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Research Article

Eigenvalues and eigenvectors for a G-frame operator

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Abstract

In this paper, we investigate eigenvalues and eigenvectors of the g-frame operator of $\{\Lambda_j P \in B(K, H_j) : j \in \mathbb{J}\}$, where $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is a g-frame for an N-dimensional Hilbert space H and P is a rank k orthogonal projection of H onto K, a closed subspace of H.

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1. Introduction

Frames for Hilbert spaces were introduced by Duffin and Schaeffer [2] in 1952. They used frames as a tool in the study of nonharmonic Fourier analysis. In 2006, g-frame as a generalization of frame was introduced and investigated by Sun [5]. For more references on g-frames, we can refer to [3, 6, 7]. The authors of [1] discussed eigenvectors of a frame operator in a finite dimensional Hilbert space. In this paper, we study eigenvalues and eigenvectors of a g-frame operator in a finite dimensional Hilbert space.
Throughout this paper, H is an N-dimensional Hilbert space and $\{H_j\}_{j\in\mathbb{J}}$ is a finite sequence of Hilbert spaces, where $\mathbb{J}$ is a finite subset of $\mathbb{N}$. We denote the space of all bounded linear operators from H into H_j by $B(H, H_j)$.

Definition 1.1 ([5])**.** A sequence of operators $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is called a g-frame for H with respect to $\{H_j\}_{j\in\mathbb{J}}$, if there exist two constants $0 < A \le B < \infty$, such that

$$A||f||^2 \le \sum_{j\in\mathbb{J}} ||\Lambda_j f||^2 \le B||f||^2, \quad f \in H, \tag{1.1}$$

A and B are called the lower and upper g-frame bounds, respectively.
We call $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ a tight g-frame if $A = B$ and a Parseval g-frame if

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$A = B = 1$. If the right hand inequality of (1.1) holds for all $f \in H$ then we say that Λ is a g-Bessel sequence. Let us consider the space

$$\widehat{H} = \left\{ \{f_j\}_{j\in\mathbb{J}} : \ f_j \in H_j, \ j \in \mathbb{J}, \sum_{j\in\mathbb{J}} \|f_j\|^2 < +\infty \right\}$$

with the inner product given by $\langle \{f_j\}_{j\in\mathbb{J}}, \{g_j\}_{j\in\mathbb{J}} \rangle = \sum_{j\in\mathbb{J}} \langle f_j, g_j \rangle$. It is easy to show that $\widehat{H}$ is a Hilbert space with respect to the pointwise operations. It is proved in [4], if $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is a g-Bessel sequence for H then the operator

$$T_\Lambda : \widehat{H} \longrightarrow H, \quad T_\Lambda(\{f_j\}_{j\in\mathbb{J}}) = \sum_{j\in\mathbb{J}} \Lambda_j^*(f_j), \tag{1.2}$$

is well defined and bounded and its adjoint is

$$T_\Lambda^* : H \longrightarrow \widehat{H}, \quad T_\Lambda^* f = \{\Lambda_j f\}_{j\in\mathbb{J}}.$$

Also, a sequence $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is a g-frame for H if and only if the operator T_Λ defined by (1.2) is bounded and onto. We call the operators T_Λ and T_Λ^*, the synthesis and analysis operators of Λ, respectively. If $\Lambda = \{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is a g-frame for H, then

$$S_\Lambda : H \longrightarrow H, \quad S_\Lambda f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f,$$

is a bounded invertible positive operator [5]. S_Λ is called the g-frame operator of Λ and

$$f = \sum_{j\in\mathbb{J}} S_\Lambda^{-1} \Lambda_j^* \Lambda_j f = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j S_\Lambda^{-1} f, \quad f \in H.$$

Here we state the following lemma and omit its proof.

Lemma 1.2. *If $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is a g-frame for H and P is an orthogonal projection of H onto K, a closed subspace of H, then $\{\Lambda_j P \in B(K, H_j) : j \in \mathbb{J}\}$ is a g-frame for K.*

2. Main results

In this section, we give some classifications of eigenvalues and eigenvectors of the g-frame operator of $\{\Lambda_j P \in B(H, H_j) : j \in \mathbb{J}\}$ where $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ is a g-frame for H and P is an orthogonal projection on H.

