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THE EXPLICIT RELATION BETWEEN THE DKP EQUATION AND THE KLEIN-GORDON EQUATION

DJAHIDA BOUCHEFRA*, AND BADREDINE BOUDJEDAA**

*DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, INSTITUTE OF SCIENCES AND TECHNOLOGY, UNIVERSITY CENTER ABDELHAFID BOUSSOUF-MILA, 43000 MILA, ALGERIA.

LABORATORY OF MATHEMATICS AND THEIR INTERACTIONS (MELILAB).
D.BOUCHEFRA@CENTRE-UNIV-MILA.DZ.

**DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, INSTITUTE OF SCIENCES AND TECHNOLOGY, UNIVERSITY CENTER ABDELHAFID BOUSSOUF-MILA, 43000 MILA, ALGERIA.

Abstract. DKP equation describes spin-0 and spin-1 relativistic particles. Many researchers have been interested in the DKP equation. In this work, we give an explicit relation between the DKP and the KG equations for both the spin-0 particle in $(1+3)$ dimensions and spin-1 particle in $(1+1)$ dimensions. From the DKP equation in its explicit form, we get another system generated by the KG equation, which gives us the equivalence between the DKP equation and the KG equation. Using this equivalence, the Volkov-like solution of the DKP equation for the spin-0 particle in the field of an electromagnetic plane wave, is calculated.

1. Introduction

Relativistic quantum mechanics is the branch of quantum mechanics that deals with the motion of relativistic particles. Among the most known equations in the relativistic quantum mechanics are: the Klein-Gordon equation (KG equation), which describes spinless particles, i.e. the spin-0 particles (e.g. the Higgs boson ...), the Dirac equation, which describes the spin-$\frac{1}{2}$ particles (e.g. electron, positron and neutrinos ...) and the Proca equation, which describes the spin-1 particles (e.g. the photon ...). Petiau [1], Duffin [2] and Kemmer [3] were motivated by Dirac's work for the spin-$\frac{1}{2}$ relativistic particle; they gave an equation (DKP equation) that describes spin-0 and spin-1 relativistic particles, which is an equation similar to the Dirac equation and in which the gamma matrices (γ) are replaced by the beta matrices (β), where the (β) matrices are 5×5 matrices for the spin-0 particle

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and 10×10 matrices for the spin-1 particle, which satisfies a different algebra from the algebra of the (γ) matrices for the Dirac equation.

In recent decades, many researchers have been interested in the DKP equation. Fischbach et al. [4], Krajcik et al. [5] have been interested in the equivalence of the DKP equation with the KG and the Proca equations. Nedjadi et al. [6] have studied some properties of the DKP equation and have also addressed the unresolved problem of the spinless DKP boson in a central field. Fainberg et al. [7] provided an equivalence between DKP and KG theories. They established this equivalence via the matrix S and the reduction formula LSZ (Lehmann-Symanzik-Zimmermann). They used the in and out asymptotic solutions and different diagrams generated by the generating function. Lunardi et al. [8] have discussed two problems relative to the electromagnetic coupling of DKP theory: the presence of an anomalous term in the Hamiltonian form of the theory and the apparent difference between the interaction terms in DKP and KG Lagrangians. Chetouani et al. [9] have solved the DKP equation in the presence of step potential. Merad [10] solved the DKP equation for spin-0 and spin-1 with smooth potential and position dependent-mass where the solution is given in terms of the Heun function. Boutabia-Chéraitia et al. [11] presented a calculation of the Green's function of the DKP equation in the case of scalar and vectorial particles interacting with a square barrier potential and its relation to the KG equation. Recently, Lunardi [12] has shown that the supposed spin-1 sector of the theory restricted to $(1 + 1)$ space-time dimensions actually is unitarily equivalent to its spin-0 sectors.

This work is organized as follows: In Section 2, we give the DKP equation for the spin-0 with 5×5 beta matrices and 10×10 beta matrices for the spin-1. In Section 3, we give an explicit relation, which is a direct equivalence, between the DKP and the KG equations for the spin-0 particle in $(1 + 3)$ dimensions and for the spin-1 particle in $(1 + 1)$ dimensions. The equivalence for the spin-0 particle is established not only in the free case but even in the presence of any interaction. In Section 4, using this relation, we calculate the Volkov-like solution of the DKP equation for the spin-0 particle, i.e. in the same form as the Volkov solution of the KG equation [13], in the field of an electromagnetic plane wave.

