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PITCHFORK DOMINATION AND IT'S INVERSE FOR CORONA AND JOIN OPERATIONS IN GRAPHS

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Abstract. Let G be a finite simple and undirected graph without isolated vertices. A subset D of V is a pitchfork dominating set if every vertex $v \in D$ dominates at least j and at most k vertices of $V - D$, where j and k are non-negative integers .The domination number of G, denoted by $\gamma_{pf}(G)$ is a minimum cardinality over all pitchfork dominating sets in G. A subset D^{-1} of $V - D$ is an inverse pitchfork dominating set if D^{-1} is a pitchfork dominating set. The inverse domination number of G, denoted by $\gamma_{pf}^{-1}(G)$ is a minimum cardinality over all inverse pitchfork dominating sets in G. In this paper, the pitchfork domination and the inverse pitchfork domination are determined when $j = 1$ and $k = 2$ for some graphs that obtained from graph operations corona and join.

1. Introduction

Let $G = (V, E)$ be a graph without isolated vertices with vertex set V of order n and edge set E of size m. The complement $\bar{G}$ of a simple graph G with vertex set $V(G)$ is the graph in which two vertices are adjacent if and only if they are not adjacent in G. The join $G_1 + G_2$ between two graphs G_1 and G_2 is a graph contains all edges and vertices of both graphs and every vertex of G_1 joined by edges with all vertices of G_2. The corona $G_1 \odot G_2$ between two graphs G_1 and G_2 is a graph has one copy of G_1 and $|V(G_1)|$ copies of G_2 such that the i^{th} vertex of G_1 joined by edges with all vertices of the i^{th} copy of G_2. For graph theoretic terminology we refer to [6] and [10]. For a detailed survey of domination one can see [7] and [8]. A set $D \subseteq V$ is a dominating set if every vertex in $V - D$ is adjacent to a vertex in D. If no proper subset of D is a dominating set then D is said to be minimal . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set D of G. There are many papers deals with different types of domination, such as [3, 4, 5, 9].

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Here, a new model of domination in graphs called the pitchfork domination and it's inverse, which were studied in [1, 2], are applied on some graphs formed by using two types of operations.

Theorem 1.1. [2] *The cycle graph C_n with $n \geq 3$ has an inverse pitchfork domination such that:* $\gamma_{pf}^{-1}(C_n) = \gamma_{pf}(C_n) = \lceil \frac{n}{3} \rceil$.

Proposition 1.2. [1] *Let $G = K_n$ the complete graph with $n \geq 3$, then* $\gamma_{pf}(K_n) = n - 2$.

Proposition 1.3. [2] *The complete graph K_n has an inverse pitchfork domination if and only if $n = 3, 4$ and* $\gamma_{pf}^{-1}(K_n) = n - 2$.

Theorem 1.4. [1] *Let G be the complete bipartite graph, then:*

$$\gamma_{pf}(K_{n,m}) = \begin{cases} m, & if\, n = 2 \wedge m < 3 \quad or \quad n = 1 \wedge m > 2 \\ m - 1, & if\, n = 2, m \geq 3 \\ n + m - 4, & if\, n, m > 2. \end{cases}$$

Theorem 1.5. [2] *The complete bipartite graph $K_{n,m}$ has an inverse pitchfork domination if and only if $K_{n,m} \equiv K_{1,2}, K_{2,2}, K_{2,3}, K_{2,4}, K_{3,3}, K_{3,4}$ or $K_{4,4}$ such that:*

$$\gamma_{pf}^{-1}(K_{n,m}) = \begin{cases} 2 & for\ K_{1,2} \\ n + m - 4 & if\, n,\, m\, =\, 2,\, 3,\, 4 \end{cases}$$

Proposition 1.6. [1] *For any graph G having a pitchfork domination set, if G has a support vertex, that is adjacent to more than two pendents then all it's pendents belong to the pitchfork dominating set.*

Note 1.7. [2] *If $\gamma_{pf}(G) > \frac{n}{2}$ then G has no inverse pitchfork domination.*

Proposition 1.8. [2] *Let G be a graph which has a support vertex adjacent to more than two pendent vertices, then G has no inverse pitchfork domination.*

2. The Main Results

The pitchfork domination and the inverse pitchfork domination are studied here for some graphs constructed by corona or join operations.

