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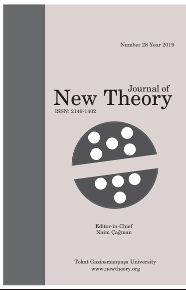

On μsp-Continuous Maps in Topological Spaces

Selvaraj Ganesan[1], Rajamanickam Selva Vinayagam[2], Balakrishnan Sarathkumar[3]

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Abstract − In this paper, we introduce a new class of continuous maps called μsp-continuous maps and study their properties in topological spaces.

Keywords − *Topological space, μsp-closed set, μsp-continuous map, μsp-irresolute map*

1. Introduction and Preliminaries

Several authors [1–7] working in the field of general topology have shown more interest in studying the concepts of generalizations of continuous maps. A weak form of continuous maps called g-continuous maps were introduced by Balachandran et al. [8]. As generalizations of closed sets, μsp-closed sets were introduced and studied by the same author [9]. In this paper, we first introduce μsp-continuous maps and study their relations with various generalized continuous maps. We also discuss some properties of μsp-continuous maps. We introduce μsp-irresolute maps in topological spaces and discuss some of their properties. Various properties and characterizations of such maps are discussed by using μsp-closure and μsp-interior under certain conditions. Throughout this paper, (X, τ), (Y, σ), and (Z, η) (or X, Y, and Z) represent topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), $cl(A)$, $int(A)$, and A^C denote the closure of A, the interior of A, and complement of A, respectively.
We recall the following definitions which are useful in the sequel.

Definition 1.1. A subset A of a space (X, τ) is called:

1. α-open set [10] if $A \subseteq int(cl(int(A)))$.
2. semi-open set [11] if $A \subseteq cl(int(A))$.
3. pre-open set [5] if $A \subseteq int(cl(A))$.
4. β-open set [1] (= semi-pre-open set [12]) if $A \subseteq cl(int(cl(A)))$.

[1]sgsgsgsgsg77@gmail.com (Corresponding Author); [2]rsvrrc@gmail.com; [3]sarathkumarsk6696@gmail.com
[1,3]PG & Research Department of Mathematics, Raja Doraisingam Government Arts College, Sivagangai-630561, Tamil Nadu, India
[2]PG & Research Department of Computer Science, Raja Doraisingam Government Arts College, Sivagangai-630561, Tamil Nadu, India

The complements of the above mentioned open sets are called their respective closed sets. The α-closure [10](resp. semi-closure [13], pre-closure [14], semi-pre-closure [12]) of a subset A of X, denoted by $\alpha cl(A)$ (resp. $scl(A)$, $pcl(A)$, $spcl(A)$) is defined to be the intersection of all α-closed (resp. semi-closed, pre-closed, semi-pre-closed) sets of (X, τ) containing A.

Definition 1.2. A subset A of a space (X, τ) is called:

1. a generalized closed (briefy g-closed) set [15] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of g-closed set is called g-open set.
2. a generalized semi-closed (briefly gs-closed) set [16] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of gs-closed set is called gs-open set.
3. an α-generalized closed (briefly αg-closed) set [17] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of αg-closed set is called αg-open set.
4. a generalized α-closed (briefly $g\alpha$-closed) set [18] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of $g\alpha$-closed set is called $g\alpha$-open set.
5. a $g^{\#}$-closed set [19] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open in (X, τ). The complement of $g^{\#}$-closed set is called $g^{\#}$-open set.
6. a generalized semi-preclosed (briefly gsp-closed) set [20] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ). The complement of gsp-closed set is called gsp-open set.
7. a $\hat{g}$-closed set [7] (= ω-closed set [6]) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is semi-open in (X, τ). The complement of $\hat{g}$-closed set is called $\hat{g}$-open set.
8. a $^{*}g$-closed set [21] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in (X, τ). The complement of $^{*}g$-closed set is called $^{*}g$-open set.
9. a $\#g$-semi-closed(briefly $\#gs$-closed) set [22] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $*g$-open in (X, τ). The complement of $\#gs$-closed set is called $\#gs$-open set.
10. a $g\alpha^{*}$-closed set [18, 23] if $\alpha cl(A) \subseteq int(U)$ whenever $A \subseteq U$ and U is α-open in (X, τ). The complement of $g\alpha^{*}$-closed set is called $g\alpha^{*}$-open set.
11. a μ-closed set [24] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^{*}$-open in (X, τ). The complement of μ-closed set is called μ-open set.
12. a μp-closed set [25] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^{*}$-open in (X, τ). The complement of μp-closed set is called μp-open set.
13. a μs-closed set [26] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^{*}$-open in (X, τ). The complement of μs-closed set is called μs-open set.
14. a μsp-closed set [9] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is $g\alpha^{*}$-open in (X, τ). The complement of μsp-closed set is called μsp-open set.

