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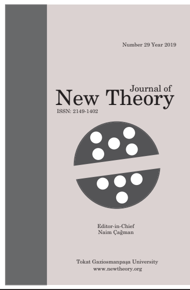

Energy Decay of Solutions for a System of Higher-Order Kirchhoff Type Equations

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Abstract − In this work, we considered a system of higher-order Kirchhoff type equations with initial and boundary conditions in a bounded domain. Under suitable conditions, we proved an energy decay result by Nakao's inequality techniques.

Keywords − *Kirchhoff type equation, energy decay, damping term*

1. Introduction

The Kirchhoff equation is the famous wave equations model which describe the small-amplitude vibrations of elastic strings introduced by Kirchhoff [1]. In one dimensional space it take th following form

$$\rho h\frac{\partial^2 u}{\partial t^2}+\delta\frac{\partial u}{\partial t}-\left\{\rho_0+\frac{Eh}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right\}\frac{\partial^2 u}{\partial x^2}=0,\quad (0<x<L,\ t\geq 0)$$

where $u\,(x,t)$ is the vertical displacement, E the Young modulus, ρ the mass density, h the cross-section area, L the length, ρ_0 the initial axial tension, δ the resistance modulus, and f and g the external forces.

In this work, we consider the following nonlinear wave equations of Kirchhoff type

$$\begin{cases} u_{tt}+M\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2\right)Au+\int_0^t g(t-s)Au(s)ds+|u_t|^{p-1}u_t=f_1, (x,t)\in\Omega\times[0,\infty) \\ v_{tt}+M\left(\left\|A^{\frac{1}{2}}u\right\|^2+\left\|A^{\frac{1}{2}}v\right\|^2\right)Av+\int_0^t h(t-s)Av(s)ds+|v_t|^{q-1}v_t=f_2,\ (x,t)\in\Omega\times[0,\infty) \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), \quad x\in\Omega \\ v(x,0)=v_0(x),\ v_t(x,0)=v_1(x), \quad x\in\Omega \\ \frac{\partial^i u}{\partial v^i}=\frac{\partial^i v}{\partial v^i}=0,\ i=0,1,2,...,m-1, \quad x\in\partial\Omega\times(0,\infty) \end{cases} \tag{1}$$

where Ω is a bounded domain in R^n $(n=1,2,3)$ with a smooth boundary $\partial\Omega$, and $g,h:R^+\to R^+$, $f_i(.,.):R^2\to R$ $(i=1,2)$ are given functions which will be specified later. Also, $A=(-\Delta)^m$, $m\geq 1$ is a positive integer and $p,q\geq 1$ are real numbers.

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When $m = 1$, the system

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{p-1}u_t = f_1(u,v) \\ v_{tt} - M\left(\|\nabla u\|^2 + \|\nabla v\|^2\right)\Delta v + \int_0^t h(t-s)\Delta v(s)ds + |v_t|^{q-1}v_t = f_2(u,v) \end{cases} \tag{2}$$

was investigated by Wu [2], here the author proved a decay and blow-up of solutions.

When $M(s) \equiv 1$, (2) become the following system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + |u_t|^{p-1}u_t = f_1(u,v) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(s)ds + |v_t|^{q-1}v_t = f_2(u,v) \end{cases} \tag{3}$$

Many authors studied the existence, blow up, lower bound for the blow up time and decay of solutions of (3) (see [3–7]).

Ye [8] considered the following system

$$\begin{cases} u_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta u + |u_t|^{p-1}u_t = f_1(u,v) \\ v_{tt} - M(\|\nabla u\|^2 + \|\nabla v\|^2)\Delta v + |v_t|^{q-1}v_t = f_2(u,v) \end{cases}$$

with initial-boundary conditions. The author proved the global existence and energy decay results. Primarily, many authors studied the higher-order wave equation ($m > 1$) (see [9–18]).

Motivated by the above paper, in this work, we prove the global existence and energy decay of solutions of the system (1). This work generalises earlier results in the literature which about the higher order wave equation ($m > 1$).

The present work is organised as follows: In the next section, we give some assumptions and lemmas. Section 3 is devoted to proving the global existence and energy decay of solutions.

2. Preliminaries

We use the standard Lebesque space $L^p(\Omega)$ and Sobolev space $H_0^m(\Omega)$. Also we will use the embedding $H_0^m \hookrightarrow L^p(\Omega)$, for $2 \le p \le \frac{2(n-m)}{n-2m}$ $(n > 2m)$ or $2 \le p$ $(n \le 2m)$,

$$\|u\|_p \le C_* \left\| A^{\frac{1}{2}} u \right\|$$

(see [19, 20], for details about Sobolev spaces).

