqcup \big((\lambda,\mu),B\big)\Big)^c = \big((\alpha,\beta),A\big)^c \sqcap \big((\lambda,\mu),B\big)^c$
4) $\Big(\big((\alpha,\beta),A\big) \sqcap \big((\lambda,\mu),B\big)\Big)^c = \big((\alpha,\beta),A\big)^c \sqcup \big((\lambda,\mu),B\big)^c$

Proposition 3.10. (Distributive Laws)

If $\big((\alpha,\beta),A\big)$, $\big((\lambda,\mu),B\big)$ and $\big((\gamma,\delta),C\big)$ be any LODFS-sets (ALODFS-sets) over U, then the following conditions hold

1) $((\alpha,\beta),A) \sqcup \big(((\lambda,\mu),B) \sqcup_\varepsilon ((\gamma,\delta),C)\big) = \big(((\alpha,\beta),A) \sqcup ((\lambda,\mu),B)\big) \sqcup_\varepsilon \big(((\alpha,\beta),A) \sqcup ((\gamma,\delta),C)\big)$ if $A \subseteq C$.
2) $((\alpha,\beta),A) \sqcup \big(((\lambda,\mu),B) \sqcap_\varepsilon ((\gamma,\delta),C)\big) = \big(((\alpha,\beta),A) \sqcup ((\lambda,\mu),B)\big) \sqcap_\varepsilon \big(((\alpha,\beta),A) \sqcup ((\gamma,\delta),C)\big)$ if $A \subseteq C$.
3) $((\alpha,\beta),A) \sqcap \big(((\lambda,\mu),B) \sqcup_\varepsilon ((\gamma,\delta),C)\big) = \big(((\alpha,\beta),A) \sqcap ((\lambda,\mu),B)\big) \sqcup_\varepsilon \big(((\alpha,\beta),A) \sqcap ((\gamma,\delta),C)\big)$ if $A \subseteq C$.
4) $((\alpha,\beta),A) \sqcap \big(((\lambda,\mu),B) \sqcap_\varepsilon ((\gamma,\delta),C)\big) = \big(((\alpha,\beta),A) \sqcap ((\lambda,\mu),B)\big) \sqcap_\varepsilon \big(((\alpha,\beta),A) \sqcap ((\gamma,\delta),C)\big)$ if $A \subseteq C$.
5) $((\alpha,\beta),A) \sqcup \big(((\lambda,\mu),B) \sqcap ((\gamma,\delta),C)\big) = \big(((\alpha,\beta),A) \sqcup ((\lambda,\mu),B)\big) \sqcap \big(((\alpha,\beta),A) \sqcup ((\gamma,\delta),C)\big)$
6) $((\alpha,\beta),A) \sqcap \big(((\lambda,\mu),B) \sqcup ((\gamma,\delta),C)\big) = \big(((\alpha,\beta),A) \sqcap ((\lambda,\mu),B)\big) \sqcup \big(((\alpha,\beta),A) \sqcap ((\gamma,\delta),C)\big)$

Let A and B be ordered sets, then σ be a partial order on $A \times B$ defined in such a way that, for (x, y), $(x', y') \in A \times B$ such that $(x, y) \leq (x', y')$ if and only if $x \leq_A x'$ and $y \leq_B y'$. From now to onward we will use σ for partial order relation on $A \times B$.

Proposition 3.11. Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be LODFS-sets (ALODFS-sets), then their union-product is also a LODFS-set (ALODFS-set).