Remark 1.3. The collection of all g-closed (resp. gs-closed, αg-closed, $g\alpha$-closed, $g^{\#}$-closed, gsp-closed, $\hat{g}$-closed, $^{*}g$-closed, $\#gs$-closed, $g\alpha^{*}$-closed, μ-closed, μp-closed, μs-closed, μsp-closed) sets is denoted by $gc(\tau)$ (resp. $gsc(\tau)$, $\alpha gc(\tau)$, $g\alpha c(\tau)$, $g^{\#}c(\tau)$, $gspc(\tau)$, $\hat{g}c(\tau)$, $^{*}gc(\tau)$, $\#gsc(\tau)$, $g\alpha^{*}c(\tau)$, $\mu c(\tau)$, $\mu pc(\tau)$, $\mu sc(\tau)$,$\mu spc(\tau)$).

We denote the power set of X by $P(X)$.

Definition 1.4. A map $f : (X, \tau) \to (Y, \sigma)$ is called:

1. α-continuous [27] if $f^{-1}(V)$ is a α-closed set of (X, τ) for every closed set V of (Y, σ).
2. semi-continuous [11] if $f^{-1}(V)$ is a semi-closed set of (X, τ) for every closed set V of (Y, σ).
3. pre-continuous [5] if $f^{-1}(V)$ is a pre-closed set of (X, τ) for every closed set V of (Y, σ).

4. β-continuous [1] if $f^{-1}(V)$ is a β-closed set of (X,τ) for every closed set V of (Y,σ).
5. g-continuous [8] if $f^{-1}(V)$ is a g-closed set of (X,τ) for every closed set V of (Y,σ).
6. gs-continuous [2] if $f^{-1}(V)$ is a gs-closed set of (X,τ) for every closed set V of (Y,σ).
7. αg-continuous [28] if $f^{-1}(V)$ is a αg-closed set of (X,τ) for every closed set V of (Y,σ).
8. $g\alpha$-continuous [28] if $f^{-1}(V)$ is a $g\alpha$-closed set of (X,τ) for every closed set V of (Y,σ).
9. $g^{\#}$-continuous [19] if $f^{-1}(V)$ is a $g^{\#}$-closed set of (X,τ) for every closed set V of (Y,σ).
10. gsp-continuous [20] if $f^{-1}(V)$ is a gsp-closed set of (X,τ) for every closed set V of (Y,σ).
11. $\hat{g}$-continuous [7] if $f^{-1}(V)$ is a $\hat{g}$-closed set of (X,τ) for every closed set V of (Y,σ).
12. $^{*}g$-continuous [21] if $f^{-1}(V)$ is a $^{*}g$-closed set of (X,τ) for every closed set V of (Y,σ).
13. $\#g$-semi-continuous [22] if $f^{-1}(V)$ is a $\#g$-semi-closed set of (X,τ) for every closed set V of (Y,σ).
14. μ-continuous [24] if $f^{-1}(V)$ is a μ-closed set of (X,τ) for every closed set V of (Y,σ).
15. μp-continuous [25] if $f^{-1}(V)$ is a μp-closed set of (X,τ) for every closed set V of (Y,σ).
16. μs-continuous [26] if $f^{-1}(V)$ is a μs-closed set of (X,τ) for every closed set V of (Y,σ).

Definition 1.5. [9] For a space (X,τ), the following hold:

1. $T_{\mu sp}$-space if every μsp-closed set is closed.
2. $\mu T_{\mu sp}$-space if every μsp-closed set is μ-closed.
3. $pT_{\mu sp}$-space if every μsp-closed set is pre-closed.
4. $spT_{\mu sp}$-space if every μsp-closed set is semi-preclosed.
5. $\alpha T_{\mu sp}$-space if every μsp-closed set is α-closed.
6. $g\alpha T_{\mu sp}$-space if every μsp-closed set is $g\alpha$-closed.

Result 1.6.