Now, we make the following assumptions:

(A1) $M(s)$ is a non-negative function for $s \ge 0$ satisfying

$$\begin{cases} m_0, \alpha \ge 0, \ \gamma > 0 \\ M(s) = m_0 + \alpha s^\gamma \end{cases} \tag{4}$$

(A2) If g and h are defined in C^1, for $s \ge 0$

$$\begin{cases} g(s) \ge 0, \ m_0 - \int_0^\infty g(s)ds = \ell > 0, g'(s) \le 0 \\ h(s) \ge 0, \ m_0 - \int_0^\infty h(s)ds = k > 0, h'(s) \le 0 \end{cases}$$

concerning the function $f_1(u,v)$ and $f_2(u,v)$ with $a, b > 0, \forall (u,v) \in R^2$,

$$\begin{cases} f_1(u,v) = (r+1)(a|u+v|^{r-1}(u+v) + b|u|^{\frac{r-3}{2}}|v|^{\frac{r+1}{2}}u \\ f_2(u,v) = (r+1)(a|u+v|^{r-1}(u+v) + b|v|^{\frac{r-3}{2}}|u|^{\frac{r+1}{2}}v \end{cases} \tag{5}$$

We can easily verify that

$$uf_1(u,v) + vf_2(u,v) = (r+1)F(u,v)$$

where

$$F(u,v) = a|u+v|^{r+1} + 2b|uv|^{\frac{r+1}{2}} \tag{6}$$

(A3) r satisfies the following requirements:

$$\begin{cases} \text{If } r > 1 \text{ then } n = 1, 2 \\ \text{If } 1 < r \le 3 \text{ then } n = 3 \end{cases} \tag{7}$$

Lemma 1.1 [4]. There exist two positive constants c_0 and c_1 such that

$$C_0(|u|^{r+1}+|v|^{r+1}) \leq F(u,v) \leq C_1(|u|^{r+1}+|v|^{r+1})$$

Lemma 1.2 [4]. Assume that (7) holds. Then there exists $\tau > 0$ such that

$$\|u+v\|_{r+1}^{r+1} + 2\|uv\|_{\frac{r+1}{2}}^{\frac{r+1}{2}} \leq \tau\left(\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2\right)^{\frac{r+1}{2}}$$

Lemma 1.3 [4]. For $g \in C^1$ and $\phi \in H_0^1(0,T)$, we have

$$-2\int_0^t\int_\Omega g(t-s)\phi\phi_t dxds = \frac{d}{dt}((g\diamond\phi)(t) - \int_0^t g(s)ds\,\|\phi\|^2) + g(t)\,\|\phi\|^2 - (g'\diamond\phi)(t)$$

where

$$(g\diamond\phi)(t) = \int_0^t g(t-s)\int_\Omega |\phi(s)-\phi(t)|^2\,dxds$$

Lemma 1.4 [21] (Nakao inequality). Let $\phi(t)$ be nonincreasing and nonnegative function defined on $[0,T]\,,\ T>1$, satisfying

$$\phi^{1+\alpha}(t) \leq w_0\left(\phi(t) - \phi(t+1)\right),\ t\in[0,T]$$

for $w_0 > 0$ and $\alpha \geq 0$. Then we have, for each $t \in [0,T]\,,$

$$\begin{cases} \phi(t) \leq \phi(0)\,e^{-w_1[t-1]^+}, & \alpha = 0 \\ \phi(t) \leq \left(\phi(0)^{-\alpha} + w_0^{-1}\alpha\,[t-1]^+\right)^{-\frac{1}{\alpha}}, & \alpha > 0 \end{cases}$$

where $[t-1]^+ = \max\{t-1,0\}$ and $w_1 = \ln\left(\frac{w_0}{w_0-1}\right)$.

3. Global Existence and Energy Decay

In this part, we state and prove the existence and energy decay of the solution for the problem (1).

We define the following functionals

$$\begin{aligned} I_1(t) \equiv\ & I_1(u(t),v(t)) = (m_0 - \int_0^t g(s)ds)\left\|A^{\frac{1}{2}}u\right\|^2 \\ & + (m_0 - \int_0^t h(s)ds)\left\|A^{\frac{1}{2}}v\right\|^2 + (g\diamond A^{\frac{1}{2}}u)(t) \\ & + (h\diamond A^{\frac{1}{2}}v)(t) - (r+1)\int_\Omega F(u,v)\,dx \end{aligned} \tag{8}$$

$$\begin{aligned} I_2(t) \equiv\ & I_2(u(t),v(t)) = (m_0 - \int_0^t g(s)ds)\left\|A^{\frac{1}{2}}u\right\|^2 \\ & + (m_0 - \int_0^t h(s)ds)\left\|A^{\frac{1}{2}}v\right\|^2 + \alpha\left(\left\|A^{\frac{1}{2}}u\right\|^2 + \left\|A^{\frac{1}{2}}u\right\|^2\right)^{\gamma+1} \\ & + (g\diamond A^{\frac{1}{2}}u)(t) + (h\diamond A^{\frac{1}{2}}v)(t) - (r+1)\int_\Omega F(u,v)\,dx \end{aligned} \tag{9}$$

$$\begin{aligned} J(t) \equiv\ & J(u(t),v(t)) = \frac{1}{2}(m_0 - \int_0^t g(s)ds)\left\|A^{\frac{1}{2}}u\right\|^2 \\ & + \frac{1}{2}(m_0 - \int_0^t h(s)ds)\left\|A^{\frac{1}{2}}v\right\|^2 \\ & + \frac{\alpha}{2(\gamma+1)}\left(\left\|A^{\frac{1}{2}}u\right\|^2 + \left\|A^{\frac{1}{2}}v\right\|^2\right)^{\gamma+1} \\ & + \frac{1}{2}(g\diamond A^{\frac{1}{2}}u)(t) + \frac{1}{2}(h\diamond A^{\frac{1}{2}}v)(t) - \int_\Omega F(u,v)\,dx \end{aligned} \tag{10}$$

and

$$E(t) \equiv E(u(t), v(t)) = \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + J(t) \tag{11}$$

