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# Fuzzy Parameterized Intuitionistic Fuzzy Soft Sets and Their Application to a Performance-Based Value Assignment Problem

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**Abstract** — Soft sets have been successfully applied to many different fields to cope with uncertainties. Recently, to increase the success of the applications, these sets have been combined with other theories, such as fuzzy sets and intuitionistic fuzzy sets. In this study, we propose the concept of fuzzy parameterized intuitionistic fuzzy soft sets (*fpifs*-sets). We then apply these sets to a performance-based value assignment (PVA) problem. Finally, we give suggestions for further research.

***Keywords*** — *Fuzzy sets, intuitionistic fuzzy sets, soft sets, intuitionistic fuzzy soft sets, fpifs-sets*

## 1. Introduction

Researchers in many scientific fields make an effort to model problems containing uncertain data. However, the classical methods are not always successful in describing uncertainties. In 1965, therefore, fuzzy sets were developed by Zadeh [1] to overcome the uncertainties. In 1986, these sets have been generalised to intuitionistic fuzzy sets (*if*-sets) by Atanassov [2]. In 1999, Molodtsov [3] proposed the concept of soft sets as a general mathematical tool to model the problems with uncertainties.

So far, many novel concepts based on the soft sets, fuzzy sets, and *if*-sets have been propounded. These concepts can be classified as follows:

- Fuzzy soft sets [4],
- Intuitionistic fuzzy soft sets [5],
- Fuzzy parameterized soft sets [6],
- Fuzzy parameterized fuzzy soft set [7],
- Fuzzy parameterized intuitionistic fuzzy soft sets [*In this study*],
- Intuitionistic fuzzy parameterized soft sets [8],
- Intuitionistic fuzzy parameterized fuzzy soft sets [9],
- Intuitionistic fuzzy parameterized intuitionistic fuzzy soft sets [10],

In the present paper, as it is pointed out above, we define parameterized intuitionistic fuzzy soft sets (*fpifs*-sets) by using fuzzy sets and *if*-sets. We then apply this concept to a decision-making problem. Finally, we discuss *fpifs*-sets and give suggestions for their further research.

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## 2. Preliminaries

This section presents some of the basic definitions of soft sets [3], fuzzy sets [1], and *if*-sets [2].

### 2.1. Soft Sets

In this subsection, we introduce some of the basic definitions and properties of soft sets provided in [3, 11, 12].

**Definition 2.1.** Let $U$ be a universal set, $P(U)$ be the power set of $U$, and $X$ be a set of parameters. Then, a soft set $S$ over $U$ is defined as a set of ordered pairs

$$S = \{(x, s(x)) : x \in X\} \text{ where } s : X \to P(U)$$

Here, $s$ is called approximate function of the soft set $S$ and the elements $(x, \emptyset)$ is not displayed in $S$.

Hereafter, the soft sets are denoted by $S, S_1, S_2, \dots$ and their approximate functions by $s, s_1, s_2, \dots$, respectively. The set of all soft sets over $U$ is denoted by $\mathbb{S}$.

**Definition 2.2.** Let $S \in \mathbb{S}$. Then,
$S$ is called **empty soft set**, denoted by $S_\emptyset$, if $s(x) = \emptyset$ for all $x \in X$, and
$S$ is called **universal soft set**, denoted by $S_U$, if $s(x) = U$ for all $x \in X$.

**Example 2.3.** Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a universal set and $X = \{x_1, x_2, x_3, x_4\}$ be a set of parameters. If $s(x_1) = \{u_1, u_2, u_4, u_6\}$, $s(x_2) = \emptyset$, $s(x_3) = \{u_1, u_3, u_5\}$, and $s(x_4) = U$, then the soft set $S$ is written by

$$S = \{(x_1, \{u_1, u_2, u_4, u_6\}), (x_3, \{u_1, u_3, u_5\}), (x_4, U)\}$$

**Definition 2.4.** Let $S_1, S_2 \in \mathbb{S}$. Then,
$S_1$ and $S_2$ are called **equal**, denoted by $S_1 = S_2$, if $s_1(x) = s_2(x)$ for all $x \in X$, and
$S_1$ is called **soft subset** of soft set $S_2$, denoted by $S_1 \subseteq S_2$, if $s_1(x) \subseteq s_2(x)$ for all $x \in X$.

**Definition 2.5.** Let $S, S_1, S_2 \in \mathbb{S}$. Then,
the **complement** of $S$ is defined by $S^c = \{(x, U \setminus s(x)) : x \in X\}$,
the **union** of $S_1$ and $S_2$ is defined by $S_1 \cup S_2 = \{(x, s_1(x) \cup s_2(x)) : x \in X\}$, and
the **intersection** of $S_1$ and $S_2$ is defined by $S_1 \cap S_2 = \{(x, s_1(x) \cap s_2(x)) : x \in X\}$.

**Proposition 2.6.** If $S \in \mathbb{S}$, then

i) $S \cup S = S$
ii) $S \cap S = S$
iii) $S \cup S_\emptyset = S$
iv) $S \cap S_\emptyset = S_\emptyset$
v) $S \cup S_U = S_U$
vi) $S \cap S_U = S$

**Proposition 2.7.** If $S_1, S_2, S_3 \in \mathbb{S}$, then

i) $S_1 \cup S_2 = S_2 \cup S_1$
ii) $S_1 \cap S_2 = S_2 \cap S_1$
iii) $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$
iv) $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$
v) $S_1 \cup (S_2 \cup S_3) = (S_1 \cup S_2) \cup S_3$
vi) $S_1 \cap (S_2 \cap S_3) = (S_1 \cap S_2) \cap S_3$
vii) $S_1 \cup (S_2 \cap S_3) = (S_1 \cup S_2) \cap (S_1 \cup S_3)$
viii) $S_1 \cap (S_2 \cup S_3) = (S_1 \cap S_2) \cup (S_1 \cap S_3)$

### 2.2. Fuzzy Sets

This subsection provides some of the basic definitions and properties of fuzzy sets presented in [1]. For more details, see [13–15].

