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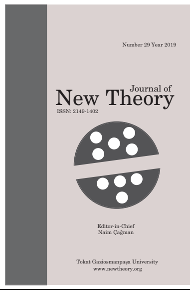

A Note on Rhotrices Ring

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Abstract — In this paper, we define algebraic operations on 3-dimensional rhotrices over an arbitrary ring R and show that the set of 3-dimensional rhotrices over an arbitrary ring R is a ring according to these operations. We investigate the properties of a rhotrices ring. Furthermore, we characterize the ideals of a rhotrices ring. Also, maximal ideals and prime ideals of a rhotrices ring are investigated. An example of these concepts is presented.

Keywords — *Rhotrix, rhotrices ring, ideals of a rhotrices ring*

1. Introduction

The concept of the rhotrix is a mathematical structure in the rhombodial form of real numbers defined by Atanasov and Shannon [1], inspired by the concepts of matrix tertion and matrix netrion. In 2003, Ajibade [2] defined an object that lies between 2×2 dimensional matrices and 3×3 dimensional matrices called rhotrix as follows:

Definition 1.1. [2] Let a, b, c, d, e be real numbers. Then a mathematical rhombodial form

$$R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

is called $3-dimensional\ rhotrix$ over real numbers. The entry c in rhotrix R is called the heart of R denoted by $h(R)$.

The set of all $3-dimensional\ rhotrices$ is denoted by $\mathcal{R}$.

$$\mathcal{R} = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \middle| a, b, c, d, e \in \mathbb{R} \right\}$$

On operations over $\mathcal{R}$ are as follows:

Let $R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & h & j \\ & k & \end{array} \right\rangle$ be in $\mathcal{R}$. Then,

$$R = Q \Leftrightarrow a = f, b = g, c = h, d = j, e = k$$

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The addition of two rhotrices R and Q was defined as

$$R+Q=\left\langle \begin{array}{ccc} & a+f & \\ b+g & c+h & d+j \\ & e+k & \end{array} \right\rangle$$

It is reported in [3] that the set of all 3-dimensional rhotrices is a commutative group w.r.t $'+'$. This group is denoted by $\langle \mathcal{R},+\rangle$. The notion $(-R)$ was given as additional inverse of rhotrix R and was defined as follows:

$$-R=\left\langle \begin{array}{ccc} & -a & \\ -b & -c & -d \\ & -e & \end{array} \right\rangle$$

$e_{\mathcal{R}}=\left\langle \begin{array}{ccc} & 0_R & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle$ was given identity element of rhotrices group $\mathcal{R}_3$. Let $\alpha \in R$ and $R \in \mathcal{R}$. The scalar multiplication of α and R was defined by

$$\alpha R=\left\langle \begin{array}{ccc} & \alpha a & \\ \alpha b & \alpha c & \alpha d \\ & \alpha e & \end{array} \right\rangle$$

Definition 1.2. Let $R=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q=\left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$ be in $\mathcal{R}$. The multiplication of R and Q is as follows:

$$RoQ=\left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+jh(R) \\ & eh(Q)+kh(R) & \end{array} \right\rangle$$

In [3] it has been shown that the set of all three-dimensional real rhotrices together with the operations addition (+) and multiplication (o) is a commutative ring with identity $I=\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$.

Definition 1.3. Let $R=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ be in $\mathcal{R}$. If $RoQ= I$ such that there exists $Q\in\mathcal{R}$ then Q is called the inverse of R, denoted by R^{-1}, and

$$Q=R^{-1}=\frac{-1}{h(R)^2}\left\langle \begin{array}{ccc} & a & \\ b & -h(R) & d \\ & e & \end{array} \right\rangle \text{ where } h(R)\neq 0.$$

Other multiplication of rhotrices called row-column multiplication was proposed by Sani [4]. This multiplication is as follows:

Definition 1.4. Let $R=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q=\left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle$ be in $\mathcal{R}$.

$$R\bullet Q=\left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & h(R)h(Q) & aj+dk \\ & bf+ek & \end{array} \right\rangle$$

Studies on this subject has progressed quickly after the rhotrix definition. Several authors have obtained interesting results on 3-dimensional rhotrices. See [5] for a comprehensive survey of the literature on these developments.