This paper is the full length paper of the AIP extended abstract [14].

2. The DKP equation

The DKP equation (for $\hbar = 1$, $c = 1$) interacting with an electromagnetic field A_μ is given by

$$[i\beta^\mu(\partial_\mu + ieA_\mu) - m]\psi = 0^1, \tag{2.1}$$

where m is the particle's mass and β^μ are square matrices satisfying the following algebra

$$\beta^\mu\beta^\nu\beta^\lambda + \beta^\lambda\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\lambda + g^{\nu\lambda}\beta^\mu, \tag{2.2}$$

[1]We use the following notations $\partial_\mu = (\partial_0, \nabla)$, $A_\mu = (A_0, -\mathbf{A})$ with the convention $\sum_\mu a^\mu b_\mu = a^\mu b_\mu$.

$g^{\mu\nu}$ is the metric tensor of Minkowski as $g^{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$. The β^μ are 5×5 matrices for the spin-0 particle and 10×10 matrices for the spin-1 particle.

For the spin-0, the β^μ matrices are given by

$$\beta^0 = \begin{pmatrix} \theta & \bar{0} \\ \bar{0}^T & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} \tilde{0} & \rho_i \\ -\rho_i^T & \mathbf{0} \end{pmatrix}, \quad i = 1, 2, 3 \tag{2.3}$$

where

$$\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.4}$$

$$\rho_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.5}$$

$\bar{0}, \tilde{0}$ and $\mathbf{0}$ are 2×3, 2×2 and 3×3 zero matrices, respectively, and ρ^T denotes the transpose of matrix ρ.

For the spin-1, the β^μ matrices are given by

$$\beta^0 = \begin{pmatrix} 0 & \bar{0} & \bar{0} & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \bar{0}^T & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad \beta^i = \begin{pmatrix} 0 & \bar{0} & e_i & \bar{0} \\ \bar{0}^T & \mathbf{0} & \mathbf{0} & -is_i \\ -e_i^T & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \bar{0}^T & -is_i & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad i = 1, 2, 3 \tag{2.6}$$

where e_i and $\bar{0}$ are given by

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1), \quad \bar{0} = (0, 0, 0). \tag{2.7}$$

$\mathbf{1}$ denoting the 3×3 unity matrix. The s_i being the standard non-relativistic 3×3 spin-1 matrices

$$s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, s_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, s_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.8}$$

3. The explicit relation between the DKP equation and the KG equation

3.1. **Spin-0 particle.** As described above, the DKP equation for the spin-0 particle in $(1+3)$ dimensions is given by equation (2.1) where the β^μ are 5×5 matrices given by the relation (2.3).

For $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^T$, which is a solution of the equation (2.1). Then the equation (2.1) can be written in its compact form as

$$(I)\begin{cases} iD^\mu \psi_{\mu+2} - m\psi_1 = 0, & (3.1)\\ iD_\mu \psi_1 - m\psi_{\mu+2} = 0, \qquad \mu = 0,1,2,3, & (3.2)\end{cases}$$

where $D_\mu = (\partial_\mu + ieA_\mu)$.

From the equations (3.2), it's easy to see that each component ψ_2, ψ_3, ψ_4 and ψ_5 depends on ψ_1 as

$$(II) \qquad \psi_{\mu+2} = \frac{i}{m} D_\mu \psi_1, \qquad \mu = 0,1,2,3.$$

Replacing each component ψ_2, ψ_3, ψ_4 and ψ_5 of the system (II) in the equation (3.1), of the system (I), we obtain

$$(D^\mu D_\mu + m^2)\psi_1 = 0, \tag{3.3}$$

which is KG equation for ψ_1.

In other words, the DKP equation (2.1) (i.e. system (I)) and the KG equation (3.3), for the spin-0 particle, are equivalent in the sense that if ψ_1 is a solution of the KG equation (3.3) , the solution of the DKP equation (2.1) is given by ψ where ψ_2, ψ_3, ψ_4 and ψ_5 are given by the system (II), and conversely, if ψ is a solution of the DKP equation (2.1) (i.e. solution of the system (I)), then ψ_1 is the solution of the KG equation (3.3).