**Definition 2.8.** Let $X$ be a universal set. Then, a fuzzy set $F$ over $X$ is defined by

$$F = \{x^{f(x)} : x \in X\} \text{ where } f : X \to [0, 1]$$

Here $f$ is called the membership function of $F$, the elements $x^0$ is not displayed in $F$, and the elements $x^1$ is displayed as $x$ in $F$. Moreover, the value $f(x)$ is called the degree of membership of $x \in X$ and represents the degree of belonging of $x$ to the fuzzy set $F$.

From now on, the fuzzy sets are denoted by $F, F_1, F_2, \dots$ and their membership functions by $f, f_1, f_2, \dots$ respectively. The set of all fuzzy sets over $X$ is denoted by $\mathbb{F}$.

**Definition 2.9.** Let $F \in \mathbb{F}$. Then,
$F$ is called **empty fuzzy set**, denoted by $F_\emptyset$, if $f(x) = 0$ for all $x \in X$.
$F$ is called **universal fuzzy set**, denoted by $F_X$, if $f(x) = 1$ for all $x \in X$.

**Example 2.10.** Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $f(x_1) = 0.7$, $f(x_2) = 0.5$, $f(x_3) = 0.2$, $f(x_4) = 0$, $f(x_5) = 0.7$, and $f(x_6) = 1$, then the fuzzy set $F$ is as follows:

$$F = \left\{x_1^{0.7}, x_2^{0.5}, x_3^{0.2}, x_5^{0.7}, x_6\right\}$$

**Definition 2.11.** Let $F_1, F_2 \in \mathbb{F}$. Then,
$F_1$ and $F_2$ are called **equal**, denoted by $F_1 = F_2$, if $f_1(x) = f_2(x)$ for all $x \in X$, and
$F_1$ is called **fuzzy subset** of $F_2$, denoted by $F_1 \subseteq F_2$, if $f_1(x) \le f_2(x)$ for all $x \in X$.

**Definition 2.12.** Let $F, F_1, F_2 \in \mathbb{F}$. Then,
the **complement** of $F$ is defined by $F^c = \{x^{1-f(x)} : x \in X\}$,
the **union** of $F_1$ and $F_2$ is defined by $F_1 \cup F_2 = \{x^{\max\{f_1(x), f_2(x)\}} : x \in X\}$, and
the **intersection** of $F_1$ and $F_2$ is defined by $F_1 \cap F_2 = \{x^{\min\{f_1(x), f_2(x)\}} : x \in X\}$.

**Proposition 2.13.** If $F \in \mathbb{F}$, then

i) $F \cup F = F$
ii) $F \cap F = F$
iii) $F \cup F_\emptyset = F$
iv) $F \cap F_\emptyset = F_\emptyset$
v) $F \cup F_X = F_X$
vi) $F \cap F_X = F$

**Proposition 2.14.** If $F_1, F_2, F_3 \in \mathbb{F}$, then

i) $F_1 \cup F_2 = F_2 \cup F_1$
ii) $F_1 \cap F_2 = F_2 \cap F_1$
iii) $(F_1 \cup F_2)^c = F_1^c \cap F_2^c$
iv) $(F_1 \cap F_2)^c = F_1^c \cup F_2^c$
v) $F_1 \cup (F_2 \cup F_3) = (F_1 \cup F_2) \cup F_3$
vi) $F_1 \cap (F_2 \cap F_3) = (F_1 \cap F_2) \cap F_3$
vii) $F_1 \cup (F_2 \cap F_3) = (F_1 \cup F_2) \cap (F_1 \cup F_3)$
viii) $F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3)$

## 2.3. Intuitionistic Fuzzy Sets

This subsection features some of the basic definitions and properties of *if*-sets provided in [2]. For more details, see [16, 17].

**Definition 2.15.** Let $U$ be a universal set. An intuitionistic fuzzy set (*if*-set) $I$ over $U$ is defined by

$$I = \left\{u^{\mu(u);\nu(u)} : u \in U\right\}$$

where $\mu : U \to [0,1]$ and $\nu : U \to [0,1]$ such that $0 \le \mu(u) + \nu(u) \le 1$ for all $u \in U$. Here, $\mu$ and $\nu$ are called membership and non-membership function of $I$ and the elements $u^{0;1}$ is not displayed in $I$. Moreover, the values $\mu(u)$ and $\nu(u)$ denote the membership degree and non-membership degree of the $u \in U$, respectively.

Hereafter, the *if*-sets are denoted by $I, I_1, I_2, \dots$ and their membership and non-membership functions by $\mu, \mu_1, \mu_2, \dots$ and $\nu, \nu_1, \nu_2, \dots$, respectively. The set of all *if*-sets over $U$ is denoted by $\mathbb{I}$.

**Definition 2.16.** Let $I \in \mathbb{I}$. Then,
$I$ is called **empty *if*-set**, denoted by $I_\emptyset$, if $\mu(u) = 0$ and $\nu(u) = 1$ for all $u \in U$, and
$I$ is called **universal *if*-set**, denoted by $I_U$, if $\mu(u) = 1$ and $\nu(u) = 0$ for all $u \in U$.