2. Rhotrices Ring on Arbitrary Ring R

The definition of n-dimensional rhotrix over an arbitrary ring was firstly given by Mohammed in [6] and he gave the set of all rhotrices over an arbitrary ring is a ring together with the operations of rhotrix addition and row-cloum rhotrix multiplication.

In this section, it has been shown that the set of 3-dimensional rhotrices over an arbitrary ring is a ring with the operations rhotrix addition and **"hearty multiplication"** as different from multiplication in Mohammed's work [6]. Also we investigate the basic properties of the rhotrices ring.

Definition 2.1. Let $(R,+,.)$ be a ring with identity. By a $3-dimensional\ rhotrix$ over the ring R, we mean a rhomboidal array defined by

$$A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$$

where a,b,c,d,e are in the ring R. The entry c of A is called heart of A denoted by $h(A)$.

The set of all 3-dimensional rhotrices over the ring R denoted by $\mathcal{R}_3(R)$**,**

$$\mathcal{R}_3(R) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \middle| a,b,c,d,e \in R \right\}$$

.

We define two binary operations addition ($\widehat{+}$) and multiplication($\odot$) on $\mathcal{R}_3(R)$ by

$$\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \widehat{+} \left\langle \begin{array}{ccc} & a' & \\ b' & c' & d' \\ & e' & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+a' & \\ b+b' & c+c' & d+d' \\ & e+e' & \end{array} \right\rangle \quad (1)$$

$$\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & a' & \\ b' & c' & d' \\ & e' & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a.c' + c.a' & \\ b.c' + c.b' & c.c' & d.c' + c.d' \\ & e.c' + c.e' & \end{array} \right\rangle \quad (2)$$

for all $\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & a' & \\ b' & c' & d' \\ & e' & \end{array} \right\rangle \in \mathcal{R}_3(R)$. It is easy to check that these operations are well defined, since " + " and "." in R are well-defined.

Theorem 2.2. The set of all $3-dimensional\ rhotrices$ $\mathcal{R}_3(R)$ over the ring R is a ring with respect to operations "$\widehat{+}$" and " $\odot$ ".

Proof. It's easy to see that $(\mathcal{R}_3(R), \widehat{+})$ is a commutative group. Now let's show that the triple $(\mathcal{R}_3(R), \widehat{+}, \odot)$ is a ring.

For all $P = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle, Q = \left\langle \begin{array}{ccc} & a\prime & \\ b\prime & c\prime & d\prime \\ & e\prime & \end{array} \right\rangle, S = \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \in \mathcal{R}_3(R)$

$$\begin{aligned} (P \odot Q) \odot S &= \left\langle \begin{array}{ccc} & a.c' + c.a' & \\ b.c' + c.b' & c.c' & d.c' + c.d' \\ & e.c' + c.e' & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & a.c'.z + c.a'.z + c.c'.x & \\ b.c'.z + c.b'.z + c.c'.y & c.c'.z & d.c'.z + c.d'.z + c.c'.t \\ & e.c'.z + c.e'.z + c.c'.u & \end{array} \right\rangle \\ &= P \odot (Q \odot S) \end{aligned}$$

The operation " $\odot$ " is an associative in $\mathcal{R}_3(R)$.