Lemma 2.1. Suppose that (A1), (A2) and (A3) hold. For $\forall t \geq 0$

$$\begin{aligned} E'(t) &= -\|u_t(t)\|_{p+1}^{p+1} - \|v_t(t)\|_{q+1}^{q+1} + \int_0^t \int_\Omega g(t-s) A^{\frac{1}{2}} u(s) A^{\frac{1}{2}} u_t dx ds \\ &\quad + \int_0^t \int_\Omega h(t-s) A v(s)^{\frac{1}{2}} A v_t^{\frac{1}{2}} dx ds \leq 0 \end{aligned} \tag{12}$$

Proof. Multiplying the first equation (1) by u_t and the second equation (1) by v_t, respectively, integrating over Ω, summing up and then using integration by parts, we obtain (12).

Lemma 2.2. Suppose that (A1), (A2) and (A3) hold. Assume further that $I_1(0) > 0$ and

$$\alpha_1 = (r+1)\eta \left(\frac{2(r+1)}{r-1} E(0) \right)^{\frac{m-1}{2}} < 1 \tag{13}$$

then

$$I_1(t) > 0 \tag{14}$$

Proof. Since $I_1(0) > 0$, then by continuity there exists a maximal time $t_{\max} > 0$,(possible $t_{\max} = T$) such that $I_1(0) > 0$, for $t \in [0, t_{\max}]$, which implies that, for $t \in [0, t_{\max}]$

$$\begin{aligned} J(t) &\geq \frac{r-1}{2(r+1)} \left[(m_0 - \int_0^t g(s)ds) \left\| A^{\frac{1}{2}} u \right\|^2 + (m_0 - \int_0^t h(s)ds) \left\| A^{\frac{1}{2}} v \right\|^2 \right] \\ &\quad + \frac{r-1}{2(r+1)} \left(((g \diamond A^{\frac{1}{2}} u)(t) + (h \diamond A^{\frac{1}{2}} v)(t) \right) + \frac{1}{r+1} I_1(t) \\ &\geq \frac{r-1}{2(r+1)} \left[(m_0 - \int_0^t g(s)ds) \left\| A^{\frac{1}{2}} u \right\|^2 + (m_0 - \int_0^t h(s)ds) \left\| A^{\frac{1}{2}} v \right\|^2 \right] \\ &\quad + \frac{r-1}{2(r+1)} \left(((g \diamond A^{\frac{1}{2}} u)(t) + (h \diamond A^{\frac{1}{2}} v)(t) \right) \\ &\geq \frac{r-1}{2(r+1)} \left(\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 \right) \end{aligned} \tag{15}$$

where

$$\begin{cases} \ell = m_0 - \int_0^t g(s)ds \\ k = m_0 - \int_0^t h(s)ds \end{cases}$$

Using (15), (11), and (12), we have

$$\begin{aligned} \ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 &\leq \frac{2(r+1)}{(r-1)} J(t) \\ &\leq \frac{2(r+1)}{(r-1)} E(t) \\ &\leq \frac{2(r+1)}{(r-1)} E(0) \end{aligned} \tag{16}$$

By (4), (16), (13), and from the (A2), we get

$$\begin{aligned} &(r+1) \int_\Omega F(u,v)\, dx \leq (r+1)\eta \left(\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 \right)^{\frac{r+1}{2}} \\ &\leq (r+1)\eta \left(\frac{2(r+1)}{r-1} E(0) \right)^{\frac{r-1}{2}} \left(\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2 \right) \\ &= \alpha_1 (\ell \left\| A^{\frac{1}{2}} u \right\|^2 + k \left\| A^{\frac{1}{2}} v \right\|^2) \\ &< \left[\left(m_0 - \int_0^t g(s)ds \right) \left\| A^{\frac{1}{2}} u \right\|^2 + \left(m_0 - \int_0^t h(s)ds \right) \left\| A^{\frac{1}{2}} v \right\|^2 \right] \end{aligned} \tag{17}$$

Thus,

$$\begin{aligned} I_1(t) &= \left(m_0 - \int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \left(m_0 - \int_0^t h(s)ds\right)\left\|A^{\frac{1}{2}}v\right\|^2 \\ &\quad + (g \diamond A^{\frac{1}{2}}u)(t) + (h \diamond A^{\frac{1}{2}}v)(t) - (r+1)\int_\Omega F(u,v)\, dx \\ &> 0 \end{aligned}$$

By repeating these steps and using the fact that

$$\lim_{t\to t_{\max}} (r+1)\eta\left(\frac{2(r+1)}{r-1}E(t)\right)^{\frac{m-1}{2}} \le \alpha_1 < 1$$

This implies that we can take $t_{\max} = T$.

Lemma 2.3. Under the conditions of Lemma 2.2. Then there exists $0 < \eta_1 < 1$ such that

$$\begin{aligned} (r+1)\int_\Omega F(u,v)\, dx &\le (1-\eta_1)\left[\left(m_0 - \int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 \right. \\ &\quad \left. + \left(m_0 - \int_0^t h(s)ds\right)\left\|A^{\frac{1}{2}}v\right\|^2\right] \end{aligned} \tag{18}$$

where $\eta_1 = 1 - \alpha_1$.