Proposition 2.1. *Let $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H. Let $e_1 \in H$, $||e_1|| = 1$ and let P be the orthogonal projection of H onto span$\{e_1\}$. Then the following statements are equivalent:*
(1) *e_1 is an eigenvector for S_Λ with the eigenvalue λ_1.*
(2) *$\sum_{j\in\mathbb{J}} ||\Lambda_j e_1||^2 = \lambda_1$ and $\sum_{j\in\mathbb{J}} \langle \Lambda_j e_1, \Lambda_j f \rangle = 0$ for all $f \in (I - P)H$.*

Proof. Assume that (1) holds; we have

$$\sum_{j\in\mathbb{J}} ||\Lambda_j e_1||^2 = \left\langle \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j e_1, e_1 \right\rangle = \langle S_\Lambda e_1, e_1 \rangle = \lambda_1.$$

If $f \in (I - P)H$, then

$$\sum_{j\in\mathbb{J}} \langle \Lambda_j e_1, \Lambda_j f \rangle = \langle S_\Lambda e_1, f \rangle = \lambda_1 \langle e_1, f \rangle = 0.$$

Conversely, let (2) hold; choose $\{e_i\}_{i=2}^N$ so that $\{e_i\}_{i=1}^N$ is an orthonormal basis for H. Then

$$\begin{aligned} S_\Lambda e_1 &= \sum_{i=1}^N \langle S_\Lambda e_1, e_i\rangle e_i = \sum_{i=1}^N \sum_{j\in\mathbb{J}} \langle \Lambda_j e_1, \Lambda_j e_i\rangle e_i \\ &= \sum_{j\in\mathbb{J}} \Big(||\Lambda_j e_1||^2\Big) e_1 + \sum_{i=2}^N \Big(\sum_{j\in\mathbb{J}} \langle \Lambda_j e_1, \Lambda_j e_i\rangle\Big) e_i = \lambda_1 e_1. \end{aligned}$$

Therefore e_1 is an eigenvector for S_Λ with the eigenvalue λ_1. □

Proposition 2.2. *Let $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for H and e_i be an eigenvector of S_Λ with the eigenvalue λ_i, for $i = 1, 2, ..., N$. Then*

$$\sum_{j\in\mathbb{J}} ||\Lambda_j f||^2 = \sum_{i=1}^N \lambda_i |\langle f, e_i\rangle|^2, \quad f \in H$$

and

$$\sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Lambda_j g\rangle = \sum_{i=1}^N \lambda_i \langle f, e_i\rangle \overline{\langle g, e_i\rangle}, \quad f, g \in H.$$

Proof. We have

$$\begin{aligned} \sum_{j\in\mathbb{J}} ||\Lambda_j f||^2 &= \langle S_\Lambda f, f\rangle \\ &= \Big\langle S_\Lambda\big(\sum_{i=1}^N \langle f, e_i\rangle e_i\big), \sum_{j=1}^N \langle f, e_j\rangle e_j \Big\rangle \\ &= \sum_{i=1}^N \sum_{j=1}^N \lambda_i \langle f, e_i\rangle \overline{\langle f, e_j\rangle} \langle e_i, e_j\rangle \\ &= \sum_{i=1}^N \lambda_i |\langle f, e_i\rangle|^2, \end{aligned}$$

for all $f \in H$. On the other hand

$$\begin{aligned} \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \Lambda_j g\rangle &= \langle S_\Lambda f, g\rangle \\ &= \Big\langle S_\Lambda\big(\sum_{i=1}^N \langle f, e_i\rangle e_i\big), \sum_{j=1}^N \langle g, e_j\rangle e_j \Big\rangle \\ &= \sum_{i=1}^N \sum_{j=1}^N \lambda_i \langle f, e_i\rangle \overline{\langle g, e_j\rangle} \langle e_i, e_j\rangle \\ &= \sum_{i=1}^N \lambda_i \langle f, e_i\rangle \overline{\langle g, e_i\rangle}, \end{aligned}$$

for all $f, g \in H$. □

Theorem 2.3. *Let $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H. Fix $1 \leq k \leq N$ and let P be a rank k orthogonal projection of H onto a subspace K. Let S_1 be the g-frame operator of $\{\Lambda_j P \in B(K, H_j) : j \in \mathbb{J}\}$ and choose orthonormal family $\{f_i\}_{i=1}^k$ in K. Then the following statements are equivalent:*

(1) $\{f_i\}_{i=1}^k$ *is a family of eigenvectors of* S_1 *with eigenvalues* $\{\mu_i\}_{i=1}^k$.
(2) *We have*

$$(i)\ \sum_{j\in\mathbb{J}}||\Lambda_j f_i||^2=\mu_i,\quad 1\le i\le k,$$

$$(ii)\ \sum_{j\in\mathbb{J}}\langle\Lambda_j f_i,\Lambda_j f_l\rangle=0,\quad 1\le i\ne l\le k.$$