$$
\begin{aligned}
(P\widehat{+}Q)\odot S &= \left\langle \begin{array}{ccc} & a+a' & \\ b+b' & c+c' & d+d' \\ & e+e' & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} & (a+a').z+(c+c').x & \\ (b+b').z+(c+c').y & (c+c').z & (d+d').z+(c+c').t \\ & (e+e').z+(c+c').u & \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} & a.z+a'.z+c.x+c'.x & \\ b.z+b'.z+c.y+c'.y & c.z+c'.z & d.z+d'.z+c.t+c'.t \\ & e.z+e'.z+c.u+c'.u & \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} & a.z+c.x & \\ b.z+c.y & c.z & d.z+c.t \\ & e.z+c.u & \end{array} \right\rangle \widehat{+} \left\langle \begin{array}{ccc} & a'.z+c'.x & \\ b'.z+c'.y & c'.z & d'.z+c'.t \\ & e'.z+c'.u & \end{array} \right\rangle \\
&= (P\odot S)\widehat{+}(Q\odot S)
\end{aligned}
$$

and similarly it is easy to check that $P\odot(S\widehat{+}Q)=(P\odot S)\widehat{+}(P\odot Q)$

Thus $<\mathcal{R}_3(R),\widehat{+},\odot>$ is a ring.

Furthermore if R is a commutative ring, then $\mathcal{R}_3(R)$ is a commutative ring and if R is a ring with identity 1_R, then $\mathcal{R}_3(R)$ to be a ring with identity $1_{\mathcal{R}_3(R)} = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & 1_R & 0_R \\ & 0_R & \end{array} \right\rangle$.

Example 2.3. Let $R=\mathbb{Z}_2$. $\mathcal{R}_3(R)$ is a rhotrix ring and since $\mathbb{Z}_2$ is a commutative ring, $\mathcal{R}_3(\mathbb{Z}_2)$ is a commutative ring.

The following theorem give us the characteristic of the ring $\mathcal{R}_3(R)$ depends on the characteristic of the ring R.

Theorem 2.4. The characteristic of the ring $\mathcal{R}_3(R)$ is equal to characteristic of the ring R.

Proof. Let R be a ring with $CharR=k$. Then the characteristic of the ring $\mathcal{R}_3(R)$ is k. Let $Char\mathcal{R}_3(R)=t$, we show that $k=t$.

$$
\begin{aligned}
Char\mathcal{R}_3(R)=t \quad &\Rightarrow \quad t.\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = 0_{\mathcal{R}_3(R)},\ for\ all\ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \\
&\Rightarrow \quad t.a=t.b=t.c=t.d=t.e=0_R,\ for\ all\ \ a,b,c,d,e\in R \\
&\Rightarrow \quad k|t
\end{aligned}
$$

$$
\begin{aligned}
CharR=k \quad &\Rightarrow \quad k.a=0_R,\ for\ all\ \ a\in R \\
&\Rightarrow \quad \left\langle \begin{array}{ccc} & k.a & \\ k.b & k.c & k.d \\ & k.e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle,\ for\ all\ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \\
&\Rightarrow \quad k.\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle = 0_{\mathcal{R}_3(R)},\ for\ all\ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \\
&\Rightarrow \quad t|k
\end{aligned}
$$

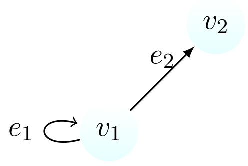
thus $k=t$.

Note: In the ring $\mathcal{R}_3(R)$, the multiplication of nonzero rhotrices A and B is equal to zero. Hence $\mathcal{R}_3(R)$ has zero divisors and $\mathcal{R}_3(R)$ is not integral domain.

The following theorem characterize idempotent elements and nilpotent elements in a ring $\mathcal{R}_3(R)$. Firstly we recall that definitions of idempotent and nilpotent elements in any ring. Let $(R,+,.)$ be a ring. An element $a\in R$ is called idempotent if $a^2=a$ and nilpotent if $a^n=0$ for some positive integer n.

Theorem 2.5. Let R be a ring with identity 1_R and c be an idempotent element in R. Then $\left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c & 0_R \\ & 0_R & \end{array} \right\rangle$ is an idempotent element in $\mathcal{R}_3(R)$.