Proof. Thanks to (17), we obtain

$$(r+1)\int_\Omega F(u,v)\, dx \le \alpha_1\left[\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2\right]$$

Let $\alpha_1 = 1 - \eta_1$ and using (A2), we obtain (18).

We are now ready to state and prove our main result.

Teorem 2.1. Assume that (A1), (A2) and (A3) hold. Let $u_0, v_0 \in H_0^m(\Omega) \cap H^{2m}(\Omega)$ and $u_1, v_1 \in H_0^m(\Omega)$ be given which satisfy $I_1(0) > 0$ and (13). Then the solution of problem (1) is global and bounded. Also, if

$$m_0 > \frac{5+2\eta_1}{2\eta_1}\max\left\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\right\} \tag{19}$$

then we have the following decay estimates for $\forall t \ge 0$,

(i) if $p = q = 1$

$$E(t) \le E(0)e^{-\varrho_1 t}$$

(ii) if $\max\{p,q\} > 1$

$$E(t) \le \left[E(0)^{-\max\{\frac{p-1}{2},\frac{q-1}{2}\}} + \varrho_2 \max\{\frac{p-1}{2}, \frac{q-1}{2}\}[t-1]^+\right]^{-\frac{2}{\max\{p,q\}-1}}$$

where $\varrho_1 = \varrho_1(m_0, \alpha, \gamma)$ and $\varrho_2 = \varrho_2(m_0, \alpha, \gamma, E(0))$ are positive constants.

Proof. (Global existence) Firstly, we prove $T = \infty$, it is sufficient to show that

$$\|u_t\|^2 + \|v_t\|^2 + \ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2$$

is bounded independently of t. We use (11) and (15), we obtain

$$\begin{aligned} E(0) &\ge E(t) = \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + J(t) \\ &\ge \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{r-1}{2(r+1)}\left(\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2\right) \end{aligned}$$

Therefore

$$\|u_t\|^2 + \|v_t\|^2 + \ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2 \le \alpha_2 E(0)$$

where $\alpha_2 = \left\{2, \frac{2(r+1)}{r-1}\right\}$. Therefore, we have the global existence result.

(Energy decay) We will derive the energy decay of the problem (1), by the Lemma 2.1, we get

$$\begin{aligned} \frac{d}{dt}E(t) &= -\|u_t(t)\|_{p+1}^{p+1} + \frac{1}{2}(g' \diamond A^{\frac{1}{2}}u)(t) - \frac{1}{2}g(t)\left\|A^{\frac{1}{2}}u\right\|^2 \\ &\quad -\|v_t(t)\|_{q+1}^{q+1} + \frac{1}{2}(h' \diamond A^{\frac{1}{2}}u)(t) - \frac{1}{2}h(t)\left\|A^{\frac{1}{2}}v\right\|^2 \\ &< 0 \end{aligned}$$

By integrating over $[t,\ t+1]$, we obtain

$$\begin{aligned} E(t) - E(t+1) &= \int_t^{t+1} \|u_t(t)\|_{p+1}^{p+1} ds - \frac{1}{2}\int_t^{t+1} (g' \diamond A^{\frac{1}{2}}u)(s)ds \\ &\quad + \frac{1}{2}\int_t^{t+1} g(s)\left\|A^{\frac{1}{2}}u\right\|^2 ds + \int_t^{t+1} \|v_t(t)\|_{q+1}^{q+1} ds \\ &\quad - \frac{1}{2}\int_t^{t+1} (h' \diamond A^{\frac{1}{2}}v)(s)ds + \frac{1}{2}\int_t^{t+1} h(s)\left\|A^{\frac{1}{2}}v\right\|^2 ds \\ &= D_1^{p+1}(t) + D_2^{q+1}(t) \end{aligned} \tag{20}$$

where

$$\begin{cases} D_1^{p+1}(t) = \int_t^{t+1} \|u_t(t)\|_{p+1}^{p+1} ds - \frac{1}{2}\int_t^{t+1} (g' \diamond A^{\frac{1}{2}}u)(s)ds + \frac{1}{2}\int_t^{t+1} g(s)\left\|A^{\frac{1}{2}}u\right\|^2 ds \\ D_2^{q+1}(t) = \int_t^{t+1} \|v_t(t)\|_{q+1}^{q+1} ds - \frac{1}{2}\int_t^{t+1} (h' \diamond A^{\frac{1}{2}}v)(s)ds + \frac{1}{2}\int_t^{t+1} h(s)\left\|A^{\frac{1}{2}}v\right\|^2 ds \end{cases} \tag{21}$$

By virtue of (21) and Hölder inequality, we observe that

$$\int_t^{t+1}\int_\Omega |u_t|^2 dxdt + \int_t^{t+1}\int_\Omega |v_t|^2 dxdt \le c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \tag{22}$$

where $c_1(\Omega) = vol(\Omega)^{\frac{p-1}{p+1}}$ and $c_2(\Omega) = vol(\Omega)^{\frac{q-1}{q+1}}$. By the mean value theorem, there exist $t_1 \in \left[t,\ t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4},\ t+1\right]$ such that

$$\|u_t(t_i)\|^2 + \|v_t(t_i)\|^2 \le 4c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \tag{23}$$