Proof. Since $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ are LODFS-sets so we must prove $\big((\alpha,\beta),A\big) \vee \big((\lambda,\mu),B\big)$ is LODFS-set. Now by definition of union-product we have $\big((\alpha,\beta),A\big) \vee \big((\lambda,\mu),B\big) = \big((H_1,H_2),D\big)$ where $D = A \times B$, is a poset. Now $A, B \subseteq E$, so both A and B have taken some partial ordered from E. Then, for all $x_1, x_2 \in A$ such that $x_1 \leq_A x_2$ implies $\alpha(x_1) \subseteq \alpha(x_2)$, $\beta(x_1) \supseteq \beta(x_2)$ and for all $y_1, y_2 \in B$ such that $y_1 \leq_B y_2$ implies $\lambda(y_1) \subseteq \lambda(y_2)$, $\mu(y_1) \supseteq \mu(y_2)$. Now σ be a porelation between the element of $D = A \times B$ in such a way $(x_1,y_1)\sigma_D(x_2,y_2)$, where $(x_1,y_1),(x_2,y_2) \in A \times B$, we note that this order induced by order of A and B. Since $(x_1,y_1)\sigma(x_2,y_2)$ and $\alpha(x_1) \subseteq \alpha(x_2)$, $\beta(x_1) \supseteq \beta(x_2)$ and $\lambda(y_1) \subseteq \lambda(y_2)$, $\mu(y_1) \supseteq \mu(y_2)$, then $\alpha(x_1) \cup \lambda(y_1) \subseteq \alpha(x_2) \cup \lambda(y_2)$ and $\beta(x_1) \cap \mu(y_1) \supseteq \beta(x_2) \cap \mu(y_2)$ implies that $H_1(x_1,y_1) \subseteq H_1(x_2,y_2)$ and $H_2(x_1,y_1) \supseteq H_2(x_2,y_2)$ implies $\big((\alpha,\beta),A\big) \vee \big((\lambda,\mu),B\big)$ is LODFS-set.

Similarly, we can prove for ALODFS-set.

Proposition 3.12. Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be LODFS-sets (ALODFS-sets), then their intersection-product is also a LODFS-set (ALODFS-set).

Proof. By using the definition of intersection-product, we can prove like Proposition 3.11.

Proposition 3.13. (De Morgan Laws) Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be LODFS-set (ALODFS-set) over U. Then,

(1) $\Big(\big((\alpha,\beta),A\big) \vee \big((\lambda,\mu),B\big)\Big)^c = \big((\alpha,\beta),A\big)^c \wedge \big((\lambda,\mu),B\big)^c$

(2) $\Big(\big((\alpha,\beta),A\big) \wedge \big((\lambda,\mu),B\big)\Big)^c = \big((\alpha,\beta),A\big)^c \vee \big((\lambda,\mu),B\big)^c$.

Proposition 3.14. (Distributive Laws)

If $\big((\alpha,\beta),A\big)$, $\big((\lambda,\mu),B\big)$ and $\big((\gamma,\delta),C\big)$ be any LODFS-sets (ALODFS-sets) over U. Then, the following conditions hold

(1) $\big((\alpha,\beta),A\big) \vee \Big(\big((\lambda,\mu),B\big) \wedge \big((\gamma,\delta),C\big)\Big)$
$= \Big(\big((\alpha,\beta),A\big) \vee \big((\lambda,\mu),B\big)\Big) \wedge \Big(\big((\alpha,\beta),A\big) \vee \big((\gamma,\delta),C\big)\Big)$.

(2) $\big((\alpha,\beta),A\big) \wedge \Big(\big((\lambda,\mu),B\big) \vee \big((\gamma,\delta),C\big)\Big)$
$= \Big(\big((\alpha,\beta),A\big) \wedge \big((\lambda,\mu),B\big)\Big) \vee \Big(\big((\alpha,\beta),A\big) \wedge \big((\gamma,\delta),C\big)\Big)$.

Proposition 3.15. If $\big((\alpha,\beta),A\big)$, $\big((\lambda,\mu),B\big)$ and $\big((\gamma,\delta),C\big)$ be any two LODFS-sets (ALODFS-sets) over U. Then, followings are LODFS-set (ALODFS-set),

(1) $\big((\alpha,\beta),A\big) \vee \Big(\big((\lambda,\mu),B\big) \sqcup_{\mathcal{E}} \big((\gamma,\delta),C\big)\Big)$.

(2) $\big((\alpha,\beta),A\big) \vee \Big(\big((\lambda,\mu),B\big) \sqcap_{\mathcal{E}} \big((\gamma,\delta),C\big)\Big)$.

