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Research Article

Delta operation on modules, prime and radical submodules and primary decomposition

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Abstract

Let R be a commutative ring with identity and M be an R-module. In this paper, in order to study prime submodules, radical submodules and primary decompositions in finitely generated free R-modules, we introduce and study an operation $\Delta : (M \oplus R)^2 \to M$ defined by $\Delta(m + r, m' + r') = r'm - rm'$. In particular, using this operation we give a characterization of prime submodules of $M \oplus R$, in terms of prime submodules of M. As an application, we present a characterization of prime submodules of finitely generated free modules. Also we present a formula for the prime radical of submodules of $M \oplus R$. Moreover, we state some conditions under which primary decompositions of submodules of M lift to $M \oplus R$.

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1. Introduction

In this paper all rings are commutative with identity, all modules are unitary, R denotes a ring and M denotes an R-module. Also by $\mathbb{N}$ we mean the set of positive integers. We indicate the relation of containment and strict containment by $\subseteq$ and $\subset$, respectively. Furthermore $N \leq M$ (resp., $N < M$) means that N is a submodule (resp., proper submodule) of M.

Prime ideals of rings play an important role in commutative ring theory; hence many have tried to generalize this concept to modules. A proper submodule P of M is called *prime*, when from $rm \in P$ for some $r \in R$ and $m \in M$, we can conclude either $m \in P$ or $rM \subseteq P$ (see for example [1, 3, 4, 8, 14, 22]). Let $(P : M)$ be the set of all $r \in R$ such that $rM \subseteq P$. If P is a prime submodule, then $\mathfrak{p} = (P : M)$ is a prime ideal of R and we say that P is $\mathfrak{p}$-prime.

If N is a submodule of M, the intersection of prime submodules of M containing N is called the *radical of* N and we denote it by $\mathrm{rad}_M(N)$ (or $\mathrm{rad}(N)$ if there is no subtlety). If there is no prime submodule containing N, we set $\mathrm{rad}(N) = M$. Many researchers have studied and tried to give formulations for the prime radical of submodules, see for example [2, 3, 5, 6, 9–13, 16–21].

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As prime and primary submodules and radical of submodules behave well under taking quotients, it is important to characterize such submodules in free modules. In [8] a characterization of prime submodules in the R-module $F = R \oplus R$ is given. Also some criteria on submodules of F is stated for having a primary decomposition.

The main tool in proving these results is a function $\Delta : F^2 \to R$ defined as $\Delta((x_1, x_2), (y_1, y_2)) = x_1 y_2 - y_1 x_2$. In particular, they prove that for a submodule N of F containing neither $(1, 0)$ nor $(0, 1)$, being prime is equivalent to $(N : F) = \mathfrak{p}$ being a prime ideal and $N = \mathfrak{p} \oplus \mathfrak{p}$ or $N = \Delta^{\mathfrak{p}}(a, b)$ with $Ra + Rb \nsubseteq \mathfrak{p}$, where $\Delta^{\mathfrak{p}}(a, b) = \{(x, y) \in F | \Delta((x, y), (a, b)) \in \mathfrak{p}\}$. In [15], by replacing this Δ function with minors of certain matrices, the results of [8] are generalized to finitely generated free modules.

The Δ function mentioned above had been previously proved to be useful in studying prime and radical submodules. For example the Λ operation defined by Man in [11, 12] and used to characterize domains satisfying specific formulas on radical of submodules, is indeed $\Lambda(a, b) = \Delta^0(a, b)$.

The main aim of this research is to generalize these results to every finitely generated free module. To this end we first investigate the following generalization of the Δ operation of [8].

Definition 1.1. For an R-module M let $\widetilde{M} = M \oplus R$ and define $\Delta_{M,R} : \widetilde{M}^2 \to M$ by $\Delta_{M,R}(m + r, m' + r') = r'm - rm'$ where $m + r$ denotes the element of $\widetilde{M}$ with $m \in M$ and $r \in R$. Also for $N \leq M$ and $\widetilde{m} \in \widetilde{M}$, we set $\Delta^N_{M,R}(\widetilde{m}) = (\Delta_{M,R}(\widetilde{m}, \cdot))^{-1}(N)$, the preimage of N under the map $\Delta_{M,R}(\widetilde{m}, \cdot) : \widetilde{M} \to M$. Moreover, for any $\widetilde{A} \subseteq \widetilde{M}$ and $N \leq M$ by $\Delta^N_{M,R}(\widetilde{A})$ we mean $\bigcap_{\widetilde{a} \in \widetilde{A}} \Delta^N_{M,R}(\widetilde{a})$. When there is no confusion we drop the subscripts M, R and write Δ or Δ^N.

Here first in Section 2, we state some basic properties of this Δ operation. Then in Section 3, using the Δ operation we present a characterization of prime submodules and radical of submodules of $\widetilde{M}$ in terms of prime and radical submodules of M. We use this to state a characterization of prime submodules of finitely generated free modules. Finally in Section 4, we study when a primary decomposition of an $A \leq M$ 'lifts' to one for $\Delta^A(\widetilde{N})$ where $\widetilde{N} \leq \widetilde{M}$. We end this introduction with the following notations.