Now, multiplying the first equation (1) by u and the second equation (1) by v,respectively, and integrating over $\Omega \times [t_1, t_2]$,using integration by parts, Hölder inequality and adding them together, we have

$$\begin{aligned} \int_{t_1}^{t_2} I_2(t) &\le \sum_{i=1}^{2}\|u_t(t_i)\|\,\|u(t_i)\| + \sum_{i=1}^{2}\|v_t(t_i)\|\,\|v(t_i)\| + \int_{t_1}^{t_2}(\|u_t\|^2 + \|v_t\|^2)dt \\ &\quad - \int_{t_1}^{t_2}\int_\Omega (|u_t|^{p-1}u_t u + |v_t|^{q-1}v_t v)dxdt \\ &\quad + \int_{t_1}^{t_2}(g \diamond A^{\frac{1}{2}}u)(t) + (h \diamond A^{\frac{1}{2}}v)(t)dt \\ &\quad + \int_{t_1}^{t_2}\int_\Omega\int_0^t g(t-s)A^{\frac{1}{2}}u)(t)[A^{\frac{1}{2}}u(s) - A^{\frac{1}{2}}u(t)]dsdxdt \\ &\quad + \int_{t_1}^{t_2}\int_\Omega\int_0^t h(t-s)A^{\frac{1}{2}}v)(t)[A^{\frac{1}{2}}v(s) - A^{\frac{1}{2}}v(t)]dsdxdt \end{aligned} \tag{24}$$

Since

$$\begin{aligned}\int_\Omega\int_0^t g(t-s)A^{\frac{1}{2}}u(t)[A^{\frac{1}{2}}u(s)-A^{\frac{1}{2}}u(t)]dsdx &= \frac{1}{2}\int_0^t g(t-s)\left(\left\|A^{\frac{1}{2}}u(t)\right\|^2+\left\|A^{\frac{1}{2}}u(s)\right\|^2\right)ds\\ &-\frac{1}{2}\int_0^t g(t-s)\left(\left\|A^{\frac{1}{2}}u(t)-A^{\frac{1}{2}}u(s)\right\|^2\right)ds\\ &-\int_\Omega\int_0^t g(s)\left|A^{\frac{1}{2}}u(t)\right|^2 dsdx\\ &= -\frac{1}{2}\int_\Omega\int_0^t g(s)\left|A^{\frac{1}{2}}u(t)\right|^2 dsdx\\ &+\frac{1}{2}\int_0^t g(t-s)(\left\|A^{\frac{1}{2}}u(s)\right\|^2)ds\\ &-\frac{1}{2}(g\diamond A^{\frac{1}{2}}u)(t)\end{aligned}$$

and

$$\begin{aligned}\int_\Omega\int_0^t h(t-s)A^{\frac{1}{2}}v)(t)[A^{\frac{1}{2}}v(s)-A^{\frac{1}{2}}v(t)]dsdx &= -\frac{1}{2}\int_\Omega\int_0^t h(s)\left|A^{\frac{1}{2}}v(s)\right|^2 dsdx\\ &+\frac{1}{2}\int_0^t h(t-s)(\left\|A^{\frac{1}{2}}v(s)\right\|^2)ds\\ &-\frac{1}{2}(h\diamond A^{\frac{1}{2}}v)(t)\end{aligned}$$

hence (24) takes the form

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t) &\leq \sum_{i=1}^{2}\|u_t(t_i)\|\,\|u(t_i)\|+\sum_{i=1}^{2}\|v_t(t_i)\|\,\|v(t_i)\|+\int_{t_1}^{t_2}(\|u_t\|^2+\|v_t\|^2)dt\\ &-\int_{t_1}^{t_2}\int_\Omega(|u_t|^{p-1}u_tu+|v_t|^{q-1}v_tv)dxdt\\ &+\frac{1}{2}\int_{t_1}^{t_2}(g\diamond A^{\frac{1}{2}}u)(t)+(h\diamond A^{\frac{1}{2}}v)(t)dt\\ &+\frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|A^{\frac{1}{2}}u(t)\right\|^2 dsdt\\ &+\frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)\left\|A^{\frac{1}{2}}v(t)\right\|^2 dsdt. \qquad (25)\end{aligned}$$

Let's estimate for the first two terms on the right side of the equation (25). By Young inequality, (23) and (16)

$$\begin{aligned}\|u_t(t_i)\|\,\|u(t_i)\| &\leq C_*\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}\sup_{t_1\leq s\leq t_2}\left\|A^{\frac{1}{2}}u(s)\right\|\\ &\leq C_*\left(\frac{2(r+1)}{\ell(r-1)}\right)^{\frac{1}{2}}\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\\ &\leq C_*\left(\frac{2(r+1)}{\beta(r-1)}\right)^{\frac{1}{2}}\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}E(t)^{\frac{1}{2}} \qquad (26)\end{aligned}$$

and

$$\|v_t(t_i)\|\,\|v(t_i)\|\leq C_*\left(\frac{2(r+1)}{\beta(r-1)}\right)^{\frac{1}{2}}\sqrt{4c_1D_1(t)^2+4c_2D_2(t)^2}E(t)^{\frac{1}{2}} \qquad (27)$$