**Example 2.17.** Let $U = \{u_1, u_2, u_3, u_4\}$ be a universal set, $\mu(u_1) = 0.7$, $\nu(u_1) = 0.2$, $\mu(u_2) = 0$, $\nu(u_2) = 1$, $\mu(u_3) = 0.2$, $\nu(u_3) = 0.6$, $\mu(u_4) = 0.3$, and $\nu(u_4) = 0.7$. Then, the *if*-set $I$ is written by

$$I = \left\{u_1^{0.7;0.2}, u_3^{0.2;0.6}, u_4^{0.3;0.7}\right\}$$

**Definition 2.18.** Let $I_1, I_2 \in \mathbb{I}$. Then,
$I_1$ and $I_2$ is called **equal**, denoted by $I_1 = I_2$, if $\mu_1(u) = \mu_2(u)$ and $\nu_1(u) = \nu_2(u)$ for all $u \in U$, and
$I_1$ is called ***if*-subset** of $I_2$, denoted by $I_1 \subseteq I_2$, if $\mu_1(u) \leq \mu_2(u)$ and $\nu_2(u) \leq \nu_1(u)$ for all $u \in U$.

**Definition 2.19.** Let $I, I_1, I_2 \in \mathbb{I}$. Then,
the **complement** of $I$ is defined by $I^c = \{u^{\nu(u);\mu(u)} : u \in U\}$,
the **union** of $I_1$ and $I_2$ is defined by $I_1 \cup I_2 = \{u^{max\{\mu_1(u),\mu_2(u)\};min\{\nu_1(u),\nu_2(u)\}} : u \in U\}$, and
the **intersection** of $I_1$ and $I_2$ is defined by $I_1 \cap I_2 = \{u^{min\{\mu_1(u),\mu_2(u)\};max\{\nu_1(u),\nu_2(u)\}} : u \in U\}$.

**Proposition 2.20.** If $I \in \mathbb{I}$, then

i) $I \cup I = I$

ii) $I \cap I = I$

iii) $I \cup I_\emptyset = I$

iv) $I \cap I_\emptyset = I_\emptyset$

v) $I \cup I_U = I_U$

vi) $I \cap I_U = I$

**Proposition 2.21.** If $I_1, I_2, I_3 \in \mathbb{I}$, then

i) $I_1 \cup I_2 = I_2 \cup I_1$

ii) $I_1 \cap I_2 = I_2 \cap I_1$

iii) $(I_1 \cup I_2)^c = I_1^c \cap I_2^c$

iv) $(I_1 \cap I_2)^c = I_1^c \cup I_2^c$

v) $I_1 \cup (I_2 \cup I_3) = (I_1 \cup I_2) \cup I_3$

vi) $I_1 \cap (I_2 \cap I_3) = (I_1 \cap I_2) \cap I_3$

vii) $I_1 \cup (I_2 \cap I_3) = (I_1 \cup I_2) \cap (I_1 \cup I_3)$

viii) $I_1 \cap (I_2 \cup I_3) = (I_1 \cap I_2) \cup (I_1 \cap I_3)$

## 3. Fuzzy Parameterized Intuitionistic Fuzzy Soft Sets

In this section, we define fuzzy parameterized intuitionistic fuzzy soft sets (*fpifs*-sets) as a new concept of the soft sets. We then present some of their basic properties.

**Definition 3.1.** Let $U$ be a universal set and $X$ be a set of parameters. If $F = \left\{x^{f(x)} : x \in X\right\}$ is a fuzzy set over $X$ and $p : X \to \mathbb{I}$, $p(x) = \left\{u^{\mu_x(u);\nu_x(u)} : u \in U\right\}$ is an *if*-set over $U$ for $x \in X$, then

$$P = \left\{\left(x^{f(x)}, p(x)\right) : x \in X\right\}$$

is called an *fpifs*-set over $U$. Here, $p$ is called approximate function of $P$ and the elements $(x^0, I_\emptyset)$ is not displayed in $P$.

Throughout this paper, the *fpifs*-sets are denoted by $P, P_1, P_2, \ldots$ and their approximate functions by $p, p_1, p_2, \ldots$, respectively. The set of all *fpifs*-sets over $U$ is denoted by $\mathbb{P}$.

**Definition 3.2.** Let $P \in \mathbb{P}$. Then,
$P$ is called **empty *fpifs*-sets**, denoted by $P_\emptyset$, if $f(x) = 0$ and $p(x) = I_\emptyset$ for all $x \in X$, and
$P$ is called **universal *if*-set**, denoted by $P_U$, if $f(x) = 1$ and $p(x) = I_U$ for all $x \in X$.

**Example 3.3.** Let $U = \{u_1, u_2, u_3\}$, $X = \{x_1, x_2, x_3, x_4\}$, $F = \left\{x_1^{0.7}, x_2^{0.4}, x_4^{0.5}\right\}$, and

$$\begin{aligned}
p(x_1) &= \left\{u_1^{0.7;0.2}, u_3^{0.5;0.2}\right\}, \\
p(x_2) &= \left\{u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}, \\
p(x_3) &= I_\emptyset, \\
p(x_4) &= \left\{u_1^{0.6;0.2}, u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
P = &\ \{(x_1^{0.7}, p(x_1)), (x_2^{0.4}, p(x_2)), (x_4^{0.5}, p(x_4))\} \\
= &\ \left\{\left(x_1^{0.7}, \left\{u_1^{0.7;0.2}, u_3^{0.5;0.2}\right\}\right), \left(x_2^{0.4}, \left\{u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}\right), \left(x_4^{0.5}, \left\{u_1^{0.6;0.2}, u_2^{0.5;0.3}, u_3^{0.8;0.1}\right\}\right)\right\}
\end{aligned}$$

is an *fpifs*-set over $U$.