PROOF. $\left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c & 0_R \\ & 0_R & \end{array} \right\rangle^2 = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c^2 & 0_R \\ & 0_R & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & c & 0_R \\ & 0_R & \end{array} \right\rangle$. But all idempotents elements in the ring $\mathcal{R}_3(R)$ is not this form. For example; let R be a ring $L_K(E)$, where $L_K(E)$ is a Leavitt Path Algebra [7] and

E: 

Let $A = \left\langle \begin{array}{ccc} & e_2 & \\ 0 & v_1 & 0 \\ & 0 & \end{array} \right\rangle$ be in $\mathcal{R}_3(L_K(E))$. Since in a ring $L_K(E)$, $v_1.v_1 = v_1$, $v_1.v_2 = v_2.v_1 = 0$, $e_2.v_1 = v_2.e_2 = 0, v_1.e_2 = e_2.v_2 = e_2$, $v_1.e_1 = e_1.v_1 = e_1, v_2.e_1 = e_1.v_2 = 0$. Therefore,

$$A^2 = \left\langle \begin{array}{ccc} & e_2.v_1 + v_1.e_2 & \\ 0 & v_1.v_1 & 0 \\ & 0 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & e_2 & \\ 0 & v_1 & 0 \\ & 0 & \end{array} \right\rangle = A$$

Theorem 2.6. Let R be a ring with identity 1_R and c be a nilpotent element in the ring R. Then $\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ is a nilpotent element in a ring $\mathcal{R}_3(R)$.

PROOF. We give any $A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R)$ and let be c a nilpotent element in a ring R. Since c is a nilpotent element, there exits $n \in \mathbb{Z}^+$ such that $c^n = 0_R$. Then, $h(A^n) = c^n = 0_R$ and $A^{2n} = A^n.A^n = 0_{\mathcal{R}_3(R)}$.

In particularly; If R is a commutative ring. Then $c^n = 0_R$ implies that $A^{n+1} = 0_{\mathcal{R}_3(R)}$.

3. Ideal of Rhotrix Ring

In this section, ideals of rhotrices ring have been investigated. Furthermore characterizations of maximal ideals and prime ideals have been given.

Theorem 3.1. Let R be a ring and I be an ideal of R. Then, $M = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : c \in I \right\}$ is an ideal in $\mathcal{R}_3(R)$.

PROOF. Since I is an ideal of R, M is a subset of $\mathcal{R}_3(R)$ and $M \neq \emptyset$. We give any $A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$, $B = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & c_1 & d_1 \\ & e_1 & \end{array} \right\rangle \in M$, and $C = \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \in \mathcal{R}_3(R)$. Then c, $c_1 \in I$ and $z \in R$, $c + (-c_1)$, $z.c$, $c.z \in I$. Hence $A \widehat{+} (-B) \in M$ and $A \odot C$, $C \odot A \in M$. Thus, M is an ideal in $\mathcal{R}_3(R)$.

Theorem 3.2. Let R be a ring and $\mathcal{R}_3(R)$ be a ring of rhotrices.

$$I \text{ is an ideal of } R \Leftrightarrow \mathcal{R}_3(I) \text{ is an ideal of } \mathcal{R}_3(R)$$

PROOF. ($\Rightarrow$) Let I be an ideal of R. Then $I \subseteq R$ and $I \neq \emptyset$. Thus $\mathcal{R}_3(I) \subseteq \mathcal{R}_3(R)$ and $\mathcal{R}_3(I) \neq \emptyset$. For any $A \in \mathcal{R}_3(I)$, since $h(A) \in I$, $\mathcal{R}_3(I)$ is an ideal in $\mathcal{R}_3(R)$ from Theorem 3.1.