where $\beta = \min\{\ell, k\}$. Also from the Hölder inequality (16)

$$\begin{aligned}\left|\int_{t_1}^{t_2}\int_{\Omega}(|u_t|^{p-1}u_t u dx dt\right| &\leq \int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^p\|u\|_{p+1}dt \\ &\leq C_*\int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^p\left\|A^{\frac{1}{2}}u\right\|dt \\ &\leq C_*\left(\frac{2(r+1)}{\ell(r-1)}\right)^{\frac{1}{2}}\sup_{t_1\leq s\leq t_2}E(s)^{\frac{1}{2}}\int_{t_1}^{t_2}\|u_t(t)\|_{p+1}^p dt \\ &\leq C_*\left(\frac{2(r+1)}{\ell(r-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D_1(t)^p \end{aligned} \tag{28}$$

and similarly

$$\left|\int_{t_1}^{t_2}\int_{\Omega}(|v_t|^{q-1}v_t v dx dt\right| \leq C_*\left(\frac{2(r+1)}{\beta(r-1)}\right)^{\frac{1}{2}}E(t)^{\frac{1}{2}}D_2(t)^q \tag{29}$$

Employing Young's inequality for convolution ($\|\phi*\psi\|_q \leq \|\phi\|_r\|\psi\|_s$ with $\frac{1}{q}=\frac{1}{r}+\frac{1}{s}-1, 1\leq q,r,s$), (25) the last two terms of inequality

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|A^{\frac{1}{2}}u(s)\right\|^2 dsdt &\leq \int_{t_1}^{t_2}g(t)dt\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}u(t)\right\|^2 dt \\ &\leq (m_0-\ell)\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}u(t)\right\|^2 dt \\ &\leq (m_0-\beta)\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}u(t)\right\|^2 dt \end{aligned} \tag{30}$$

and

$$\begin{aligned}\int_{t_1}^{t_2}\int_0^t h(t-s)\left\|A^{\frac{1}{2}}v(t)\right\|^2 dsdt &\leq \int_{t_1}^{t_2}h(t)dt\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}v(t)\right\|^2 dt \\ &\leq (m_0-\beta)\int_{t_1}^{t_2}\left\|A^{\frac{1}{2}}v(t)\right\|^2 dt \end{aligned} \tag{31}$$

Adding (29) and (30) together and nothing that, we see

$$\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2 \leq \frac{1}{\eta_1}I_2(t) \tag{32}$$

From (9) and the definition of $I_2(t)$ and also by (18), we have

$$\begin{aligned}&\frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)(\left\|A^{\frac{1}{2}}u(s)\right\|^2 dsdt + \frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)(\left\|A^{\frac{1}{2}}v(t)\right\|^2 dsdt \\ \leq\ &\frac{m_0-\beta}{2\beta}\int_{t_1}^{t_2}(\ell\left\|A^{\frac{1}{2}}u\right\|^2 + k\left\|A^{\frac{1}{2}}v\right\|^2)dt \leq \frac{m_0-\beta}{2\beta\eta_1}\int_{t_1}^{t_2}I_2(t)dt \end{aligned} \tag{33}$$

We use (30)-(32) to estimate the last two terms on the right-hand side of (25), we get

$$\begin{aligned}\frac{1}{2}\int_{t_1}^{t_2}(g\diamond A^{\frac{1}{2}}u)(t)+(h\diamond A^{\frac{1}{2}}u)(t)dt &= \frac{1}{2}\int_{t_1}^{t_2}\int_0^t g(t-s)\left\|A^{\frac{1}{2}}u(s)-A^{\frac{1}{2}}u(t)\right\|^2 dsdt \\ &\quad +\frac{1}{2}\int_{t_1}^{t_2}\int_0^t h(t-s)\left\|A^{\frac{1}{2}}v(t)-A^{\frac{1}{2}}v(t)\right\|^2 dsdt \\ &\leq \int_{t_1}^{t_2}\int_0^t g(t-s)(\left\|A^{\frac{1}{2}}u(t)\right\|^2+\left\|A^{\frac{1}{2}}u(t)\right\|^2)dsdt \\ &\quad +\int_{t_1}^{t_2}\int_0^t h(t-s)(\left\|A^{\frac{1}{2}}v(t)\right\|^2+\left\|A^{\frac{1}{2}}v(t)\right\|^2)dsdt \\ &\leq \frac{2(m_0-\beta)}{\beta}\int_{t_1}^{t_2}(\ell\left\|A^{\frac{1}{2}}u\right\|^2+k\left\|A^{\frac{1}{2}}v\right\|^2)dt \\ &\leq \frac{2(m_0-\beta)}{\beta}\int_{t_1}^{t_2}I_2(t)dt. \end{aligned} \tag{34}$$