**Definition 3.4.** Let $P_1, P_2 \in \mathbb{P}$. Then, $P_1$ and $P_2$ are called **equal**, denoted by $P_1 = P_2$, if $f_1(x) = f_2(x)$ and $p_1(x) = p_2(x)$ for all $x \in X$.

**Definition 3.5.** Let $P_1, P_2 \in \mathbb{P}$. Then, $P_1$ is called ***fpifs*-subset** of $P_2$, denoted by $P_1 \tilde{\subseteq} P_2$, if $f_1(x) \leq f_2(x)$ and $p_1(x) \subseteq p_2(x)$ for all $x \in X$.

**Definition 3.6.** Let $P_1, P_2 \in \mathbb{P}$. Then, the **union** of $P_1$ and $P_2$ is defined by

$$P_1 \tilde{\cup} P_2 := \left\{(x^{\max\{f_1(x), f_2(x)\}}, p_1(x) \cup p_2(x)) : x \in X\right\}$$

**Definition 3.7.** Let $P_1, P_2 \in \mathbb{P}$. Then, the **intersection** of $P_1$ and $P_2$ is defined by

$$P_1 \tilde{\cap} P_2 := \left\{(x^{\min\{f_1(x), f_2(x)\}}, p_1(x) \cap p_2(x)) : x \in X\right\}$$

**Definition 3.8.** Let $P \in \mathbb{P}$. Then, the **complement** of $P$ is defined by

$$P^{\tilde{c}} := \left\{(x^{1-f(x)}, p^c(x)) : x \in X\right\}$$

**Proposition 3.9.** If $P \in \mathbb{P}$, then

i) $P \tilde{\cup} P = P$
ii) $P \tilde{\cap} P = P$
iii) $P \tilde{\cup} P_\emptyset = P$
iv) $P \tilde{\cap} P_\emptyset = P_\emptyset$
v) $P \tilde{\cup} P_U = P_U$
vi) $P \tilde{\cap} P_U = P$

Proof. Let $P = \{(x^{f(x)}, p(x)) : x \in X\}$ be an *fpifs*-set over $U$. Then,

**i)** $P \tilde{\cup} P = \{(x^{\max\{f(x), f(x)\}}, p(x) \cup p(x)) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

**ii)** $P \tilde{\cap} P = \{(x^{\min\{f(x), f(x)\}}, p(x) \cap p(x)) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

**iii)** $P \tilde{\cup} P_\emptyset = \{(x^{\max\{f(x), 0\}}, p(x) \cup I_\emptyset) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

**iv)** $P \tilde{\cap} P_\emptyset = \{(x^{\min\{f(x), 0\}}, p(x) \cap I_\emptyset) : x \in X\} = \{(x^0, I_\emptyset) : x \in X\} = P_\emptyset$

**v)** $P \tilde{\cup} P_U = \{(x^{\max\{f(x), 1\}}, p(x) \cup I_U) : x \in X\} = \{(x^1, I_U) : x \in X\} = P_U$

**vi)** $P \tilde{\cap} P_U = \{(x^{\min\{f(x), 1\}}, p(x) \cap I_U) : x \in X\} = \{(x^{f(x)}, p(x)) : x \in X\} = P$

□

**Proposition 3.10.** If $P_1, P_2, P_3 \in \mathbb{P}$, then

i) $P_1 \tilde{\cup} P_2 = P_2 \tilde{\cup} P_1$
ii) $P_1 \tilde{\cap} P_2 = P_2 \tilde{\cap} P_1$
iii) $(P_1 \tilde{\cup} P_2)^{\tilde{c}} = P_1^{\tilde{c}} \tilde{\cap} P_2^{\tilde{c}}$
iv) $(P_1 \tilde{\cap} P_2)^{\tilde{c}} = P_1^{\tilde{c}} \tilde{\cup} P_2^{\tilde{c}}$
v) $P_1 \tilde{\cup} (P_2 \tilde{\cup} P_3) = (P_1 \tilde{\cup} P_2) \tilde{\cup} P_3$
vi) $P_1 \tilde{\cap} (P_2 \tilde{\cap} P_3) = (P_1 \tilde{\cap} P_2) \tilde{\cap} P_3$
vii) $P_1 \tilde{\cup} (P_2 \tilde{\cap} P_3) = (P_1 \tilde{\cup} P_2) \tilde{\cap} (P_1 \tilde{\cup} P_3)$
viii) $P_1 \tilde{\cap} (P_2 \tilde{\cup} P_3) = (P_1 \tilde{\cap} P_2) \tilde{\cup} (P_1 \tilde{\cap} P_3)$

Proof. Let $P_1 = \{(x^{f_1(x)}, p_1(x)) : x \in X\}$, $P_2 = \{(x^{f_2(x)}, p_2(x)) : x \in X\}$ and $P_3 = \{(x^{f_3(x)}, p_3(x)) : x \in X\}$ be three *fpifs*-sets over $U$. Then,

**i)**
$$\begin{aligned} P_1 \tilde{\cup} P_2 &= \{(x^{\max\{f_1(x), f_2(x)\}}, p_1(x) \cup p_2(x)) : x \in X\}, \\ &= \{(x^{\max\{f_2(x), f_1(x)\}}, p_2(x) \cup p_1(x)) : x \in X\}, \\ &= P_2 \tilde{\cup} P_1 \end{aligned}$$