($\Leftarrow$) Let $\mathcal{R}_3(I)$ be an ideal of $\mathcal{R}_3(R)$. It is easy check that $I \neq \emptyset$, $I \subseteq R$.
We give any $a, b \in I$ and $r \in R$.

i. $a \in I \Rightarrow A = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$ and $b \in I \Rightarrow B = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & b & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$. Since $\mathcal{R}_3(I)$ is an ideal of $\mathcal{R}_3(R)$, $A \widehat{+} (-B) = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a-b & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I)$ and $a - b \in I$

ii. $r \in R \Rightarrow C = \left\langle \begin{array}{ccc} & r & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(R)$. Since $\mathcal{R}_3(I)$ is a ideal of $\mathcal{R}_3(R)$,

$$A \odot C = \left\langle \begin{array}{ccc} & a.r & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I) \text{ and } a.r \in I$$

Similarly,

$$C \odot A = \left\langle \begin{array}{ccc} & r.a & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I) \text{ and } r.a \in I$$

Consequently, I is an ideal of R.

Corollary 3.3. Let K be a subset of $\mathcal{R}_3(R)$. K is an ideal in $\mathcal{R}_3(R)$ if and only if there exists an ideal I in R such that $h(A) \in I$, for all $A \in K$.

PROOF. Let $I = \{a \in R : a = h(A) \ for \ all \ A \in K\} \subseteq R$.

Since $0 \in I$ for $0_{\mathcal{R}_3(R)} \in K$, $I \neq \emptyset$. We will show that $a - b$, $a.r$, $r.a \in I$ for all $a, b \in I$ and $r \in R$. $a \in I \Rightarrow A = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \in K$, $b \in I \Rightarrow B = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & b & 0_R \\ & 0_R & \end{array} \right\rangle \in K$, and $A \widehat{+} (-B) \in K$ because K is an ideal in $\mathcal{R}_3(R)$ and so $h(A \widehat{+} (-B)) = a - b \in I$.
$C = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & r & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(R)$ for $r \in R$ and since K is an ideal in $\mathcal{R}_3(R)$, $A \odot C$, $C \odot A \in K$ and so $h(A \odot C) = a.r$, $h(C \odot A) = r.a \in I$. Thus I is an ideal in R.

Conversely, let $K = \{A \in \mathcal{R}_3(R) : h(A) \in I\} \subseteq \mathcal{R}_3(R)$.

$I \neq \emptyset$ then there exists $a \in I$ and so $A \in K$ such that $h(A) = a$ and $K \neq \emptyset$. We give any $A, B \in K$ and $C \in R$. Then $h(A), h(B) \in I$ and since I is an ideal in R, $h(A) - h(B) = h(A \widehat{+} (-B))$, $h(A).h(C) = h(A \odot C)$, $h(C).h(A) = h(C \odot A) \in I$ and so $A \widehat{+} (-B)$, $A \odot C$, $C \odot A \in K$. Thus K is an ideal in $\mathcal{R}_3(R)$.

Theorem 3.4. Let R be a commutative ring with identity and I be an ideal of R. Then

$$I \text{ is a principal ideal of } R \Leftrightarrow \mathcal{R}_3(I) \text{ is a principal ideal of } \mathcal{R}_3(R)$$

PROOF. Let I be a principal ideal of R. Then there exists $a \in R$ such that $I = (a)$.

Let $P \in \mathcal{R}_3(R)$. Since $I = (a)$, $P = \left\langle \begin{array}{ccc} & a.r_1 & \\ a.r_2 & a.r_3 & a.r_4 \\ & a.r_5 & \end{array} \right\rangle$. Then,

$$P = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & r_1 & \\ r_2 & r_3 & r_4 \\ & r_5 & \end{array} \right\rangle \in (A)$$

where $A = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle$.

Conversely, let $\mathcal{R}_3(I)$ be a principal ideal in $\mathcal{R}_3(R)$. Then there exist $P \in \mathcal{R}_3(R)$ such that $\mathcal{R}_3(I) = (P)$. We will show that $I = (h(P))$.

$$a \in I \Rightarrow \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & a & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(I) = (P)$$

$\Rightarrow a = h(P).x$, $x \in R \Rightarrow a \in (h(P))$. Thus $I \subseteq (h(P))$. Since $\mathcal{R}_3(I) = (P)$, $h(P) \in I$. Then $(h(P)) \subseteq I$. Thus $I = (h(P))$.