Proof. Let (1) hold. Then

$$\sum_{j\in\mathbb{J}}||\Lambda_j f_i||^2=\sum_{j\in\mathbb{J}}||\Lambda_j Pf_i||^2=\langle S_1 f_i,f_i\rangle=\mu_i||f_i||^2=\mu_i,$$

for $i=1,2,...,k$. Similarly,

$$\sum_{j\in\mathbb{J}}\langle\Lambda_j f_i,\Lambda_j f_l\rangle=\langle S_1 f_i,f_l\rangle=\mu_i\langle f_i,f_l\rangle=0,\quad 1\le i\ne l\le k.$$

Now, assume that (2) is true. Then choose $\{f_i\}_{i=k+1}^N$ so that $\{f_i\}_{i=1}^N$ is an orthonormal basis for H. Then

$$\begin{aligned}S_1 f_i&=\sum_{l=1}^N\langle S_1 f_i,f_l\rangle f_l=\sum_{l=1}^N\Big\langle\sum_{j\in\mathbb{J}}P\Lambda_j^*\Lambda_j Pf_i,f_l\Big\rangle f_l\\&=\sum_{l=1}^k\sum_{j\in\mathbb{J}}\langle\Lambda_j f_i,\Lambda_j f_l\rangle f_l\\&=\Big(\sum_{j\in\mathbb{J}}\langle\Lambda_j f_i,\Lambda_j f_i\rangle\Big)f_i+\sum_{\substack{l=1\\l\ne i}}^k\Big(\sum_{j\in\mathbb{J}}\langle\Lambda_j f_i,\Lambda_j f_l\rangle\Big)f_l\\&=\Big(\sum_{j\in\mathbb{J}}||\Lambda_j f_i||^2\Big)f_i+\sum_{\substack{l=1\\l\ne i}}^k\Big(\sum_{j\in\mathbb{J}}\langle\Lambda_j f_i,\Lambda_j f_l\rangle\Big)f_l\\&=\mu_i f_i,\end{aligned}$$

so, f_i is an eigenvector of S_1 with eigenvalue μ_i for $i=1,2,...,k$. □

Theorem 2.4. *Let* $\{\Lambda_j\in B(H,H_j):j\in\mathbb{J}\}$ *be a g-frame for* H. *Let* $\{e_i\}_{i=1}^N$ *be an orthonormal basis for* H *and* e_i *be an eigenvector of* S_Λ *with the eigenvalue* λ_i, *for* $i=1,2,...,N$. *Fix* $1\le k\le N$ *and let* P *be a rank* k *orthogonal projection of* H *onto* K *subspace of* H *and let* S_1 *be the g-frame operator of* $\{\Lambda_j P\in B(K,H_j):j\in\mathbb{J}\}$ *and choose orthonormal family* $\{f_i\}_{i=1}^k$ *in* K . *Then the following statements are equivalent:*
(1) $\{f_i\}_{i=1}^k$ *is a family of eigenvectors of* S_1 *with eigenvalues* $\{\mu_i\}_{i=1}^k$.
(2) *We have*

$$(i)\ \sum_{n=1}^N\lambda_n|\langle f_i,e_n\rangle|^2=\mu_i,\quad 1\le i\le k,$$

$$(ii)\ \sum_{n=1}^N\lambda_n\langle f_i,e_n\rangle\overline{\langle f_j,e_n\rangle}=0,\quad 1\le i\ne j\le k.$$

Proof. It is clear by Proposition 2.2 and Theorem 2.3. □

Proposition 2.5. *Let $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H. Let $\{e_i\}_{i=1}^N$ be an orthonormal basis for H and e_i be an eigenvector of S_Λ with eigenvalue λ_i for $i = 1, 2, ..., N$. Let $\{\sigma_j\}_{j=1}^k$ be a partition of $\{1, 2, ..., N\}$ and for every $1 \le j \le k$ let $f_j = \sum_{i\in\sigma_j} a_i e_i$ with $||f_j|| = \sum_{i\in\sigma_j} |a_i|^2 = 1$. Let P be the orthogonal projection of H onto $K = span\{f_j\}_{j=1}^k$ and let S_1 be the g-frame operator for $\{\Lambda_j P \in B(K, H_j) : j \in \mathbb{J}\}$. Then $\{f_j\}_{j=1}^k$ is an orthonormal basis for K and f_j is an eigenvector of S_1 with eigenvalue $\mu_j = \sum_{i\in\sigma_j} \lambda_i |a_i|^2$ for $j = 1, 2, ..., k$.*