**ii)**
$$\begin{aligned} P_1 \tilde{\cap} P_2 &= \{(x^{\min\{f_1(x), f_2(x)\}}, p_1(x) \cap p_2(x)) : x \in X\}, \\ &= \{(x^{\min\{f_2(x), f_1(x)\}}, p_2(x) \cap p_1(x)) : x \in X\}, \\ &= P_2 \tilde{\cap} P_1 \end{aligned}$$

**iii)** $(P_1 \cup P_2)^c = \{(x^{1-\max\{f_1(x),f_2(x)\}}, (p_1(x) \cup p_1(x))^c) : x \in X\},$
$= \{(x^{\min\{1-f_1(x),1-f_2(x)\}}, p_1^c(x) \cap p_2^c(x)) : x \in X\},$
$= P_1^c \cap P_2^c$

**iv)** $(P_1 \cap P_2)^c = \{(x^{1-\min\{f_1(x),f_2(x)\}}, (p_1(x) \cap p_2(x))^c) : x \in X\},$
$= \{(x^{\max\{1-f_1(x),1-f_2(x)\}}, p_1^c(x) \cup p_2^c(x)) : x \in X\},$
$= P_1^c \cup P_2^c$

**v)** $P_1 \cup (P_2 \cup P_3) = \{(x^{\max\{f_1(x),\max\{f_2(x),f_3(x)\}\}}, p_1(x) \cup (p_2(x) \cup p_3(x))) : x \in X\}$
$= \{(x^{\max\{\max\{f_1(x),f_2(x)\},f_3(x)\}}, (p_1(x) \cup p_2(x)) \cup p_3(x)) : x \in X\}$
$= (P_1 \cup P_2) \cup P_3$

**vi)** $P_1 \cap (P_2 \cap P_3) = \{(x^{\min\{f_1(x),\min\{f_2(x),f_3(x)\}\}}, p_1(x) \cap (p_2(x) \cap p_3(x))) : x \in X\}$
$= \{(x^{\min\{\min\{f_1(x),f_2(x)\},f_3(x)\}}, (p_1(x) \cap p_2(x)) \cap p_3(x)) : x \in X\}$
$= (P_1 \cap P_2) \cap P_3$

**vii)** $P_1 \cup (P_2 \cap P_3) = \{(x^{\max\{f_1(x),\min\{f_2(x),f_3(x)\}\}}, p_1(x) \cup (p_2(x) \cap p_3(x))) : x \in X\}$
$= \{(x^{\min\{\max\{f_1(x),f_2(x)\},\max\{f_1(x),f_3(x)\}\}}, (p_1(x) \cup p_2(x)) \cap (p_1(x) \cup p_3(x))) : x \in X\}$
$= (P_1 \cup P_2) \cap (P_1 \cup P_3)$

**viii)** $P_1 \cap (P_2 \cup P_3) = \{(x^{\min\{f_1(x),\max\{f_2(x),f_3(x)\}\}}, p_1(x) \cap (p_2(x) \cup p_2(x))) : x \in X\}$
$= \{(x^{\max\{\min\{f_1(x),f_2(x)\},\min\{f_1(x),f_3(x)\}\}}, (p_1(x) \cap p_2(x)) \cup (p_1(x) \cap p_3(x))) : x \in X\}$
$= (P_1 \cap P_2) \cup (P_1 \cap P_3)$

□

## 4. A Soft Decision-Making Method Proposed on *fpifs*-sets

In this section, we suggest a soft decision-making method that assigns a performance-based value to the alternatives via *fpifs*-sets. Thus, we can choose the optimal elements among the alternatives.

*The Proposed Algorithm Steps*

**Step 1.** Construct an *fpifs*-set $P$ such that $P = \left\{\left(x^{f(x)}, \left\{u^{\mu_x(u);\nu_x(u)} : u \in U\right\}\right) : x \in X\right\}$

**Step 2.** Obtain the values $\omega(u) = \frac{1}{|E|} \sum_{x \in X} f(x)(\mu_x(u) - \nu_x(u))$, for all $u \in U$

**Step 3.** Obtain the decision set $\{u_k^{d(u_k)} | u_k \in U\}$ such that $d(u_k) = \frac{\omega(u_k) + |\min_i \omega(u_i)|}{\max_i \omega(u_i) + |\min_i \omega(u_i)|}$

## 5. An Application of the Proposed Method to a Performance-Based Value Assignment Problem

In this section, we apply the proposed method to the performance-based value assignment (PVA) problem for seven filters used in image denoising, namely Decision Based Algorithm (DBA) [18], Modified Decision Based Unsymmetrical Trimmed Median Filter (MDBUTMF) [19], Based on Pixel Density Filter (BPDF) [20], Noise Adaptive Fuzzy Switching Median Filter (NAFSMF) [21], A New Adaptive Weighted Mean Filter (AWMF) [22], Different Applied Median Filter (DAMF) [23], and Adaptive Riesz Mean Filter (ARmF) [24]. Hereafter, let $U = \{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ be the set of the alternatives such that

$u_1 =$ "DBA", $u_2 =$ "MDBUTMF", $u_3 =$ "BPDF", $u_4 =$ "NAFSMF", $u_5 =$ "AWMF", $u_6 =$ "DAMF", and $u_7 =$ "ARmF"

Moreover, let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$ be a parameter set determined by a decision-maker such that

$x_1 =$ "*noise density* 10%", $x_2 =$ "*noise density* 20%", $x_3 =$ "*noise density* 30%",

$x_4 =$ "*noise density* 40%", $x_5 =$ "*noise density* 50%", $x_6 =$ "*noise density* 60%", $x_7 =$ "*noise density* 70%", $x_8 =$ "*noise density* 80%", and $x_9 =$ "*noise density* 90%".