Theorem 3.5. Let R be a ring and I be an ideal of R. Then,

$$\mathcal{R}_3(R/I) = \left\{ \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle : a+I, b+I, c+I, d+I, e+I \in R/I \right\}$$

is a ring with as known operations "$\widehat{+}$" and "$\odot$" and $\mathcal{R}_3(R)/\mathcal{R}_3(I)$ isomorphic to ring $\mathcal{R}_3(R/I)$.

Proof. Since R/I is a ring, $\mathcal{R}_3(R/I)$ is a ring. We will show that $\mathcal{R}_3(R/I) \cong \mathcal{R}_3(R)/\mathcal{R}_3(I)$. We define $f : \mathcal{R}_3(R) \to \mathcal{R}_3(R/I)$ by

$$f\left(\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle\right) = \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle, \text{for any} \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R)$$

It is easy to see that f is a well-defined. We give any $A = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle$ and $B = \left\langle \begin{array}{ccc} & x & \\ y & z & t \\ & u & \end{array} \right\rangle \in \mathcal{R}_3(R)$

i.

$$\begin{aligned} f(A\widehat{+}B) &= \left\langle \begin{array}{ccc} & (a+x)+I & \\ (b+y)+I & (c+z)+I & (d+t)+I \\ & (e+u)+I & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle \widehat{+} \left\langle \begin{array}{ccc} & x+I & \\ y+I & z+I & t+I \\ & u+I & \end{array} \right\rangle \\ &= f(A)\widehat{+}f(B) \end{aligned}$$

and

$$\begin{aligned} f(A \odot B) = & \ f\left(\left\langle \begin{array}{ccc} & a.z+c.x & \\ b.z+c.y & c.z & d.z+c.t \\ & e.z+c.u & \end{array} \right\rangle\right) \\ = & \left\langle \begin{array}{ccc} & (a.z+c.x)+I & \\ (b.z+c.y)+I & (c.z)+I & (d.z+c.t)+I \\ & (e.z+c.u)+I & \end{array} \right\rangle \\ = & \ f(A) \odot f(B) \end{aligned}$$

Thus, f is a ring homomorphism.

ii.

$$\begin{aligned} Kerf &= \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) : f\left(\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle\right) = 0_{\mathcal{R}_3(R/I)} \right\} \\ &= \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in I \right\} \\ &= \mathcal{R}_3(I) \end{aligned}$$

iii.

$$\begin{aligned} Imf &= \left\{ f\left(\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle\right) : \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \in \mathcal{R}_3(R) \right\} \\ &= \left\{ \left\langle \begin{array}{ccc} & a+I & \\ b+I & c+I & d+I \\ & e+I & \end{array} \right\rangle : a,b,c,d,e \in R \right\} \\ &= \mathcal{R}_3(R/I) \end{aligned}$$

Thus, f is a surjective.

Consequently $\mathcal{R}_3(R)/\mathcal{R}_3(I)$ is isomorphic to $\mathcal{R}_3(R/I)$ by the first isomorphism theorem.

Theorem 3.6. Let R be any ring, $\mathcal{R}_3(R)$ be a ring of 3-dimensional rhotrices over R. If $\mathcal{R}_3(M)$ is a maximal ideal of $\mathcal{R}_3(R)$, then M is a maximal ideal of R.