Theorem 2.1. *If G is a graph of order n, then:*
1- $\gamma_{pf}(G \odot K_2) = \gamma_{pf}(\overline{G} \odot K_2) = \gamma_{pf}(G \odot \overline{K}_2) = \gamma_{pf}(\overline{G} \odot \overline{K}_2) = n$.
2- $\gamma_{pf}(G + K_2) = \gamma_{pf}(\overline{G} + K_2) = \gamma_{pf}(G + \overline{K}_2) = \gamma_{pf}(\overline{G} + \overline{K}_2) = n$.
3- $\gamma_{pf}(G \odot \overline{K}_1) = \gamma_{pf}(\overline{G} \odot \overline{K}_1) = n$.

Proof. Let $D \subseteq V$. 1 and 2: Since every $v \in G$ is adjacent to two vertices of K_2 or $\overline{K}_2$, then $v \in D$. Therefore, every $v \in D$ dominates exactly two vertices. Thus, D is $\gamma_{pf}-$set and $D = V(G)$ with order n . Others cases are proved by the same way. 3: Since every support vertex or it's leaf belongs to D, then $D = V(G)$ is a $\gamma_{pf}-$set. □

Theorem 2.2. *If G is a graph of order n, then:*
1- $\gamma_{pf}^{-1}(G \odot K_2) = \gamma_{pf}^{-1}(\overline{G} \odot K_2) = n$.
2- $\gamma_{pf}^{-1}(G \odot \overline{K}_2) = \gamma_{pf}^{-1}(\overline{G} \odot \overline{K}_2) = 2n$.
3- $\gamma_{pf}^{-1}(G \odot \overline{K}_1) = \gamma_{pf}^{-1}(\overline{G} \odot \overline{K}_1) = n$.

Proof. Let $D \subseteq V$. 1- There are n cycles of order three and $\gamma_{pf}^{-1}(C_3) = 1$ according to Theorem 1.1. The result is obtained.
2- Every vertex of G or $\overline{G}$ is a support vertex and is adjacent to two (non-adjacent) vertices of $\overline{K_2}$. So that, D contains all vertices of G or $\overline{G}$ according to Theorem 2.1 part 1. Therefore, $D^{-1} = V - D$ which has all vertices of the copies of $\overline{K}_2$. Hence, $\gamma_{pf}^{-1} = 2n$.
3- Similar to proof in Theorem 2.1 case 3. □

Theorem 2.3. *$G + K_2$, $\overline{G} + K_2$, $G + \overline{K}_2$ and $\overline{G} + \overline{K}_2$, have an inverse pitchfork domination if and only if $n \leq 2$ such that:*
1- $\gamma_{pf}^{-1}(G + K_2) = \gamma_{pf}^{-1}(\overline{G} + K_2) = n$.
2- $\gamma_{pf}^{-1}(G + \overline{K}_2) = \gamma_{pf}^{-1}(\overline{G} + \overline{K}_2) = 2$.

Proof. 1- If $n = 1$ then $G + K_2 = \overline{G} + K_2 = C_3$ which has $\gamma_{pf}^{-1}(C_3) = 1$ by Theorem 1.1. If $n = 2$ then $D = V(G)$ by Theorem 2.1. So, $D^{-1} = V(K_2)$ which is a $\gamma_{pf}^{-1}-$set of order 2.
2- Since every $v \in G$ or $\overline{G}$ is adjacent to two vertices of $\overline{K}_2$ and $v \in D$ from Theorem 2.1, then we have $D^{-1} = V(\overline{K}_2)$. Hence, D^{-1} dominates all vertices of the graph and it is an inverse pitchfork dominating set. Every $w \in D^{-1}$ dominates exactly two vertices of G or $\overline{G}$. Therefore, D^{-1} is a $\gamma_{pf}^{-1}-$set of order 2. Now, If $n \geq 3$ then the graph has no inverse pitchfork domination by Note 1.7 since $\gamma_{pf} > \frac{n+2}{2}$. □

Theorem 2.4. *For K_m with $m \geq 3$ and G of order n, we have:*
1- $\gamma_{pf}(G \odot K_m) = \gamma_{pf}(\overline{G} \odot K_m) = n(m-1)$.
2- $\gamma_{pf}(G \odot \overline{K}_m) = \gamma_{pf}(\overline{G} \odot \overline{K}_m) = nm$.