Further, let bold numbers in a table point out the best scores therein.

We first present the results of the filters in [24] by Structural Similarity (SSIM) [25] for the image Cameraman in Table 1. Hereinafter, let $\mu_x(u)$ corresponds to the SSIM/MSSIM results of the image/images for filter $u$ and noise density $x$. Moreover, let $\nu_x(u) = 1 - \mu_x(u)$, for all $x \in X$ and $u \in U$.

**Table 1.** The SSIM results of the filters for the Cameraman image.

| **Filters** | **10%** | **20%** | **30%** | **40%** | **50%** | **60%** | **70%** | **80%** | **90%** |
|---|---|---|---|---|---|---|---|---|---|
| **DBA** | 0.9938 | 0.9847 | 0.9710 | 0.9520 | 0.9222 | 0.8843 | 0.8283 | 0.7584 | 0.6645 |
| **MDBUTMF** | 0.9897 | 0.9278 | 0.7945 | 0.7964 | 0.8844 | 0.9158 | 0.8962 | 0.8056 | 0.4451 |
| **BPDF** | 0.9910 | 0.9783 | 0.9588 | 0.9306 | 0.8934 | 0.8406 | 0.7700 | 0.6665 | 0.4990 |
| **NAFSMF** | 0.9798 | 0.9636 | 0.9484 | 0.9329 | 0.9164 | 0.8954 | 0.8696 | 0.8335 | 0.7288 |
| **AWMF** | 0.9872 | 0.9839 | 0.9798 | 0.9748 | 0.9667 | 0.9541 | 0.9345 | 0.9015 | 0.8346 |
| **DAMF** | 0.9960 | 0.9906 | 0.9833 | 0.9749 | 0.9638 | 0.9492 | 0.9293 | 0.8973 | 0.8294 |
| **ARmF** | **0.9969** | **0.9933** | **0.9885** | **0.9824** | **0.9735** | **0.9600** | **0.9395** | **0.9059** | **0.8376** |

The application of the soft decision-making method proposed in Section 4 is as follows:

**Step 1.** Suppose that the success at high noise densities is more important than in the presence of other densities. In this case, the values in Table 1 can be represented with *fpifs*-set as follows:

$$P_1 = \Big\{\big({x_1}^{0.1}, \{{u_1}^{0.9938;0.0062}, {u_2}^{0.9897;0.0103}, {u_3}^{0.9910;0.0090}, {u_4}^{0.9798;0.0202}, {u_5}^{0.9872;0.0128}, {u_6}^{0.9960;0.0040}, {u_7}^{0.9969;0.0031}\}\big), \big({x_2}^{0.2}, \{{u_1}^{0.9847;0.0153}, {u_2}^{0.9278;0.0722}, {u_3}^{0.9783;0.0217}, {u_4}^{0.9636;0.0364}, {u_5}^{0.9839;0.0161}, {u_6}^{0.9906;0.0094}, {u_7}^{0.9933;0.0067}\}\big), \big({x_3}^{0.3}, \{{u_1}^{0.9710;0.0290}, {u_2}^{0.7945;0.2055}, {u_3}^{0.9588;0.0412}, {u_4}^{0.9484;0.0516}, {u_5}^{0.9798;0.0202}, {u_6}^{0.9833;0.0167}, {u_7}^{0.9885;0.0115}\}\big), \big({x_4}^{0.4}, \{{u_1}^{0.9520;0.0480}, {u_2}^{0.7964;0.2036}, {u_3}^{0.9306;0.0694}, {u_4}^{0.9329;0.0671}, {u_5}^{0.9748;0.0252}, {u_6}^{0.9749;0.0251}, {u_7}^{0.9824;0.0176}\}\big), \big({x_5}^{0.5}, \{{u_1}^{0.9222;0.0778}, {u_2}^{0.8844;0.1156}, {u_3}^{0.8934;0.1066}, {u_4}^{0.9164;0.0836}, {u_5}^{0.9667;0.0333}, {u_6}^{0.9638;0.0362}, {u_7}^{0.9735;0.0265}\}\big), \big({x_6}^{0.6}, \{{u_1}^{0.8843;0.1157}, {u_2}^{0.9158;0.0842}, {u_3}^{0.8406;0.1594}, {u_4}^{0.8954;0.1046}, {u_5}^{0.9541;0.0459}, {u_6}^{0.9492;0.0508}, {u_7}^{0.9600;0.0400}\}\big), \big({x_7}^{0.7}, \{{u_1}^{0.8283;0.1717}, {u_2}^{0.8962;0.1038}, {u_3}^{0.7700;0.2300}, {u_4}^{0.8696;0.1304}, {u_5}^{0.9345;0.0655}, {u_6}^{0.9293;0.0707}, {u_7}^{0.9395;0.0605}\}\big), \big({x_8}^{0.8}, \{{u_1}^{0.7584;0.2416}, {u_2}^{0.8056;0.1944}, {u_3}^{0.6665;0.3335}, {u_4}^{0.8335;0.1665}, {u_5}^{0.9015;0.0985}, {u_6}^{0.8973;0.1027}, {u_7}^{0.9059;0.0941}\}\big), \big({x_9}^{0.9}, \{{u_1}^{0.6645;0.3355}, {u_2}^{0.4451;0.5549}, {u_3}^{0.4990;0.5010}, {u_4}^{0.7288;0.2712}, {u_5}^{0.8346;0.1654}, {u_6}^{0.8294;0.1706}, {u_7}^{0.8376;0.1624}\}\big)\Big\}$$