Proof. By Theorem 3.2, M is an ideal of R. Let J be an ideal of R such that $M \subseteq J \subseteq R$. We will show that $M = J$ or $J = R$. $M \subseteq J \subseteq R$ implies that $\mathcal{R}_3(M) \subseteq \mathcal{R}_3(J) \subseteq \mathcal{R}_3(R)$. Since $\mathcal{R}_3(M)$ is a maximal ideal in $\mathcal{R}_3(R)$, $\mathcal{R}_3(M) = \mathcal{R}_3(J)$ or $\mathcal{R}_3(J) = \mathcal{R}_3(R)$. Hence $M = J$ or $J = R$. Thus M is a maximal ideal in R.

The converse of the above theorem is not true, as shown by the following example.

Example 3.7. Let R be a ring and M be an maximal ideal of R and

$$K = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle \middle| a,b,d,e \in R \ and \ c \in M \right\}$$

K is an ideal of $\mathcal{R}_3(R)$ and $\mathcal{R}_3(M) \subseteq K \subseteq \mathcal{R}_3(R)$. Thus M is a maximal ideal in R but $\mathcal{R}_3(M)$ is not a maximal ideal in $\mathcal{R}_3(R)$.

Theorem 3.8. Let K be an ideal in $\mathcal{R}_3(R)$ and $M = \{a \in R : a = h(A), A \in K\}$ be a subset of R. If M is a maximal ideal in R then K is a maximal ideal in $\mathcal{R}_3(R)$.

Proof. Suppose that K is not a maximal ideal in $\mathcal{R}_3(R)$. Then there exists an ideal J in $\mathcal{R}_3(R)$ such that $K \subseteq J \subseteq \mathcal{R}_3(R)$.

Since J is an ideal, there exists an ideal I in R such that $h(A) \in I$ for arbitrary $A \in J$ and since $K \subseteq J$, $M \subseteq I$ but $I \subsetneq M$ because for every $A \in J$, $h(A) \notin M$. Therefore there exists an ideal I in R. However, this gives a contradiction since M is a maximal ideal of R.

Theorem 3.9. Let R be a ring and $\mathcal{R}_3(P)$ be a prime ideal of ring $\mathcal{R}_3(R)$. Then P is a prime ideal of R.

Proof. Since $\mathcal{R}_3(P)$ is an ideal in $\mathcal{R}_3(R)$, P is an ideal in R by Theorem 3.2.

We give any $a, b \in R$ and let $aRb \subseteq P$. Then for any $x \in R$, $axb \in P$. Hence, $\left\langle \begin{array}{ccc} & axb & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle = A \odot X \odot B \in \mathcal{R}_3(P)$, where $A = \left\langle \begin{array}{ccc} & a & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle$, $X = \left\langle \begin{array}{ccc} & 0_R & \\ 0_R & x & 0_R \\ & 0_R & \end{array} \right\rangle$, and $B = \left\langle \begin{array}{ccc} & b & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle$. Since, $\mathcal{R}_3(P)$ is a prime ideal, either $\left\langle \begin{array}{ccc} & a & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(P)$ or $\left\langle \begin{array}{ccc} & b & \\ 0_R & 0_R & 0_R \\ & 0_R & \end{array} \right\rangle \in \mathcal{R}_3(P)$. Hence either $a \in P$ or $b \in P$. Therefore P is a prime ideal in R.

The converse of the above theorem is not true, as shown by the following example.

Example 3.10. Although $3\mathbb{Z}$ is a prime ideal in the ring $\mathbb{Z}$, $\mathcal{R}_3(3\mathbb{Z})$ is not a prime ideal in the ring $\mathcal{R}_3(\mathbb{Z})$. Indeed,

$$\begin{aligned} A \odot B &= \left\langle \begin{array}{ccc} & -2 & \\ 5 & 3 & 1 \\ & 2 & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & 4 & \\ 1 & 6 & -1 \\ & 2 & \end{array} \right\rangle \\ &= \left\langle \begin{array}{ccc} & 0 & \\ 33 & 18 & 3 \\ & 18 & \end{array} \right\rangle \in \mathcal{R}_3(3\mathbb{Z}) \end{aligned}$$

but $A \notin \mathcal{R}_3(3\mathbb{Z})$ and $B \notin \mathcal{R}_3(3\mathbb{Z})$.