Remark. *This equivalence, i.e. the relation between the DKP equation and the KG equation is established in the same manner for both* $(1+2)$ *dimensions and* $(1+1)$ *dimensions.*

3.2. **Spin-1 particle.** In the same way, as in the previous part, we can find an explicit relation between the DKP equation and the KG equation for the spin-1 particle in $(1+1)$ dimensions. The DKP equation for the spin-1 particle in $(1+1)$ dimensions is given by

$$[i\beta^0(\partial_0 + ieA_0) + i\beta^1(\partial_1 - ieA_1) - m]\psi = 0, \tag{3.4}$$

where the β^0 and β^1 are 10×10 matrices given by the relation (2.6). Then if ψ is written as

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8, \psi_9, \psi_{10})^T, \tag{3.5}$$

the equation (3.4), written in its explicit form, takes the following form

$$
(III)\begin{cases}
(i\partial_1 + eA_1)\psi_5 - m\psi_1 = 0, & (3.6)\\
(i\partial_0 - eA_0)\psi_5 - m\psi_2 = 0, & (3.7)\\
(i\partial_0 - eA_0)\psi_6 - (i\partial_1 + eA_1)\psi_{10} - m\psi_3 = 0, & (3.8)\\
(i\partial_0 - eA_0)\psi_7 + (i\partial_1 + eA_1)\psi_9 - m\psi_4 = 0, & (3.9)\\
(i\partial_0 - eA_0)\psi_2 - (i\partial_1 + eA_1)\psi_1 - m\psi_5 = 0, & (3.10)\\
(i\partial_0 - eA_0)\psi_3 - m\psi_6 = 0, & (3.11)\\
(i\partial_0 - eA_0)\psi_4 - m\psi_7 = 0, & (3.12)\\
-m\psi_8 = 0, & (3.13)\\
-(i\partial_1 + eA_1)\psi_4 - m\psi_9 = 0, & (3.14)\\
(i\partial_1 + eA_1)\psi_3 - m\psi_{10} = 0. & (3.15)
\end{cases}
$$

From equations (3.6), (3.7), (3.11), (3.12), (3.13), (3.14) and (3.15), we can easily see that each component depends on ψ_3, ψ_4 and ψ_5, and we get the following system

$$
(IV)\begin{cases}
\psi_1 = \frac{1}{m}(i\partial_1 + eA_1)\psi_5, & (3.16)\\
\psi_2 = \frac{1}{m}(i\partial_0 - eA_0)\psi_5, & (3.17)\\
\psi_6 = \frac{1}{m}(i\partial_0 - eA_0)\psi_3, & (3.18)\\
\psi_7 = \frac{1}{m}(i\partial_0 - eA_0)\psi_4, & (3.19)\\
\psi_8 = 0, & (3.20)\\
\psi_9 = \frac{-1}{m}(i\partial_1 + eA_1)\psi_4, & (3.21)\\
\psi_{10} = \frac{1}{m}(i\partial_1 + eA_1)\psi_3. & (3.22)
\end{cases}
$$

If we replace each component $\psi_1, \psi_2, \psi_6, \psi_7, \psi_9$ and ψ_{10} of the system (IV) in equations (3.8), (3.9) and (3.10) we get

$$
(V)\begin{cases}
(D^\mu D_\mu + m^2)\psi_3 = 0, & (3.23)\\
(D^\mu D_\mu + m^2)\psi_4 = 0, \qquad (\mu = 0, 1) & (3.24)\\
(D^\mu D_\mu + m^2)\psi_5 = 0. & (3.25)
\end{cases}
$$

Equations (3.23), (3.24) and (3.25) are the KG equations interacting with an electromagnetic field A_μ for each component ψ_3, ψ_4 and ψ_5 respectively.

More precisely, the DKP equation (3.4) and the KG equations (3.23), (3.24) and (3.25) for ψ_3, ψ_4 and ψ_5 respectively are equivalent. Indeed, if ψ_3, ψ_4 and ψ_5 are solutions of the KG equations (i.e. system (V)) so ψ is a solution of the system (III) i.e. DKP equation (3.4), where $\psi_1, \psi_2, \psi_6, \psi_7, \psi_8, \psi_9$ and ψ_{10} are given by the system (IV), and conversely, if ψ is the solution of the DKP equation (3.4), ψ_3, ψ_4 and ψ_5 are solutions of the KG equations in the system (V).