1. Every closed set (resp. pre-closed set, α-closed set, semi-closed set, β-closed set) is μsp-closed but not conversely [9].
2. Every μ-closed set (resp. μp-closed set, μs-closed set) is μsp-closed but not conversely [9].
3. Every $g\alpha$-closed set (resp. $g^{\#}$-closed set, $\hat{g}$-closed set) is μsp-closed but not conversely [9].
4. Every open set is μsp-open set but not conversely.

2. μsp-Continuous Maps and Irresolute Maps

We introduce the following definition.

Definition 2.1. A map $f:(X,\tau)\to(Y,\sigma)$ is called μsp-continuous if $f^{-1}(V)$ is a μsp-closed set of (X,τ) for every closed set V of (Y,σ).

Proposition 2.2. Every continuous (resp. prec-continuous, α-continuous, semi-continuous, β-continuous) is μsp-continuous but not conversely.

PROOF. The proof follows from Result 1.6 (1).

Example 2.3. Let $X=Y=\{a,b,c\}$, $\tau=\{\phi,\{a\},\{a,c\},X\}$, and $\sigma=\{\phi,\{b\},X\}$. Then, $\mu spc(\tau)=\{\phi,\{b\},\{c\},\{a,c\},\{b,c\},X\}$ and $pc(\tau)=\alpha c(\tau)=sc(\tau)=spc(\tau)=\{\phi,\{b\},\{c\},\{b,c\},X\}$. Define $f:(X,\tau)\to(Y,\sigma)$ be the identity map. Then, f is μsp-continuous but not continuous (resp. prec-continuous, α-continuous, semi-continuous, semi-precontinuous), since $f^{-1}(\{a,c\})=\{a,c\}$ is not closed (resp. preclosed, α-closed, semi-closed, semi-preclosed).

Proposition 2.4. Every μ-continuous (resp. μp-continuous, μs-continuous) is μsp-continuous but not conversely.

PROOF. The proof follows from Result 1.6 (2).

Example 2.5. Let $X = Y = \{a,b,c\}$, $\tau = \{\phi, \{a\}, \{b\}, \{a,b\}, X\}$, and $\sigma = \{\phi, \{a,c\}, X\}$. Then, $\mu spc(\tau) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,c\}, \{b,c\}, X\}$ and $\mu c(\tau) = \mu pc(\tau) = \{\phi, \{c\}, \{a,c\}, \{b,c\}, X\}$. Define $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Then, f is μsp-continuous but not μ-continuous (resp. μp-continuous), since $f^{-1}(\{b\}) = \{b\}$ is not μ-closed (resp. μp-closed).

Example 2.6. Let $X = Y = \{a,b,c\}$, $\tau = \{\phi, \{a\}, \{b,c\}, X\}$, and $\sigma = \{\phi, \{b\}, X\}$. Then, $\mu spc(\tau) = P(X)$ and $\mu sc(\tau) = \{\phi, \{a\}, \{b,c\}, X\}$. Define $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Then, f is μsp-continuous but not μs-continuous, since $f^{-1}(\{a,c\}) = \{a,c\}$ is not μs-closed.

Proposition 2.7. Every $g\alpha$-continuous (resp. $g^{\#}$-continuous, $\hat{g}$-continuous) is μsp-continuous but not conversely.

PROOF. The proof follows from Result 1.6 (3).

Example 2.8. Let X, Y, τ, σ, and f be as in the Example 2.5. Then, $g\alpha c(\tau) = g^{\#}c(\tau) = \hat{g}c(\tau) = \{\phi, \{c\}, \{a,c\}, \{b,c\}, X\}$. Then, f is μsp-continuous but not $g\alpha$-continuous (resp. $g^{\#}$-continuous, $\hat{g}$-continuous), since $f^{-1}(\{b\}) = \{b\}$ is not $g\alpha$-closed (resp. $g^{\#}$-closed, $\hat{g}$-closed).

Theorem 2.9. μsp-continuity is independent of g-continuity, αg-continuity, gs-continuity, gsp-continuity, $^{*}g$-continuity, and $\#gs$-continuity.

PROOF. It follows from the following Example.

Example 2.10.