Proof. We have

$$\begin{aligned}
\sum_{n\in\mathbb{J}} ||\Lambda_n f_i||^2 &= \sum_{n\in\mathbb{J}} \langle \Lambda_n f_i, \Lambda_n f_i\rangle \\
&= \sum_{n\in\mathbb{J}} \Big\langle \Lambda_n(\sum_{j\in\sigma_i} a_j e_j), \Lambda_n(\sum_{l\in\sigma_i} a_l e_l)\Big\rangle \\
&= \sum_{n\in\mathbb{J}} \sum_{j\in\sigma_i} \sum_{l\in\sigma_i} a_j \overline{a_l} \langle \Lambda_n e_j, \Lambda_n e_l\rangle \\
&= \sum_{n\in\mathbb{J}} \sum_{j\in\sigma_i} |a_j|^2 \langle \Lambda_n e_j, \Lambda_n e_j\rangle + \sum_{n\in\mathbb{J}} \sum_{\substack{j,l\in\sigma_i \\ j\ne l}} a_j \overline{a_l} \langle \Lambda_n e_j, \Lambda_n e_l\rangle \\
&= \sum_{j\in\sigma_i} |a_j|^2 \langle S_\Lambda e_j, e_j\rangle + \sum_{\substack{j,l\in\sigma_i \\ j\ne l}} a_j \overline{a_l} \langle S_\Lambda e_j, e_l\rangle \\
&= \sum_{j\in\sigma_i} \lambda_j |a_j|^2 + 0 = \mu_i,
\end{aligned}$$

for $1 \le i \le k$. On the other hand

$$\begin{aligned}
\sum_{n\in\mathbb{J}} \langle \Lambda_n f_i, \Lambda_n f_l\rangle &= \sum_{n\in\mathbb{J}} \Big\langle \Lambda_n(\sum_{m\in\sigma_i} a_m e_m), \Lambda_n(\sum_{t\in\sigma_l} a_t e_t)\Big\rangle \\
&= \sum_{m\in\sigma_i} \sum_{t\in\sigma_l} a_m \overline{a_t} \langle S_\Lambda e_m, e_t\rangle \\
&= \sum_{m\in\sigma_i} \sum_{t\in\sigma_l} \lambda_m a_m \overline{a_t} \langle e_m, e_t\rangle = 0,
\end{aligned}$$

for $1 \le i \ne l \le k$. Therefore $\{f_j\}_{j=1}^k$ is an orthonormal basis for K and by Theorem 2.3, f_j is an eigenvector of S_1 with eigenvalue $\mu_j = \sum_{i\in\sigma_j} \lambda_i |a_i|^2$ for $j = 1, 2, ..., k$. □

Proposition 2.6. *Let $\{\Lambda_j \in B(H, H_j) : j \in \mathbb{J}\}$ be a g-frame for H with* $\dim H = 2N$. *Let $\{e_i\}_{i=1}^{2N}$ be an orthonormal basis for H and e_i be an eigenvector of S_Λ with eigenvalue λ_i, for $i = 1, 2, ..., 2N$ and $\lambda_{2N} \le \lambda_{2N-1} \le ... \le \lambda_1$. Let $\lambda_{2N-i+1} \le \mu_i \le \lambda_i$ for $i = 1, 2, ..., 2N$. Then there exists a subspace $K \subseteq H$ with* $\dim K = N$ *such that if P is the orthogonal projection of H onto K, then $\{\Lambda_j P \in B(K, H_j) : j \in \mathbb{J}\}$ is a g-frame for K with the g-frame operator S_1 having eigenvalues $\mu_1, \mu_2, ..., \mu_N$.*

Proof. For $1 \le i \le N$ choose $0 < \varepsilon_i < 1$ so that $\lambda_i \varepsilon_i^2 + \lambda_{2N-i+1}(1 - \varepsilon_i^2) = \mu_i$ and let $f_i = \varepsilon_i e_i + \sqrt{1-\varepsilon_i^2} e_{2N-i+1}$. So, $\{f_i\}_{i=1}^N$ is an orthonormal sequence in H. Let $K =$