(3) $\big((\alpha,\beta),A\big) \wedge \Big(\big((\lambda,\mu),B\big) \sqcup_{\mathcal{E}} \big((\gamma,\delta),C\big)\Big)$.

(4) $\big((\alpha,\beta),A\big) \wedge \Big(\big((\lambda,\mu),B\big) \sqcap_{\mathcal{E}} \big((\gamma,\delta),C\big)\Big)$.

(5) $\big((\alpha,\beta),A\big) \vee \Big(\big((\lambda,\mu),B\big) \sqcap \big((\gamma,\delta),C\big)\Big)$.

(6) $\big((\alpha,\beta),A\big) \wedge \Big(\big((\lambda,\mu),B\big) \sqcup \big((\gamma,\delta),C\big)\Big)$.

If $\sqcup_{\mathcal{E}}$ and $\sqcap_{\mathcal{E}}$ is LODFS-set (ALODFS-set).

4. Algebraic Structure Associated with LODFS-Set (ALODFS-Set)

In this section, we proposed the concept of algebraic structures of LODFS-set (ALODFS-set) which will help solve daily life problems. We also discussed the algebraic properties of LODFS-set (ALODFS-set).

Proposition 4.1. If $\big((\alpha,\beta),A\big)$, $\big((\lambda,\mu),B\big)$ and $\big((\gamma,\delta),C\big)$ are any LODFS-sets (ALODFS-sets), then following axioms hold

(1) $\Big(\big((\alpha,\beta),A\big) \diamond \big((\lambda,\mu),B\big)\Big) \diamond \big((\gamma,\delta),C\big) = \big((\alpha,\beta),A\big) \diamond \Big(\big((\lambda,\mu),B\big) \diamond \big((\gamma,\delta),C\big)\Big)$ (Assoc. property)

(2) $\big((\alpha,\beta),A\big) \diamond \big((\lambda,\mu),B\big) = \big((\lambda,\mu),B\big) \diamond \big((\alpha,\beta),A\big)$ (Commutative property)

For all $\diamond \in \{\sqcup, \sqcap, \vee, \wedge\}$.

Proof. (1) Since $\big((\alpha,\beta),A\big)$, $\big((\lambda,\mu),B\big)$ and $\big((\gamma,\delta),C\big)$ are LODFS-sets, so we have for $e \in (A \cap B) \cap C$ such that

$$\Big(\big((\alpha,\beta),A\big) \sqcup \big((\lambda,\mu),B\big)\Big) \sqcup \big((\gamma,\delta),C\big) = \Big(\big((\alpha \mathbin{\tilde{\cup}} \lambda) \mathbin{\tilde{\cup}} \gamma, (\beta \mathbin{\tilde{\cap}} \mu) \mathbin{\tilde{\cap}} \delta\big), (A \cap B) \cap C\Big)$$

as

$$e \in (A \cap B) \cap C, \text{ so } e \longrightarrow \big(\alpha(e) \cup \lambda(e)\big) \cup \gamma(e) \text{ and } e \longrightarrow \big(\beta(e) \cap \mu(e)\big) \cap \delta(e),$$

implies

$$e \longrightarrow \alpha(e) \cup \big(\lambda(e) \cup \gamma(e)\big) \text{ and } e \longrightarrow \beta(e) \cap \big(\mu(e) \cap \delta(e)\big)$$

Hence,
$$\Big(\big((\alpha,\beta),A\big)\sqcup\big((\lambda,\mu),B\big)\Big)\sqcup\big((\gamma,\delta),C\big)=\Big(\big((\alpha\,\tilde{\cup}\,(\lambda\,\tilde{\cup}\,\gamma),\beta\,\tilde{\cap}\,(\mu\,\tilde{\cap}\,\delta),A\cap(B\cap C)\Big)$$
Similarly, we can prove for ALODFS-set.

(2) Straightforward.
Throughout this paper, the collection of all LODFS-sets of E over U is represented as $LODFS(U)_E$, and the collection of all LODFS-sets over U with any fixed set of parameters A is represented as $LODFS(U)_A$.