By (25) and the above inequalities

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t)dt &\leq c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \\ &+4c_3\sqrt{4c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+c_3E(t)^{\frac{1}{2}}(D_1(t)^p + D_2(t)^q) + c_4\int_{t_1}^{t_2} I_2(t)dt \end{aligned} \quad (35)$$

where $c_3 = C_*(\frac{2(r+1)}{\beta(r-1)})^{\frac{1}{2}}$ and $c_4 = \frac{5(m_0-\beta)}{2\beta\eta_1}$. Then, rewriting (35)

$$\begin{aligned}\beta_2\int_{t_1}^{t_2} I_2(t)dt &\leq c_1(\Omega)D_1(t)^2 + c_2(\Omega)D_2(t)^2 \\ &+4c_3\sqrt{4c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+c_3E(t)^{\frac{1}{2}}(D_1(t)^p + D_2(t)^q)\end{aligned}$$

where $\beta_2 = 1 - \frac{5(m_0-\beta)}{2\beta\eta_1}$ and $m_0 > \frac{5+2\eta_1}{2\eta_1}.\max\left\{\int_0^\infty g(s)ds, \int_0^\infty h(s)ds\right\}$. So $\beta_2 > 0$, thus

$$\begin{aligned}\int_{t_1}^{t_2} I_2(t)dt &\leq c_5[\sqrt{4c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+D_1(t)^2 + D_2(t)^2 + E(t)^{\frac{1}{2}}(D_1(t)^p + D_2(t)^q)]\end{aligned} \quad (36)$$

where $c_5 = \frac{\max\{c_1(\Omega), c_2(\Omega), 4c_3\}}{\beta_2}$. On the other hand, by $E(t)$ function in the definition of the equation (11), (8) and (9), we obtain

$$I_2(t) = I_1(t) + \alpha\left\|A^{\frac{1}{2}}u\right\|^2 + \left\|A^{\frac{1}{2}}v\right\|^2)^{\gamma+1}$$

$$\begin{aligned}E(t) &= \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{r-1}{2(r+1)}\left[\left(m_0 - \int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 + (m_0 - \int_0^t h(s)ds\left\|A^{\frac{1}{2}}v\right\|^2)\right] \\ &+\frac{r-1}{2(r+1)}(g\diamond A^{\frac{1}{2}}u)(t) + (h\diamond A^{\frac{1}{2}}u)(t) + \frac{\alpha}{2(\gamma+1)}(\left\|A^{\frac{1}{2}}u\right\|^2 + \left\|A^{\frac{1}{2}}v\right\|^2)^{\gamma+1} + \frac{1}{r+1}I_1(t) \\ &\leq \frac{1}{2}(\|u_t\|^2 + \|v_t\|^2) + \frac{r-1}{2(r+1)}\left[\left(m_0 - \int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 + (m_0 - \int_0^t h(s)ds\left\|Av^{\frac{1}{2}}\right\|^2)\right] \\ &+\frac{r-1}{2(r+1)}\left((g\diamond A^{\frac{1}{2}}u)(t) + (h\diamond A^{\frac{1}{2}}u)(t)\right) + \left(\frac{1}{r+1} + \frac{1}{2(\gamma+1)}\right)I_2(t)\end{aligned}$$

The (37) is integrated over (t_1, t_2) and then using (22), (32), (34), (36), we obtain

$$\begin{aligned}\int_{t_1}^{t_2} E(t)dt &\leq \frac{1}{2}\int_{t_1}^{t_2}(\|u_t\|^2 + \|v_t\|^2)dt + \frac{r-1}{2(r+1)}\int_{t_1}^{t_2}\left(m_0 - \int_0^t g(s)ds\right)\left\|A^{\frac{1}{2}}u\right\|^2 dt \\ &+\frac{r-1}{2(r+1)}\int_{t_1}^{t_2}\left(m_0 - \int_0^t h(s)ds\right)\left\|A^{\frac{1}{2}}v\right\|^2 dt \\ &+\frac{r-1}{2(r+1)}\int_{t_1}^{t_2}\left((g\diamond A^{\frac{1}{2}}u)(t) + (h\diamond A^{\frac{1}{2}}u)(t)\right)dt \\ &+\left(\frac{1}{r+1} + \frac{1}{2(\gamma+1)}\right)\int_{t_1}^{t} I_2(t)dt \\ &\leq c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2 + c_6\int_{t_1}^{t} I_2(t)dt \\ &\leq c_7[\sqrt{4c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} \\ &+D_1(t)^2 + D_2(t)^2 + E(t)^{\frac{1}{2}}(D_1(t)^p + D_2(t)^q)]\end{aligned} \quad (37)$$

where $c_6 = \frac{1}{r+1} + \frac{1}{2(\gamma+1)}\frac{r-1}{2(r+1)\eta_1} + \frac{2(r-1)(m_0-\beta)}{(r+1)\beta\eta_1}$ and $c_7 = \max\{c_1(\Omega), c_2(\Omega), c_6c_5\}$. Moreover, integrating (12) over (t_1, t_2), we obtain

$$E(t_2) \leq 2\int_{t_1}^{t_2} E(t)dt,$$

due to $t_2 - t_1 \geq \frac{1}{2}$, we get

$$\begin{aligned} E(t) &= E(t_2) + \int_t^{t_2} \|u_t\|_{p+1}^{p+1} ds - \frac{1}{2}\int_t^{t_2} (g' \diamond A^{\frac{1}{2}}u)(s)ds \\ &+ \frac{1}{2}\int_t^{t_2} g(s)\left\|A^{\frac{1}{2}}u\right\|^2 ds + \int_t^{t_2} \|v_t\|_{q+1}^{q+1} ds \\ &- \frac{1}{2}\int_t^{t_2} (h' \diamond A^{\frac{1}{2}}v)(s)ds + \frac{1}{2}\int_t^{t+1} h(s)\left\|A^{\frac{1}{2}}v\right\|^2 ds \\ &\leq 2\int_t^{t_2} E(t)dt + D_1(t)^{p+1} + D_2(t)^{q+1} \end{aligned} \tag{38}$$