Remark. *This equivalence, i.e. the relation between DKP equation and KG equation for the spin-1 particle in* $(1+1)$ *dimensions, can be established in the same way if we use* β^2 *or* β^3 *instead of* β^1*.*

4. Application

4.1. **Volkov-like solution for the spin-0 particle.** Now, we are interested in using this equivalence between the DKP equation and the KG equation (i.e. equivalence between system (I) and equation (3.3)) to calculate the Volkov-like solution of the DKP equation (2.1) for the spin-0 particle in the field of an electromagnetic plane wave.

So if the solution of the DKP equation (2.1) is

$$\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^T, \tag{4.1}$$

then by this equivalence, ψ_1 is the solution of the KG equation (3.3), ψ_2, ψ_3, ψ_4 and ψ_5 are given by the system (II).

We know that the Volkov solution ψ_1, solution of KG equation (3.3), in the field of an electromagnetic plane wave is given by

$$\psi_1 = Ce^{-ipx} F_1(\phi), \quad \text{for } \phi = kx, \tag{4.2}$$

where C is a normalisation constant, and F_1 is a solution of the differential equation

$$2i(kp)F_1'(\phi) + [-2e(pA) + e^2A^2]F_1(\phi) = 0, \tag{4.3}$$

then, we find

$$F_1'(\phi) = -i\left[\frac{e}{(kp)}(pA) - \frac{e^2}{2(kp)}A^2\right]F_1(\phi), \tag{4.4}$$

i.e.

$$F_1(\phi) = \exp\left(-i\int_0^{kx}\left[\frac{e}{(kp)}(pA) - \frac{e^2}{2(kp)}A^2\right]d\phi\right). \tag{4.5}$$

Then from the system (II), $\psi = (\psi_1, \psi_2, \psi_3, \psi_4, \psi_5)^T$ is a Volkov-like solution of the DKP equation (2.1) for the spin-0 particle in the field of an electromagnetic plane wave, i.e.

$$(VI)\begin{cases} \psi_1 = C\exp\{-iS\}, & (4.6) \\ \psi_{\mu+2} = \dfrac{1}{m}\left[k_\mu\left(\dfrac{e}{(kp)}(pA) - \dfrac{e^2}{2(kp)}A^2\right) + p_\mu - eA_\mu\right]\psi_1, \quad \mu = 0,1,2,3, & (4.7)\end{cases}$$

where $S = px + \int_0^{kx}\left[\frac{e}{(kp)}(pA) - \frac{e^2}{2(kp)}A^2\right]d\phi$, is the classical action of the system.

5. Conclusion

In relativistic quantum mechanics, the DKP equation occupies an important place in the description of the spin-0 and the spin-1 particles. The DKP equation is a very interesting equation in relativistic quantum mechanics. In the last decades, the DKP equation has attracted the attention of many researchers and has been studied in its various aspects.

In this paper, we discussed two points relating to the explicit relation between the DKP equation and the KG equation for the description of particles, one for the spin-0 particles in $(1+3)$ dimensions and the other for the spin-1 particles in $(1+1)$ dimensions. The results of this work are important and interesting, where we have shown that the equivalence of these equations is established for any wave function, not only in the free case, but even in the presence of any interaction for both the spin-0 particle in $(1+3)$ dimensions and spin-1 particle in $(1+1)$ dimensions. Moreover, we found a Volkov-like solution to the DKP equation for spin-0 particles in the presence of an electromagnetic field.

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Djahida Bouchefra,
Department of Mathematics and Computer Science, Institute of Sciences and Technology, University Center Abdelhafid Boussouf- Mila, 43000 Mila, Algeria.
Laboratory of Mathematics and their interactions (Melilab).
Email address: d.bouchefra@centre-univ-mila.dz

Badredine Boudjedaa,
Department of Mathematics and Computer Science, Institute of Sciences and Technology, University Center Abdelhafid Boussouf- Mila, 43000 Mila, Algeria.
Email address: b.boudjedaa@centre-univ-mila.dz -- badredineb@gmail.com