Corollary 3.11. Let K be an ideal in $\mathcal{R}_3(R)$. K is a prime ideal in $\mathcal{R}_3(R)$ if and only if there exists a prime ideal P in R such that $h(A) \in P$, for all $A \in K$.

Proof. Let K be a prime ideal in $\mathcal{R}_3(R)$. Then by Corollary 3.3, P is an ideal in R. We will show that P is a prime.

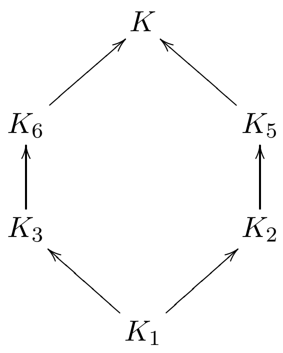
$a.R.b \subseteq P$, for all $a, b \in R$. Then $a.c.b \in P$, for all $c \in R$. By hypothesis, there exists $A \in K$ such that $h(A) = a.c.b \in P$. There exists X, Y, Z rhotrices such that $A = X \odot Y \odot Z$ and $h(X) = a$, $h(Y) = b$, $h(Z) = c$. Since K is a prime ideal in $\mathcal{R}_3(R)$ and $A \in K$, either $X \in K$ or $Z \in K$. Hence either $a \in P$ or $c \in P$. Thus P is a prime ideal in R.

Conversely, let P be a prime ideal in R. Then by Corollary 3.3 , K is a ideal in $\mathcal{R}_3(R)$. Let $X \odot \mathcal{R}_3(R) \odot Y \subseteq K$, for any $X, Y \in \mathcal{R}_3(R)$. Then $X \odot C \odot Y \in K$, for all $C \in \mathcal{R}_3(R)$. Hence $h(X \odot C \odot Y) = h(X).h(C).h(Y) \in P$ and since P is a prime ideal in R, either $h(X) \in P$ or $h(Y) \in P$. Thus either $X \in K$ or $Y \in K$ and so K is a prime ideal in $\mathcal{R}_3(R)$.

Example 3.12. Let $R = \mathbb{Z}_6 = \{\overline{0}, \ \overline{1}, \ \overline{2}, \ \overline{3}, \ \overline{4}, \ \overline{5}\}$. Then, $A = (\overline{0})$, $B = (\overline{2})$, $C = (\overline{3})$, $D = \mathbb{Z}_6$ are ideals in $\mathbb{Z}_6$. Hence, $K = \mathcal{R}_3(R)$, $K_1 = 0_{\mathcal{R}_3(R)}$, $K_2 = \mathcal{R}_3(B)$, $K_3 = \mathcal{R}_3(C)$,

$$K_4 = \left\langle \begin{array}{ccc} & R & \\ R & A & R \\ & R & \end{array} \right\rangle, \quad K_5 = \left\langle \begin{array}{ccc} & R & \\ R & B & R \\ & R & \end{array} \right\rangle, \text{ and } K_6 = \left\langle \begin{array}{ccc} & R & \\ R & C & R \\ & R & \end{array} \right\rangle$$

are ideals in $\mathcal{R}_3(\mathbb{Z}_6)$. Furthermore since B and C are prime ideals in $\mathbb{Z}_6$, K_5 ve K_6 are prime ideals in $\mathcal{R}_3(\mathbb{Z}_6)$. It is easy to see that K_5 ve K_6 are prime ideals in $\mathcal{R}_3(\mathbb{Z}_6)$.



Furthermore since B and C are maximal ideals in $\mathbb{Z}_6$, K_5 ve K_6 are maximal ideals in $\mathcal{R}_3(\mathbb{Z}_6)$. From above graphic, it is easy to see that K_5 ve K_6 are prime ideals in $\mathcal{R}_3(\mathbb{Z}_6)$.

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