As a result, by (37) and (38), we obtain

$$\begin{aligned} E(t) &\leq c_8\sqrt{4c_1(\Omega)D_1(t)^2 + 4c_2(\Omega)D_2(t)^2}E(t)^{\frac{1}{2}} + D_1(t)^2 + D_2(t)^2 \\ &+ E(t)^{\frac{1}{2}}D_1(t)^p + E(t)^{\frac{1}{2}}D_2(t)^q + D_1(t)^{p+1} + D_2(t)^{q+1} \end{aligned}$$

Hence, by Young inequality, we have

$$E(t) \leq c_9\left[D_1(t)^2 + D_2(t)^2 + D_1(t)^{2p} + D_2(t)^{2q} + D_1(t)^{p+1} + D_1(t)^{q+1}\right] \tag{39}$$

where c_8 and c_9 are positive constants.

(i) if $p = q = 1$. By (20) and (39), we have

$$E(t) \leq c_{10}\left[E(t) - E(t+1)\ \right]$$

where $c_{10} > 1$. Using Nakao's inequality, we get

$$E(t) \leq E(0)e^{-\varrho_1 t}$$

where $\varrho_1 = \ln(\frac{w_0}{w_0 - 1})$.

(ii) if $\max\{p, q\} > 1$. From (39), we get

$$E(t) \leq c_9\left[D_1(t)^2(1 + D_1(t)^{2p-2} + D_1(t)^{p-1}) + D_2(t)^2(1 + D_2(t)^{2q-2} + D_2(t)^{q-1})\right]$$

Then since

$$\begin{cases} D_1(t) \leq E(t)^{\frac{1}{p+1}} \leq E(0)^{\frac{1}{p+1}} \\ D_2(t) \leq E(t)^{\frac{1}{q+1}} \leq E(0)^{\frac{1}{q+1}} \end{cases}$$

we see from (20)

$$\begin{aligned} E(t) &\leq c_9\left[D_1(t)^2\left(1 + E(0)^{\frac{P-1}{P+1}} + E(0)^{\frac{2p-2}{p+1}}\right) + D_2(t)^2\left(1 + E(0)^{\frac{q-1}{q+1}} + E(0)^{\frac{2q-2}{q+1}}\right)\right] \\ &\leq c_9\left(D_1(t)^2 + D_2(t)^2\right)(1 + E(0)^{\frac{p-1}{p+1}} + E(0)^{\frac{2p-2}{p+1}} + E(0)^{\frac{q-1}{q+1}} + E(0)^{\frac{2q-2}{q+1}} \\ &= c_{10}E(0)\left(D_1(t)^2 + D_2(t)^2\right) \end{aligned}$$

where $\lim_{E(0)\to 0} c_{10}(E(0)) = c_9$ and $\rho = \max\left\{\frac{p-1}{2}, \frac{q-1}{2}\right\}$. Then, we get

$$\begin{aligned} E(t)^{1+\rho} &\leq \left[c_{10}\left(D_1(t)^2 + D_2(t)^2\right)\right]^{1+\rho} \\ &\leq c_{11}E(0)\left(D_1(t)^{2\rho+2} + D_2(t)^{2\rho+2}\right) \\ &= c_{11}E(0)\left(D_1(t)^{p+1}D_1(t)^{2\rho+2-p-1} + D_2(t)^{q+1}D_2(t)^{2\rho+2-q-1}\right) \\ &= c_{11}E(0)\left(D_1(t)^{p+1}D_1(t)^{2\rho-p+1} + D_2(t)^{q+1}D_2(t)^{2\rho-q+1}\right) \\ &\leq c_{11}E(0)\left(D_1(t)^{p+1}E(0)^{\frac{2\rho-p+1}{P+1}} + D_2(t)^{q+1}E(0)^{\frac{2\rho-q+1}{q+1}}\right) \\ &\leq c_{12}E(0)\left(D_1(t)^{p+1} + D_2(t)^{q+1}\right) \\ &\leq c_{12}E(0)\left(E(t) - E(t+1)\right) \end{aligned} \tag{40}$$

where

$$c_{11}(E(0)) = 2^{\rho}\left(c_{10}(E(0))\right)^{1+\rho}$$

and

$$c_{12}(E(0)) = c_{11}(E(0)) \max\left\{E(0)^{\frac{2\rho-p+1}{P+1}}, E(0)^{\frac{2\rho-q+1}{q+1}}\right\}$$

Thus, from (40) and Nakao inequality, we get

$$E(t) \leq \left(E(0)^{-\rho} + \varrho_2 \rho\,[t-1]^{+}\right)^{-\frac{1}{\rho}}$$

where $\varrho_2 = c_{12}^{-1}(E(0))$. Thus, the proof of theorem is completed.

4. Conclusion

In this work, we obtained the existence of global solutions and energy decay for a system of higher-order Kirchhoff type equations. This improves and extends many results in the literature.

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