Note that,

1) $(LODFS(U)_E,\vee)$ and $(LODFS(U)_E,\wedge)$ are monoids.
2) $(LODFS(U)_E,\vee,\wedge)$ and $(LODFS(U)_E,\wedge,\vee)$ are hemirings.
3) $(LODFS(U)_E,\sqcup)$ and $(LODFS(U)_E,\sqcap)$ are monoids.
4) $(LODFS(U)_E,\sqcup,\sqcap)$ and $(LODFS(U)_E,\sqcap,\sqcup)$ are hemirings.

Similarly, we can define for ALODFS-set.

Proposition 4.2. (Absorption Laws)

Let $\big((\alpha,\beta),A\big)$ and $\big((\lambda,\mu),B\big)$ be LODFS-sets (ALODFS-sets), then

1) $\Big(\big((\alpha,\beta),A\big)\wedge\big((\lambda,\mu),B\big)\Big)\vee\big((\lambda,\mu),B\big)=\big((\lambda,\mu),B\big)$.
2) $\Big(\big((\alpha,\beta),A\big)\vee\big((\lambda,\mu),B\big)\Big)\wedge\big((\lambda,\mu),B\big)=\big((\lambda,\mu),B\big)$.
3) $\Big(\big((\alpha,\beta),A\big)\sqcap\big((\lambda,\mu),\boldsymbol{B}\big)\Big)\sqcup\big((\lambda,\mu),\boldsymbol{B}\big)=\big((\lambda,\mu),\boldsymbol{B}\big)$.
4) $\Big(\big((\alpha,\beta),A\big)\sqcup\big((\lambda,\mu),\boldsymbol{B}\big)\Big)\sqcap\big((\lambda,\mu),\boldsymbol{B}\big)=\big((\lambda,\mu),\boldsymbol{B}\big)$.

Theorem 4.3. $\big(LODFS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is an MV-algebra.

Proof. (1-MV) $\big(LODFS(U)_A,\sqcup,{}^c,A_{(\emptyset,\mathfrak{U})}\big)$ is a commutative monoid.
(2-MV) $\left(A_{(\alpha_1,\beta_1)^c}\right)^c=A_{(\alpha_1,\beta_1)}$.
(3-MV) $A_{(\emptyset,\mathfrak{U})^c}\sqcup A_{(\alpha_1,\beta_1)}=A_{(\mathfrak{U},\emptyset)}\sqcup A_{(\alpha_1,\beta_1)}=A_{(\mathfrak{U},\emptyset)}=A_{(\emptyset,\mathfrak{U})^c}$.
(4-MV)
$$\begin{aligned}\left(A_{(\alpha_1,\beta_1)^c}\sqcup A_{(\alpha_2,\beta_2)}\right)^c\sqcup A_{(\alpha_2,\beta_2)}&=\left(A_{({\alpha_1}^c,{\beta_1}^c)^c}\sqcap A_{(\alpha_2,\beta_2)^c}\right)\sqcup A_{(\alpha_2,\beta_2)}\\&=\left(A_{(\alpha_1,\beta_1)}\sqcup A_{(\alpha_2,\beta_2)}\right)\sqcap\left(A_{({\alpha_2}^c,{\beta_2}^c)}\sqcup A_{(\alpha_2,\beta_2)}\right)\\&=\left(A_{(\alpha_1,\beta_1)}\sqcup A_{(\alpha_2,\beta_2)}\right)\sqcap A_{(\mathfrak{U},\emptyset)}\\&=\left(A_{(\alpha_1,\beta_1)}\sqcup A_{(\alpha_2,\beta_2)}\right)\sqcap\left(A_{(\alpha_1,\beta_1)^c}\sqcup A_{(\alpha_1,\beta_1)}\right)\\&=\left(A_{(\alpha_2,\beta_2)}\sqcap A_{(\alpha_1,\beta_1)^c}\right)\sqcup A_{(\alpha_1,\beta_1)}\\&=\left(\left(A_{(\alpha_2,\beta_2)^c}\sqcup A_{(\alpha_1,\beta_1)}\right)^c\right)\sqcup A_{(\alpha_1,\beta_1)}\end{aligned}$$

Theorem 4.4. $\big(LODFS(U,)_A,\sqcap,{}^c,A_{(\mathfrak{U},\emptyset)}\big)$ is an MV-algebra.