Proof. 1- $\gamma_{pf}(K_m) = m - 2$ by Proposition 1.2 then there are two vertices in every copy of K_m which are not in D. But all the vertices from every copy of K_m which are adjacent to one vertex of G. Then we must add to D one vertex from every copy of K_m. Hence, D is a pitchfork dominating set that contains $m - 1$ vertices from every copy of K_m. Since, every vertex of D dominates exactly two vertices, therefore D is a $\gamma_{pf}-$set with order $n(m-1)$.
2- Since every vertex of G becomes a support vertex and it is adjacent to $m \geq 3$ leaves of $\overline{K}_m$, then by Proposition 1.6, D consists of only the end vertices which are all vertices of n copies of K_m. Hence, $\gamma_{pf}(G \odot \overline{K}_m) = nm$. □

Theorem 2.5. *For K_m with $m \geq 3$ and G of order n, then:*
1. $G \odot K_m$ and $\overline{G} \odot K_m$ has an inverse pitchfork domination if and only if $m = 3$, 4 such that $\gamma_{pf}^{-1}(G \odot K_m) = \gamma_{pf}^{-1}(\overline{G} \odot K_m) = n(m-1)$.
2. $G \odot \overline{K}_m$ and $\overline{G} \odot \overline{K}_m$ has no inverse pitchfork domination.

Proof. 1-K_{m+1} has an inverse pitchfork domination if and only if $m + 1 = 3, 4$ where $\gamma_{pf}^{-1}(K_{m+1}) = m-1$ according to Proposition 1.3. Therefore, $\gamma_{pf}^{-1}(G \odot K_m) = \gamma_{pf}^{-1}(\overline{G} \odot K_m) = n(m-1)$.
2- Since D contains all vertices of the copies of $\overline{K}_m$ by Theorem 2.4. And for all $v \in G$ or $\overline{G}$, then v is a support vertex that joins with more than two pendents. Then $v \notin D^{-1}$ and there is no $\gamma_{pf}^{-1}-$set according to Proposition 1.8. □

Theorem 2.6. *For any graph G of order n and complete graph K_m with $m \geq 3$, we have:*

1. $\gamma_{pf}(G+K_m)=\gamma_{pf}(\overline{G}+K_m)=|V(G)|+\gamma_{pf}(K_m)=n+m-2.$
2. $G+K_m$ *and* $\overline{G}+K_m$ *has an inverse pitchfork domination if and only if* $n=1$ *and* $m=3$ *such that* $\gamma_{pf}^{-1}(G+K_m)=\gamma_{pf}^{-1}(\overline{G}+K_m)=2.$

Proof. 1- Since $\gamma_{pf}(K_m)=m-2$ by Proposition 1.2 and since every vertex in G is adjacent with all vertices of K_m, then all vertices of G must belong to the dominating set D. Hence $\gamma_{pf}(G+K_m)=n+m-2$.
2- It is clear from Proposition1.3. □

Observation 2.7. *Let G be a disconnected graph with $H_1,H_2,\cdots,H_n$ components, then:*
1- $\gamma_{pf}(G)=\sum_{i=1}^{n}\gamma_{pf}(H_i)$.
2- $\gamma_{pf}^{-1}(G)=\sum_{i=1}^{n}\gamma_{pf}^{-1}(H_i)$.

Theorem 2.8. *For a connected graph G_1 of order $n\geq 2$ and a null graph G_2 of order $m\geq 2$, we have:*

$$n+m-3\leq\gamma_{pf}(G_1+G_2)\leq n+m-2$$

Proof. Let $v_1,v_2\in V-D$ where $v_1\in G_1$ and $v_2\in G_2$. Then any vertex of G_1 which is adjacent to v_1 will dominates v_1 and v_2. Any vertex of G_1 which is not adjacent to v_1 will dominates only v_2. While all vertices of G_2 unless v_2 will dominates only v_1. Therefore, $V-D$ can not take another vertex of G_2. But $V-D$ can contain another vertex from G_1 say u (by condition: G_1 is not a complete graph and there is no vertex in G_1 adjacent with both v_1 and u). Hence $\gamma_{pf}(G_1+G_2)=n+m-3$ when the condition hold. But if the condition doesn't hold, then $\gamma_{pf}(G_1+G_2)=n+m-2$. Therefore, in general $n+m-3\leq\gamma_{pf}(G_1+G_2)\leq n+m-2$. □