**Step 2.** The values $\omega(u)$ are as follows:

$$\omega(u_1) = 0.3322,\ \omega(u_2) = 0.2790,\ \omega(u_3) = 0.2616,\ \omega(u_4) = 0.3612,\ \omega(u_5) = 0.4248,\ \omega(u_6) = 0.4220, \text{ and } \omega(u_7) = 0.4304$$

**Step 3.** The decision set is as follows:

$$\{\text{DBA}^{0.8580}, \text{MDBUTMF}^{0.7812}, \text{BPDF}^{0.7560}, \text{NAFSMF}^{0.8999}, \text{AWMF}^{0.9919}, \text{DAMF}^{0.9878}, \text{ARmF}^{1}\}$$

The results show that ARmF outperforms the others and the ranking order BPDF $\prec$ MDBUTMF $\prec$ DBA $\prec$ NAFSMF $\prec$ DAMF $\prec$ AWMF $\prec$ ARmF is valid. Moreover, the results confirm the expert's view.

The visual performances of the filters are provided in Fig. 1. The performances of the filters can not be discriminated in consideration of Fig. 1. Moreover, when a large number of data come into question, it is impossible to do so. Therefore, the proposed method has an essential role in dealing with PVA problems.

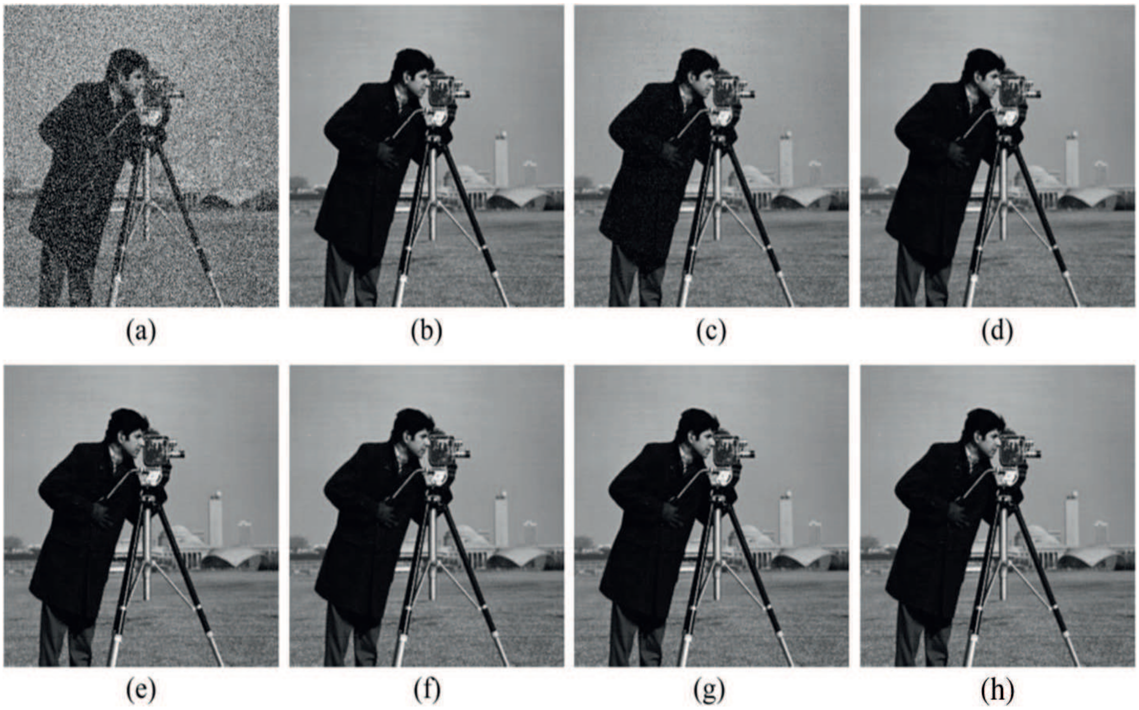


**Fig. 1.** [24] SSIM results for "Cameraman" of $512 \times 512$ with a SPN ratio of 30. (a) Noisy image 0.0550, (b) DBA 0.9710, (c) MDBUTMF 0.7945, (d) BPDF 0.9588, (e) NAFSMF 0.9484, (f) AWMF 0.9798, (g) DAMF 0.9833, and (h) ARmF 0.9885

Secondly, to better establish the success of the proposed method, we present the results of the filters in [24] by Mean Structural Similarity (MSSIM) for the 20 traditional images in Table 2.

**Table 2.** The MSSIM results of the filters for the 20 traditional images.

| **Filters** | **10%** | **20%** | **30%** | **40%** | **50%** | **60%** | **70%** | **80%** | **90%** |
|---|---|---|---|---|---|---|---|---|---|
| **DBA** | 0.9796 | 0.9584 | 0.9315 | 0.8968 | 0.8520 | 0.7949 | 0.7213 | 0.6265 | 0.4966 |
| **MDBUTMF** | 0.9774 | 0.9197 | 0.8117 | 0.7973 | 0.8399 | 0.8410 | 0.8025 | 0.7023 | 0.3566 |
| **BPDF** | 0.9783 | 0.9536 | 0.9229 | 0.8838 | 0.8323 | 0.7634 | 0.6680 | 0.5096 | 0.2585 |
| **NAFSMF** | 0.9748 | 0.9504 | 0.9248 | 0.8973 | 0.8666 | 0.8320 | 0.7910 | 0.7357 | 0.6190 |
| **AWMF** | 0.9728 | 0.9622 | 0.9484 | 0.9315 | 0.9098 | 0.8816 | 0.8437 | 0.7904 | 0.7028 |
| **DAMF** | 0.9854 | 0.9699 | 0.9516 | 0.9303 | 0.9051 | 0.8748 | 0.8368 | 0.7846 | 0.6964 |
| **ARmF** | **0.9868** | **0.9735** | **0.9581** | **0.9400** | **0.9173** | **0.8880** | **0.8491** | **0.7947** | **0.7056** |