$span\{f_i\}_{i=1}^N$. We have

$$\begin{aligned}\sum_{n=1}^{2N}\lambda_n|\langle f_i, e_n\rangle|^2 &= \langle S_\Lambda f_i, f_i\rangle \\ &= \Big\langle S_\Lambda\Big(\varepsilon_i e_i + \sqrt{1-\varepsilon_i^2}e_{2N-i+1}\Big), \varepsilon_i e_i + \sqrt{1-\varepsilon_i^2}e_{2N-i+1}\Big\rangle \\ &= \Big\langle \varepsilon_i\lambda_i e_i + \sqrt{1-\varepsilon_i^2}\lambda_{2N-i+1}e_{2N-i+1}, \varepsilon_i e_i + \sqrt{1-\varepsilon_i^2}e_{2N-i+1}\Big\rangle \\ &= \lambda_i\varepsilon_i^2 + \lambda_{2N-i+1}(1-\varepsilon_i^2) = \mu_i, \quad 1 \le i \le N.\end{aligned}$$

For $i \neq j$,

$$\begin{aligned}\sum_{n=1}^{2N}\lambda_n\langle f_i, e_n\rangle\overline{\langle f_j, e_n\rangle} &= \langle S_\Lambda f_i, f_j\rangle \\ &= \Big\langle S_\Lambda\Big(\varepsilon_i e_i + \sqrt{1-\varepsilon_i^2}e_{2N-i+1}\Big), \varepsilon_j e_j + \sqrt{1-\varepsilon_j^2}e_{2N-j+1}\Big\rangle \\ &= \Big\langle \varepsilon_i\lambda_i e_i + \sqrt{1-\varepsilon_i^2}\lambda_{2N-i+1}e_{2N-i+1}, \varepsilon_j e_j + \sqrt{1-\varepsilon_j^2}e_{2N-j+1}\Big\rangle = 0.\end{aligned}$$

So by Theorem 2.4, f_i is an eigenvector for S_1 with eigenvalue μ_i for $i = 1, 2, ..., N$. □

We close the paper by giving examples in which the conditions of Proposition 2.1 and Theorem 2.3 are satisfied.

Example 2.7. Let $H = \mathbb{C}^3$ and we define

$$\begin{aligned}\Lambda_1 &: \mathbb{C}^3 \longrightarrow \mathbb{C}, \quad \Lambda_1(x,y,z) = 2x - y, \\ \Lambda_2 &: \mathbb{C}^3 \longrightarrow \mathbb{C}, \quad \Lambda_2(x,y,z) = x + 2y, \\ \Lambda_3 &: \mathbb{C}^3 \longrightarrow \mathbb{C}, \quad \Lambda_3(x,y,z) = \sqrt{6}zi.\end{aligned}$$

Then $\{\Lambda_j \in B(\mathbb{C}^3, \mathbb{C}) : j = 1, 2, 3\}$ is a g-frame for $\mathbb{C}^3$ with the g-frame bounds $5, 6$. Therefore, the g-frame operator of $\{\Lambda_j \in B(\mathbb{C}^3, \mathbb{C}) : j = 1, 2, 3\}$ is

$$S_\Lambda : \mathbb{C}^3 \longrightarrow \mathbb{C}^3, \quad S_\Lambda(x,y,z) = (5x, 5y, 6z).$$

Then 5 is an eigenvalue of S_Λ with eigenvectors $(1,0,0)$, $(0,1,0)$ and 6 is an eigenvalue of S_Λ with eigenvector $(0,0,1)$. Let $\{e_i\}_{i=1}^3$ be the standard orthonormal basis for $\mathbb{C}^3$. Let us consider the orthogonal projection P on $\mathbb{C}^3$,

$$P : \mathbb{C}^3 \longrightarrow \text{span}\{(0,0,1)\} = \text{span}\{e_3\}.$$

Then

$$\sum_{j=1}^{3}|\Lambda_j(e_3)|^2 = 0 + 0 + |\sqrt{6}i|^2 = 6,$$

and

$$\sum_{j=1}^{3}\langle\Lambda_j(e_3), \Lambda_j(f)\rangle = \langle S_\Lambda(e_3), f\rangle = \langle 6e_3, c_1e_1 + c_2e_2\rangle = 0, \quad f \in (I-P)\mathbb{C}^3.$$

Now, by the above assumption we define the orthogonal projection P on $\mathbb{C}^3$ as follows:

$$P : \mathbb{C}^3 \longrightarrow K = span\{(1,0,0), (0,1,0)\}, \quad P(x,y,z) = (x,y,0).$$