Notation 1.2. Throughout the paper, $\widetilde{M} = M \oplus R$ and its elements are written as $m + r$ with $m \in M$ and $r \in R$. Also we consider M and R as submodules of $\widetilde{M}$ in the natural way and denote the canonical projections $\widetilde{M} \to M$ and $\widetilde{M} \to R$ by π_1 and π_2, respectively.

2. Basic properties of the delta operation

We start with the following properties of the delta operation, whose easy proofs are left to the reader. Here $(N :_M I) = \{m \in M | Im \subseteq N\}$, for $N \leq M, I \leq R$. Also by $\Delta(\widetilde{A}, \widetilde{B})$ we mean the submodule generated by $\{\Delta(\widetilde{a}, \widetilde{b}) | \widetilde{a} \in \widetilde{A}, \widetilde{b} \in \widetilde{B}\}$, for $\widetilde{A}, \widetilde{B} \subseteq \widetilde{M}$.

Lemma 2.1. *Suppose that $N, K, N_\lambda \leq M$ for each $\lambda \in \Lambda$, $I \leq R$ and $\widetilde{A} \subseteq \widetilde{B} \subseteq \widetilde{M}$. Then the following hold.*

- **(a)** Δ *is an R-bilinear map.*
- **(b)** $\Delta^N(\widetilde{A}) = \Delta^N(\langle \widetilde{A} \rangle) \leq \widetilde{M}$.
- **(c)** $\Delta^N(\widetilde{B}) \subseteq \Delta^N(\widetilde{A})$.
- **(d)** $\Delta^N(K \oplus I) = (N :_M I) \oplus (N :_R K)$.
- **(e)** $\Delta^N(\widetilde{A})$ *is the largest subset of $\widetilde{M}$ such that $\Delta(\widetilde{A}, \Delta^N(\widetilde{A})) \subseteq N$.*
- **(f)** $\Delta^{\bigcap_{\lambda \in \Lambda} N_\lambda}(\widetilde{A}) = \bigcap_{\lambda \in \Lambda} \Delta^{N_\lambda}(\widetilde{A})$.

Corollary 2.2. *For any $\widetilde{A} \subseteq \widetilde{M}$ and $N \leq M$ we have $\widetilde{A} \subseteq \Delta^N(\Delta^N(\widetilde{A}))$ and equality holds if and only if $\widetilde{A} = \Delta^N(\widetilde{B})$ for some $\widetilde{B} \subseteq \widetilde{M}$. If we set $S = \{\Delta^N(\widetilde{B}) | \widetilde{B} \subseteq \widetilde{M}\}$, then S is a lattice with respect to inclusion and $\Delta^N : S \to S$ is an order anti-automorphism.*

Proof. The inclusion $\widetilde{A} \subseteq \Delta^N(\Delta^N(\widetilde{A}))$ and also the fact that the equality holds only if $A = \Delta^N(\widetilde{B})$ are clear. Conversely if $\widetilde{A} = \Delta^N(\widetilde{B})$, then $\widetilde{B} \subseteq \Delta^N(\Delta^N(\widetilde{B}))$ and by 2.1c we deduce that $\widetilde{A} = \Delta^N(\widetilde{B}) \supseteq \Delta^N(\Delta^N(\Delta^N(\widetilde{B}))) = \Delta^N(\Delta^N(\widetilde{A}))$. Since the reverse inclusion always holds, we conclude that indeed equality holds.

Now it is clear that the map $\Delta^N : S \to S$ is an order reversing bijection whose inverse is again Δ^N. Suppose that $\widetilde{B}_1, \widetilde{B}_2 \subseteq \widetilde{M}$. It can readily be checked that $\Delta^N(\widetilde{B}_1 \cup \widetilde{B}_2) = \Delta^N(\widetilde{B}_1) \cap \Delta^N(\widetilde{B}_2)$ and $\Delta^N(\Delta^N(\Delta^N(\widetilde{B}_1) + \Delta^N(\widetilde{B}_2)))$ are respectively the greatest lower bound and the least upper bound of $\Delta^N(\widetilde{B}_1)$ and $\Delta^N(\widetilde{B}_2)$. Thus S is a lattice. □

Next we find the ideal $(\Delta^N(\widetilde{A}) : \widetilde{M})$. Note that by 2.1b, we can assume that $\widetilde{A} \leq \widetilde{M}$.

Proposition 2.3. *Suppose that $\widetilde{A} \leq \widetilde{M}$, $N \leq M$ and $K = \Delta(\widetilde{A}, \widetilde{A})$. Then*

(a) $(\Delta^N(\widetilde{A}) : \widetilde{M}) = (N : (\pi_1(\widetilde{A}) + \pi_2(\widetilde{A})M)) \supseteq (N : M)$;

(b) $K \subseteq \widetilde{A} \cap M$ *and* $(\widetilde{A} : \widetilde{M}) \subseteq \sqrt{(K : M)}$.