1. Let $X = Y = \{a,b,c\}$, $\tau = \{\phi, \{a\}, \{a,b\}, X\}$, and $\sigma = \{\phi, \{c\}, Y\}$. Then, $\mu spc(\tau) = \{\phi, \{b\}, \{c\}, \{a,b\}, \{b,c\}, X\}$, $gc(\tau) = {}^{*}gc(\tau) = \{\phi, \{c\}, \{a,c\}, \{b,c\}, X\}$, and $\alpha gc(\tau) = gsc(\tau) = gspc(\tau) = \#gsc(\tau) = \{\phi, \{b\}, \{c\}, \{a,c\}, \{b,c\}, X\}$. Define $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Then, f is μsp-continuous but not g-continuous (resp. αg-continuous, gs-continuous, gsp-continuous, $^{*}g$-continuous, and $\#gs$-continuous), since $f^{-1}(\{a,b\}) = \{a,b\}$ is not g-closed (resp. αg-closed, gs-closed, gsp-closed, $^{*}g$-closed, and $\#gs$-closed).

2. Let X and τ be defined as an Example 2.10 (1). Let $Y = \{a,b,c\}$ and $\sigma = \{\phi, \{b\}, \{a,b\}, Y\}$. Define $f : (X,\tau) \to (Y,\sigma)$ be the identity map. Then, f is g-continuous (resp. αg-continuous, gs-continuous, gsp-continuous, $^{*}g$-continuous and $\#gs$-continuous) but not μsp-continuous, since $f^{-1}(\{a,c\}) = \{a,c\}$ is not μsp-closed.

Remark 2.11. The composition of two μsp-continuous maps need not be μsp-continuous and this is shown from the following example.

Example 2.12. Let X and τ be as in Example 2.3. Let $Y = Z = \{a,b,c\}$, $\sigma = \{\phi, \{a\}, Y\}$, and $\eta = \{\phi, \{a,b\}, Z\}$. Define $f : (X,\tau) \to (Y,\sigma)$ by $f(a) = b$, $f(b) = a$, and $f(c) = c$. Define $g : (Y,\sigma) \to (Z,\eta)$ by $g(a) = b$, $g(b) = c$, and $g(c) = a$. Clearly, f and g are μsp-continuous but their $g \circ f : (X,\tau) \to (Z,\eta)$ is not μsp-continuous, because $V = \{c\}$ is closed in (Z,η) but $(g \circ f^{-1}(\{c\}) = f^{-1}(g^{-1}(\{c\})) = f^{-1}(\{b\}) = \{a\}$, which is not μsp-closed in (X,τ).

Theorem 2.13. If $f : (X,\tau) \to (Y,\sigma)$ is μsp-continuous and $g : (Y,\sigma) \to (Z,\eta)$ is continuous, then $g \circ f : (X,\tau) \to (Z,\eta)$ is μsp-continuous.

PROOF. Clearly follows from definitions.

Proposition 2.14. A map $f : (X,\tau) \to (Y,\sigma)$ is μsp-continuous if and only if $f^{-1}(U)$ is μsp-open in (X,τ) for every open set U in (Y,σ).

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be μsp-continuous and U be an open set in (Y, σ). Then, U^c is closed in (Y, σ) and since f is μsp-continuous, $f^{-1}(U^c)$ is μsp-closed in (X, τ). But $f^{-1}(U^c) = f^{-1}((U))^c$ and so $f^{-1}(U)$ is μsp-open in (X, τ).
Conversely, assume that $f^{-1}(U)$ is μsp-open in (X, τ) for each open set U in (Y, σ). Let F be a closed set in (Y, σ). Then, F^c is open in (Y, σ) and by assumption, $f^{-1}(F^c)$ is μsp-open in (X, τ). Since $f^{-1}(F^c) = f^{-1}((F))^c$, we have $f^{-1}(F)$ is closed in (X, τ) and so f is μsp-continuous.
We introduce the following definition

Definition 2.15. A map $f : (X, \tau) \to (Y, \sigma)$ is called μsp-irresolute if $f^{-1}(V)$ is a μsp-closed set of (X, τ) for every μsp-closed set V of (Y, σ).

Theorem 2.16. Every μsp-irresolute map is μsp-continuous but not conversely.

Proof. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-irresolute map. Let V be a closed set of (Y, σ). Then, by the Result 1.6 (1), V is μsp-closed. Since f is μsp-irresolute, then $f^{-1}(V)$ is a μsp-closed set of (X, τ). Therefore, f is μsp-continuous.