Proof. Similarly, we can prove like Theorem 4.3.

Theorem 4.5. $\big(LODFS(U)_A,\sqcup,\sqcap,A_{(\mathfrak{U},\emptyset)},A_{(\emptyset,\mathfrak{U})}\big)$ and $\big(LODFS(U)_A,\sqcap,\sqcup,A_{(\emptyset,\mathfrak{U})},A_{(\mathfrak{U},\emptyset)}\big)$ are bounded lattices.

Proof. Since $(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{A},\emptyset)}, A_{(\emptyset,\mathfrak{A})})$ is a hemiring and the absorption laws hold in hemiring, so $(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{A},\emptyset)}, A_{(\emptyset,\mathfrak{A})})$ is a bounded lattice with $A_{(\mathfrak{A},\emptyset)}$ and $A_{(\emptyset,\mathfrak{A})}$ as maximal and minimal elements respectively. Using the same steps, we can prove that $(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{A})}, A_{(\mathfrak{A},\emptyset)})$ is a bounded lattice.

Theorem 4.6. $(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{A},\emptyset)}, A_{(\emptyset,\mathfrak{A})})$ and $(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{A})}, A_{(\mathfrak{A},\emptyset)})$ are Boolean algebras.

Proof. Consider $((\lambda,\mu),A) \in LDFS(U)_A$, then

$$((\lambda,\mu),A) \sqcap ((\lambda,\mu),A)^c = A_{(\emptyset,\mathfrak{A})} \text{ and } ((\lambda,\mu),A) \sqcup ((\lambda,\mu),A)^c = A_{(\mathfrak{A},\emptyset)}$$

holds imply $(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{A},\emptyset)}, A_{(\emptyset,\mathfrak{A})})$ is a Boolean algebra. Using the same steps, we can prove that $(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{A})}, A_{(\mathfrak{A},\emptyset)})$ is a Boolean algebra.

Now by the previous discussion, we note that De Morgan's laws hold in $(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{A},\emptyset)}, A_{(\emptyset,\mathfrak{A})})$ so $LODFS(U)_A$ is a De Morgan's algebra.

Now for any $((\alpha,\beta),A), ((\lambda,\mu),A) \in LODFS(U)_A$ such that

$$((\alpha,\beta),A) \sqcap ((\alpha,\beta),A)^c \mathrel{\tilde{\sqsubseteq}} ((\lambda,\mu),A) \sqcup ((\lambda,\mu),A)^c$$

is hold in $LODFS(U)_A$. Then, we can say that $LODFS(U)_A$ is a Kleene algebra.

By the previous discussion, we note that $((\lambda,\mu),A) \sqcap ((\lambda,\mu),A)^c = A_{(\emptyset,\mathfrak{A})}$ and if $((\alpha,\beta),A) \sqcap ((\lambda,\mu),A) = A_{(\emptyset,\mathfrak{A})}$, then $((\alpha,\beta),A) \mathrel{\tilde{\sqsubseteq}} ((\lambda,\mu),A)^c$ and we can say that $((\lambda,\mu),A)^c$ is the pseudo complement of $((\lambda,\mu),A)$.

If $((\lambda,\mu),A)^c \sqcup \left(((\lambda,\mu),A)^c\right)^c = A_{(\mathfrak{A},\emptyset)}$ (Stone identity) is hold in $LODFS(U)_A$, then $(LODFS(U)_A, \sqcup, \sqcap, A_{(\mathfrak{A},\emptyset)}, A_{(\emptyset,\mathfrak{A})})$ is called Stone algebra.

Similarly, we can also prove that $(LODFS(U)_A, \sqcap, \sqcup, A_{(\emptyset,\mathfrak{A})}, A_{(\mathfrak{A},\emptyset)})$ is Stone algebra.

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