For any $(x,y,z) \in \mathbb{C}^3$, we have

$$\begin{aligned}(\Lambda_1P)(x,y,z) &= 2x - y, \\ (\Lambda_2P)(x,y,z) &= x + 2y, \\ (\Lambda_3P)(x,y,z) &= 0.\end{aligned}$$

Thus the g-frame operator of $\{\Lambda_j P \in B(\mathbb{C}^3, \mathbb{C}) : j = 1, 2, 3\}$ is

$$S_1 : K \longrightarrow K$$
$$S_1(x, y, z) = PS_\Lambda P(x, y, z) = (5x, 5y, 0).$$

Thus $(1, 0, 0)$ and $(0, 1, 0)$ are the eigenvectors of S_1 with the eigenvalue 5. We have

$$\sum_{j=1}^{3} |\Lambda_j(e_1)|^2 = 5,$$

and

$$\sum_{j=1}^{3} |\Lambda_j(e_2)|^2 = 5.$$

Also,

$$\sum_{j=1}^{3} \langle \Lambda_j(e_1), \Lambda_j(e_2)\rangle = \langle S_\Lambda(e_1), e_2\rangle = \langle (5, 0, 0), (0, 1, 0)\rangle = 0.$$

Example 2.8. Let $H = \mathbb{C}^N$ and we define

$$\Lambda_j : \mathbb{C}^N \longrightarrow \mathbb{C}, \quad \Lambda_j(x_1, x_2, ..., x_N) = x_j, \quad 1 \le j \le N - 1$$
$$\Lambda_N : \mathbb{C}^N \longrightarrow \mathbb{C}, \quad \Lambda_N(x_1, x_2, ..., x_N) = 2x_N.$$

Then $\{\Lambda_j \in B(\mathbb{C}^N, \mathbb{C}) : j = 1, ..., N\}$ is a g-frame for $\mathbb{C}^N$ with the g-frame bounds $1, 4$. Therefore, the g-frame operator is

$$S_\Lambda : \mathbb{C}^N \longrightarrow \mathbb{C}^N$$
$$S_\Lambda(x_1, x_2, ..., x_{N-1}, x_N) = (x_1, x_2, ..., x_{N-1}, 4x_N).$$

The eigenvalues of S_Λ are 1 and 4 and the eigenvectors of 1 are $(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 1, 0)$ and the eigenvector of 4 is $(0, 0, ..., 0, 1)$. Let $\{e_i\}_{i=1}^N$ is the standard orthonormal basis for $\mathbb{C}^N$ and consider the orthogonal projection P on $\mathbb{C}^N$ as follows:

$$P : \mathbb{C}^N \longrightarrow \text{span}\{(0, 0, ..., 0, 1)\}.$$

Then

$$\sum_{j=1}^{N} |\Lambda_j(e_N)|^2 = 4,$$

and

$$\sum_{j=1}^{N} \langle \Lambda_j(e_N), \Lambda_j(f)\rangle = \langle S_\Lambda(e_N), f\rangle = \langle 4e_N, \sum_{i=1}^{N-1} c_i e_i\rangle = 0, \quad f \in (I - P)\mathbb{C}^N.$$

Now, we consider the orthogonal projection P on $\mathbb{C}^N$,

$$P : \mathbb{C}^N \longrightarrow K = span\{(1, 0, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, 0, ..., 1, 0)\}$$
$$P(x_1, x_2, ..., x_{N-1}, x_N) = (x_1, x_2, ..., x_{N-1}, 0)$$

Then the g-frame operator of $\{\Lambda_j P \in B(\mathbb{C}^N, \mathbb{C}) : j = 1, ..., N\}$ is

$$S_1 : K \longrightarrow K$$
$$S_1(x_1, x_2, ..., x_{N-1}, x_N) = (x_1, x_2, ..., x_{N-1}, 0).$$

The eigenvalues of S_1 are 0 and 1 and the eigenvector of 0 is $(0,0,...,0,1)$ and the eigenvectors of 1 are $(1,0,0,...,0),(0,1,0,...,0),...,(0,0,...,1,0)$. We have

$$\sum_{j=1}^{N} |\Lambda_j(e_i)|^2 = 1, \quad 1 \leq i \leq N-1,$$

and

$$\sum_{j=1}^{N} \langle \Lambda_j(e_l), \Lambda_j(e_i)\rangle = \langle S_\Lambda(e_l), e_i\rangle = \langle e_l, e_i\rangle = 0, \quad 1 \leq i \neq l \leq N-1.$$

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