Example 2.17. Let X, Y, τ, σ, and f be as in the Example 2.12. $\{b\}$ is μsp-closed set of (Y, σ) but $f^{-1}(\{b\}) = \{a\}$ is not a μsp-closed set of (X, τ). Thus, f is not μsp-irresolute map. However, f is μsp-continuous map.

Theorem 2.18. Let $f : (X, \tau) \to (Y, \sigma)$ and $g : (Y, \sigma) \to (Z, \eta)$ be any two maps. Then,

1. $g \circ f$ is μsp-continuous if g is continuous and f is μsp-continuous.
2. $g \circ f$ is μsp-irresolute if both f and g are μsp-irresolute.
3. $g \circ f$ is μsp-continuous if g is μsp-continuous and f is μsp-irresolute.

Proof. Omitted.

Theorem 2.19. Let $f : (X, \tau) \to (Y, \sigma)$ be an μsp-continuous map. If (X, τ), the domain of f is an $T_{\mu sp}$-space, then f is continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $T_{\mu sp}$-space, then $f^{-1}(V)$ is a closed set of (X, τ). Therefore, f is continuous.

Theorem 2.20. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $\alpha T_{\mu sp}$-space, then f is α-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $\alpha T_{\mu sp}$-space, then $f^{-1}(V)$ is a α-closed set of (X, τ). Therefore, f is α-continuous.

Theorem 2.21. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $pT_{\mu sp}$-space, then f is pre-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $pT_{\mu sp}$-space, then $f^{-1}(V)$ is a pre-closed set of (X, τ). Therefore, f is pre-continuous.

Theorem 2.22. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $\mu T_{\mu sp}$-space, then f is μ-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $\mu T_{\mu sp}$-space, then $f^{-1}(V)$ is a μ-closed set of (X, τ). Therefore, f is μ-continuous.

Theorem 2.23. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $\mu p T_{\mu sp}$-space, then f is μp-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $\mu pT_{\mu sp}$-space, then $f^{-1}(V)$ is a μp-closed set of (X, τ). Therefore, f is μp-continuous.

Theorem 2.24. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $spT_{\mu sp}$-space, then f is β-continuous.

Proof. Let V be a closed set of (Y, σ). Then $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $spT_{\mu sp}$-space, then $f^{-1}(V)$ is a β-closed set of (X, τ). Therefore, f is β-continuous.

Theorem 2.25. Let $f : (X, \tau) \to (Y, \sigma)$ be a μsp-continuous map. If (X, τ), the domain of f is an $g\alpha T_{\mu sp}$-space, then f is $g\alpha$-continuous.

Proof. Let V be a closed set of (Y, σ). Then, $f^{-1}(V)$ is a μsp-closed set of (X, τ), since f is μsp-continuous. Since (X, τ) is an $g\alpha T_{\mu sp}$-space, then $f^{-1}(V)$ is a $g\alpha$-closed set of (X, τ). Therefore, f is $g\alpha$-continuous.

3. Characterization of μsp-Continuous Maps

In this section we introduce μsp-interior and μsp-closure of a set and obtain the characterization theorem for μsp-continuous maps under certain conditions.

Definition 3.1. For any $A \subseteq X$, $\mu sp\text{-}int(A)$ is defined as the union of all μsp-open sets contained in A, i.e., $\mu sp\text{-}int(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is } \mu sp\text{-open}\}$.

Lemma 3.2. For any $A \subseteq X$, $int(A) \subseteq \mu sp\text{-}int(A) \subseteq A$.

Proof. The proof follows from Result 1.6 (4).

The following two Propositions are easy consequences from definitions.

Proposition 3.3. For any $A \subseteq X$, the following holds.

1. $\mu sp\text{-}int(A)$ is the largest μsp-open set contained in A.
2. A is μsp-open if and only if $\mu sp\text{-}int(A) = A$.

Proposition 3.4. For any subsets A and B of (X, τ), the following holds.

1. $\mu sp\text{-}int(A \cap B) = \mu sp\text{-}int(A) \cap \mu sp\text{-}int(B)$.
2. $\mu sp\text{-}int(A \cup B) \supseteq \mu sp\text{-}int(A) \cup \mu sp\text{-}int(B)$.
3. If $A \subseteq B$, then $\mu sp\text{-}int(A) \subseteq \mu sp\text{-}int(B)$.
4. $\mu sp\text{-}int(X) = X$ and $\mu sp\text{-}int(\phi) = \phi$.