Similarly, the values in Table 2 can be represented with *fpifs*-set as follows:

$$
\begin{aligned}
P_2 = \quad & \{(x_1{}^{0.1}, \{u_1{}^{0.9796;0.0204}, u_2{}^{0.9774;0.0226}, u_3{}^{0.9783;0.0217}, u_4{}^{0.9748;0.0252}, u_5{}^{0.9728;0.0272}, u_6{}^{0.9854;0.0146}, \\
& u_7{}^{0.9868;0.0132}\}), (x_2{}^{0.2}, \{u_1{}^{0.9584;0.0416}, u_2{}^{0.9197;0.0803}, u_3{}^{0.9536;0.0464}, u_4{}^{0.9504;0.0496}, u_5{}^{0.9622;0.0378}, \\
& u_6{}^{0.9699;0.0301}, u_7{}^{0.9735;0.0265}\}), (x_3{}^{0.3}, \{u_1{}^{0.9315;0.0685}, u_2{}^{0.8117;0.1183}, u_3{}^{0.9229;0.0771}, u_4{}^{0.9248;0.0752}, \\
& u_5{}^{0.9484;0.0516}, u_6{}^{0.9516;0.0484}, u_7{}^{0.9581;0.0419}\}), (x_4{}^{0.4}, \{u_1{}^{0.8968;0.1032}, u_2{}^{0.7973;0.2027}, u_3{}^{0.8838;0.1162}, \\
& u_4{}^{0.8973;0.1027}, u_5{}^{0.9315;0.0685}, u_6{}^{0.9303;0.0697}, u_7{}^{0.9400;0.0600}\}), \Big(x_5{}^{0.5}, \Big\{u_1{}^{0.8520;0.1480}, u_2{}^{0.8399;0.1601},
\end{aligned}
$$

$u_3{}^{0.8323;0.1677}, u_4{}^{0.8666;0.1334}, u_5{}^{0.9098;0.0902}, u_6{}^{0.9051;0.0949}, u_7{}^{0.9173;0.0827}\}), (x_6{}^{0.6}, \{u_1{}^{0.7949;0.02051},$
$u_2{}^{0.8410;0.1590}, u_3{}^{0.7634;0.2366}, u_4{}^{0.8320;0.1680}, u_5{}^{0.8816;0.1184}, u_6{}^{0.8748;0.1252}, u_7{}^{0.8880;0.1120}\}), (x_7{}^{0.7},$
$\{u_1{}^{0.7213;0.2787}, u_2{}^{0.8025;0.1975}, u_3{}^{0.6680;0.3320}, u_4{}^{0.7910;0.2090}, u_5{}^{0.8437;0.1563}, u_6{}^{0.8368;0.1632}, u_7{}^{0.8491;0.1509}\}),$
$(x_8{}^{0.8}, \{u_1{}^{0.6265;0.3735}, u_2{}^{0.7023;0.2977}, u_3{}^{0.5096;0.4904}, u_4{}^{0.7357;0.2643}, u_5{}^{0.7904;0.2096}, u_6{}^{0.7846;0.2154}$
$u_7{}^{0.7947;0.2053}\}), (x_9{}^{0.9}, \{u_1{}^{0.4966;0.5034}, u_2{}^{0.3566;0.6434}, u_3{}^{0.2585;0.7415}, u_4{}^{0.6190;0.3810}, u_5{}^{0.7028;0.2972},$
$u_6{}^{0.6964;0.3036}, u_7{}^{0.7056;0.2944}\})\}$

If we apply the proposed method to the *fpifs*-set $P_2$, then the decision set is as follows:

$$\{\mathrm{DBA}^{0.7608}, \mathrm{MDBUTMF}^{0.7289}, \mathrm{BPDF}^{0.5880}, \mathrm{NAFSMF}^{0.8837}, \mathrm{AWMF}^{0.9877}, \mathrm{DAMF}^{0.9794}, \mathrm{ARmF}^{1}\}$$

The results show that ARmF outperforms the others and the following ranking order is valid.

$$\mathrm{BPDF} \prec \mathrm{MDBUTMF} \prec \mathrm{DBA} \prec \mathrm{NAFSMF} \prec \mathrm{DAMF} \prec \mathrm{AWMF} \prec \mathrm{ARmF}$$

Moreover, performance ranking order of filters obtained with the SSIM results of the filters only for the Cameraman image is the same therein. Therefore, the proposed method has been successfully applied to the PVA problem.

## 6. Conclusion

To deal with uncertainties, the soft set theory has been applied to many theoretical and practical fields. Recently, soft sets, using other theories, have been prominent. In this work, we defined fuzzy parameterized intuitionistic fuzzy soft sets (*fpifs*-sets) by using fuzzy sets, intuitionistic fuzzy sets, and soft sets. We then proposed a soft decision-making method and successfully applied it to a decision-making problem. We think that this study will be beneficial for future studies on soft sets and their applications, particularly in decision-making.

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