Theorem 2.9. *For any two connected graphs G_1 of order $n\geq 2$ with $\gamma_{pf}(G_1)$ and G_2 of order $m\geq 2$ with $\gamma_{pf}(G_2)$, then:*
1. $\gamma_{pf}(G_1+G_2)\geq\gamma_{pf}(G_1)+\gamma_{pf}(G_2)$ *and* $\gamma_{pf}(G_1+G_2)=n+m-2.$
2. G_1+G_2 *has an inverse pitchfork domination if and only if* $n=m=2$ *such that* $\gamma_{pf}^{-1}(G_1+G_2)=n+m-2.$

Proof. 1- Let $V-D$ consists of two vertices one vertex from G_1 (say v_1) and one from G_2 (say v_2). Since G_1 is a connected graph then for any vertex $u_1\in G_1$ which is adjacent to v_1, then u_1 dominates v_1 and v_2. Also, since G_2 is a connected graph then for any vertex $u_2\in G_2$ which is adjacent to v_2, it will dominate v_1 and v_2. The other vertices of G_1 dominate only v_2 and the other vertices of G_2 dominate only v_1. Therefor, all vertices except v_1 and v_2 belong to D which is a $\gamma_{pf}-$set.
2- The proof is clear when $n=m=2$. If $n+m\geq 5$ then there is no inverse pitchfork domination according to Note 1.7 since $\gamma_{pf}(G_1+G_2)>\frac{n+m}{2}$. □

Theorem 2.10. *Let G_1 and G_2 be two null graphs of order n and m respectively, then:*
1- $\gamma_{pf}(G_1+G_2)=\gamma_{pf}(K_{n,m})$.
2- G_1+G_2 *has an inverse pitchfork domination if and only if* $n=1$ *and* $m=2$ *or* n, $m=2$, 3, 4 *such that* $\gamma_{pf}^{-1}(G_1+G_2)=\gamma_{pf}^{-1}(K_{n,m})$.

Proof. Since the bipartite graph formed by joining any two null graphs, then the pitchfork domination and it's inverse given according to Theorem 1.4 and Theorem 1.5. □

Theorem 2.11. *For any two graphs G_1 and G_2, of order n and m respectively $(n, m > 2)$, then:*

$$n + m - 4 \leq \gamma_{pf}(G_1 + G_2) \leq n + m - 2$$

Proof. To prove the lower bound, suppose that G_1 and G_2 are two null graphs having as few edges as possible. Then $\gamma_{pf}(G_1 + G_2) = \gamma_{pf}(K_{n,m}) = n + m - 4$ by Theorem 1.4 and Theorem 2.10. Also, to prove the upper bound, suppose that G_1 and G_2 are two complete graphs. Then $\gamma_{pf}(G_1 + G_2) = \gamma_{pf}(K_{n+m}) = n + m - 2$ by Proposition 1.2. □

3. Conclusion

The pitchfork domination and the inverse pitchfork domination are determined when $j = 1$ and $k = 2$ for some graphs that obtained from two types of operations: corona operation and join operation.

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References

[1] M. A. Abdlhusein and M. N. Al-harere, *Pitchfork Domination in Graphs*, (2019) reprint.
[2] M. A. Abdlhusein and M. N. Al-harere, *Inverse Pitchfork Domination in Graphs*, (2019) reprint.
[3] M. N. Al-harere and A. T. Breesam, *Further Results on Bi-domination in Graphs*, AIP Conference Proceedings **2096** (2019) 020013-020013-9p.
[4] M. N. Al-harere and P. A. Khuda Bakhash, *Tadpole Domination in Graphs*, *Baghdad Science Journal* **15** (2018) 466-471.
[5] M. Chellali, T. W. Haynes, S. T. Hedetniemi, and A. M. Rae, *[1,2]-Set in graphs*, Discrete Applied Mathematic, **161** (2013) 2885- 2893.
[6] F. Harary, *Graph Theory*, Addison-Wesley, Reading Mass, (1969).
[7] T. W. Haynes, S. T. Hedetniemi and P.J. Slater, *Domination in Graphs -Advanced Topics*, Marcel Dekker Inc., (1998).
[8] S.T. Hedetneimi and R. Laskar, *Topics in domination in graphs*, Discrete Math. **86** (1990).
[9] A. A. Omran and Y. Rajihy, *Some Properties of Frame Domination in Graphs*, Journal of Engineering and Applied Sciences, **12** (2017) 8882-8885.
[10] Ore O. *Theory of Graphs*, American Mathematical Society, Provedence, R.I. (1962).

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