Definition 3.5. For every set $A \subseteq X$, we define the μsp-closure of A to be the intersection of all μsp-closed sets containing A, i.e., $\mu sp\text{-}cl(A) = \cap\{F : A \subseteq F \in \mu spc(\tau)\}$.

Lemma 3.6. For any $A \subseteq X$, $A \subseteq \mu sp\text{-}cl(A) \subseteq cl(A)$.

Proof. The proof follows from Result 1.6 (1).

Remark 3.7. Both containment relations in Lemma 3.6 may be proper as seen from the following example.

Example 3.8. Let $X = \{a, b, c\}$, $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Here $\mu spc(\tau) = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$. Let $A = \{a\}$. Then, $\mu sp\text{-}cl(\{a\}) = \{a, b\}$ and so $A \subseteq \mu sp\text{-}cl(A) \subseteq cl(A)$.

The following two Propositions are easy consequences from definitions.

Proposition 3.9. For any $A \subseteq X$, the following holds.

1. $\mu sp\text{-}cl(A)$ is the smallest μsp-closed set containing A.

2. A is μsp-closed if and only if $\mu sp\text{-}cl(A) = A$.

Proposition 3.10. For any two subsets A and B of (X, τ), the following holds.

1. If $A \subseteq B$, then $\mu sp\text{-}cl(A) \subseteq \mu sp\text{-}cl(B)$.
2. $\mu sp\text{-}cl(A \cap B) \subseteq \mu sp\text{-}cl(A) \cap \mu sp\text{-}cl(B)$.

Proposition 3.11. Let A be a subset of a space X, then the following are true.

1. $(\mu sp\text{-}int(A))^c = \mu sp\text{-}cl(A^c)$.
2. $\mu sp\text{-}int(A) = (\mu sp\text{-}cl(A^c))^c$.
3. $\mu sp\text{-}cl(A) = (\mu sp\text{-}int(A^c))^c$.

PROOF.
1. Clearly follows from definitions.
2. Follows by taking complements in (1).
3. Follows by replacing A by A^c in (1).

Definition 3.12. Let (X, τ) be a topological space. Let x be a point of X and G be a subset of X. Then, G is called an μsp-neighbourhood of x (briefly, μsp-nbhd of x) in X if there exists an μsp-open set U of X such that $x \in U \subseteq G$.

Proposition 3.13. Let A be a subset of (X, τ). Then, $x \in \mu sp\text{-}cl(A)$ if and only if for any μsp-nbhd G_x of x in (X, τ), $A \cap G_x \neq \phi$.

PROOF. Necessity. Assume $x \in \mu sp\text{-}cl(A)$. Suppose that there is an μsp-nbhd G of the point x in (X, τ) such that $G \cap A = \phi$. Since G is μsp-nbhd of x in (X, τ), by Definition 3.12, there exists an μsp-open set U_x such that $x \in U_x \subseteq G$. Therefore, we have $U_x \cap A = \phi$ and so $A \subseteq (U_x)^c$. Since $(U_x)^c$ is an μsp-closed set containing A, we have by Definition 3.5, $\mu sp\text{-}cl(A) \subseteq (U_x)^c$ and therefore $x \notin \mu sp\text{-}cl(A)$, which is a contradiction. Sufficiency. Assume for each μsp-nbhd G_x of x in (X, τ), $A \cap G_x \neq \phi$. Suppose that $x \notin \mu sp\text{-}cl(A)$. Then, by Definition 3.5, there exists an μsp-closed set F of (X, τ) such that $A \subseteq F$ and $x \notin F$. Thus, $x \in F^c$ and F^c is μsp-open in (X, τ) and hence F^c is a μsp-nbhd of x in (X, τ). But $A \cap F^c = \phi$, which is a contradiction.

In the next theorem we explore certain characterizations of μsp-continuous functions.

Theorem 3.14. Let $f : (X, \tau) \to (Y, \sigma)$ be a map from a topological space (X, τ) into a topological space (Y, σ). Then the following statements are equivalent.

1. The function f is μsp-continuous.
2. The inverse of each open set is μsp-open.
3. For each point x in (X, τ) and each open set V in (Y, σ) with $f(x) \in V$, there is an μsp-open set U in (X, τ) such that $x \in U$, $f(U) \subseteq V$.
4. The inverse of each closed set is μsp-closed.
5. For each x in (X, τ), the inverse of every neighbourhood of $f(x)$ is an μsp-nbhd of x.
6. For each x in (X, τ) and each neighbourhood N of $f(x)$, there is an μsp-nbhd G of x such that $f(G) \subseteq N$.
7. For each subset A of (X, τ), $f(\mu sp\text{-}cl(A)) \subseteq cl(f(A))$.
8. For each subset B of (Y, σ), $\mu sp\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Proof. (1) ⇔ (2). This follows from Proposition 2.14.

(1) ⇔ (3). Suppose that (3) holds and let V be an open set in (Y, σ) and let $x \in f^{-1}(V)$. Then, $f(x) \in V$ and thus there exists an μsp-open set U_x such that $x \in U_x$ and $f(U_x) \subseteq V$. Now, $x \in U_x \subseteq f^{-1}(V)$ and $f^{-1}(V) = \cup_{x \in f^{-1}(V)} U_x$. By assumption, $f^{-1}(V)$ is μsp-open in (X, τ) and therefore f is μsp-continuous.

Conversely, Suppose that (1) holds and let $f(x) \in V$. Then, $x \in f^{-1}(V) \in \mu sp(\tau)$, since f is μsp-continuous. Let $U = f^{-1}(V)$. Then, $x \in U$ and $f(U) \subseteq V$.

(2) ⇔ (4). This result follows from the fact if A is a subset of (Y, σ), then $f^{-1}(A^c) = (f^{-1}(A))^c$.

(2) ⇔ (5). For x in (X, τ), let N be a neighbourhood of $f(x)$. Then, there exists an open set U in (Y, σ) such that $f(x) \in U \subseteq N$. Consequently, $f^{-1}(U)$ is an μsp-open set in (X, τ) and $x \in f^{-1}(U) \subseteq f^{-1}(N)$. Thus, $f^{-1}(N)$ is an μsp-nbhd of x.

(5) ⇔ (6). Let $x \in X$ and let N be a neighbourhood of $f(x)$. Then, by assumption, $G = f^{-1}(N)$ is an μsp-nbhd of x and $f(G) = f(f^{-1}(N)) \subseteq N$.

(6) ⇔ (3). For x in (X, τ), let V be an open set containing $f(x)$. Then, V is a neighborhood of $f(x)$. So by assumption, there exists an μsp-nbhd G of x such that $f(G) \subseteq V$. Hence, there exists an μsp-open set U in (X, τ) such that $x \in U \subseteq G$ and so $f(U) \subseteq f(G) \subseteq V$.

(7)⇔ (4). Suppose that (4) holds and let A be a subset of (X, τ). Since $A \subseteq f^{-1}(A)$, we have $A \subseteq f^{-1}(cl(f(A)))$. Since $cl(f(A))$ is a closed set in (Y, σ), by assumption $f^{-1}(cl(f(A)))$ is an μsp-closed set containing A. Consequently, $\mu sp\text{-}cl(A) \subseteq f^{-1}(cl(f(A)))$. Thus, $f(\mu sp\text{-}cl(A)) \subseteq f(f^{-1}(cl(f(A)))) \subseteq cl(f(A))$.

Conversely, suppose that (7) holds for any subset A of (X, τ). Let F be a closed subset of (Y, σ). Then, by assumption, $f(\mu sp\text{-}cl(f^{-1}(F))) \subseteq cl(f(f^{-1}(F))) \subseteq cl(F) = F$, i.e., $\mu sp\text{-}cl(f^{-1}(F)) \subseteq f^{-1}(F)$ and so $f^{-1}(F)$ is μsp-closed.

(7) ⇔ (8). Suppose that (7) holds and B be any subset of (Y, σ). Then, replacing A by $f^{-1}(B)$ in (7), we obtain $f(\mu sp\text{-}cl(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) \subseteq cl(B)$, i.e., $\mu sp\text{-}cl(f^{-1}(B)) \subseteq f^{-1}cl(B)$.

Conversely, suppose that (8) holds. Let $B = f(A)$ where A is a subset of (X, τ). Then, we have, $\mu sp\text{-}cl(A) \subseteq \mu sp\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(cl(f(A))$ and so $f(\mu sp\text{-}cl(A)) \subseteq cl(f(A))$.

This completes the proof of the theorem.

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