Proof. (a) Let $i \in I = (\Delta^N(\widetilde{A}) : \widetilde{M})$ and $\tilde{a} \in \widetilde{A}$ with $\pi_j(\tilde{a}) = a_j$ for $j = 1, 2$. For each $m \in M$ we have $-ia_2m = 0a_1 - ia_2m = \Delta(\tilde{a}, im) \in \Delta(\widetilde{A}, \Delta^N(\widetilde{A})) \subseteq N$. So $I\pi_2(\widetilde{A})M \subseteq N$, that is, $I \subseteq (N : \pi_2(\widetilde{A})M)$. Similarly $ia_1 = \Delta(\tilde{a}, i(0+1)) \subseteq N$ and hence $I \subseteq (N : \pi_1(\widetilde{A}))$.

Conversely, if $i \in (N : (\pi_1(\widetilde{A}) + \pi_2(\widetilde{A})M))$, $\tilde{m} \in \widetilde{M}$ and $\tilde{a} \in \widetilde{A}$, then

$$\Delta(\tilde{a}, i\tilde{m}) = i(\pi_2(\tilde{m})\pi_1(\tilde{a}) - \pi_2(\tilde{a})\pi_1(\tilde{m})) \in i\pi_1(\widetilde{A}) + i\pi_2(\widetilde{A})M \subseteq N.$$

Thus by definition of Δ^N, we deduce that $i\tilde{m} \in \Delta^N(\widetilde{A})$, which means $i \in I$, as required.

(b) Obviously $K \subseteq M$. Let $\tilde{a}, \tilde{a}' \in \widetilde{A}$ with $\pi_j(\tilde{a}) = a_j$, $\pi_j(\tilde{a}') = a'_j$. Then

$$\Delta(\tilde{a}, \tilde{a}') = a'_2a_1 - a_2a'_1 = a'_2(a_1 + a_2) - a_2(a'_1 + a'_2) = a'_2\tilde{a} - a_2\tilde{a}' \in \widetilde{A}.$$

Therefore, $K \subseteq \widetilde{A}$. Now suppose that $r \in (\widetilde{A} : \widetilde{M})$. Then for each $m \in M$ we have $rm \in \widetilde{A}$ and also $r = r(0 + 1) \in \widetilde{A}$, whence $r^2m = \Delta(rm, r) \in K$ and the result follows. □

Later we will need the following lemmas which show how Δ behaves under localization and taking quotients.

Lemma 2.4. *Suppose that $K \leq N \leq M$, $\widetilde{A} \leq \widetilde{M}$, $I \subseteq (K : M)$ and $\widehat{A}$ denotes the image of $\widetilde{A}$ under the canonical projection from $\widetilde{M} \to \widehat{M} = \frac{M}{K} \oplus \frac{R}{I} = \frac{\widetilde{M}}{K \oplus I}$. Then*

$$\Delta^{\frac{N}{K}}_{\frac{M}{K}, \frac{R}{I}}(\widehat{A}) = \frac{\Delta^N_{M,R}(\widetilde{A})}{K \oplus I}.$$

In particular, $N \oplus (N : M) \subseteq \Delta^N(\widetilde{A}) = \Delta^N(\widetilde{A} + (N \oplus (N : M)))$.

Proof. Suppose that "$\bar{\cdot}$" denotes the image of submodules of M or R in $\frac{M}{K}$ or $\frac{R}{I}$. As $\widetilde{A} \subseteq M \oplus R$ it follows 2.1c and d, that $K \oplus I \subseteq N \oplus (N : M) \subseteq \Delta^N_{M,R}(\widetilde{A})$. Let $\tilde{m} \in \widetilde{M}$ and $\hat{m} = \tilde{m} + (K \oplus I)$. Then $\hat{m} \in \Delta^{\overline{N}}_{\overline{M},\overline{R}}(\widehat{A}) \Leftrightarrow \Delta_{\overline{M},\overline{R}}(\hat{a}, \hat{m}) \in \overline{N}$ for each $\hat{a} \in \widehat{A}$. But if $\tilde{a} \in \widetilde{A}$ is a preimage of $\hat{a}$, then $\Delta_{\overline{M},\overline{R}}(\hat{a}, \hat{m}) = \Delta_{M,R}(\tilde{a}, \tilde{m}) + K$. Hence $\hat{m} \in \Delta^{\overline{N}}_{\overline{M},\overline{R}}(\widehat{A}) \Leftrightarrow \Delta_{M,R}(\tilde{a}, \tilde{m}) \in N$ for each $\tilde{a} \in \widetilde{A} \Leftrightarrow \tilde{m} \in \Delta^N_{M,R}(\widetilde{A})$. The "in particular" statement follows by setting $K = N$ and $I = (N : M)$ in the main statement. □

Lemma 2.5. *Assume that S is a multiplicatively closed subset of R, $\widetilde{A} \leq \widetilde{M}$ and K is an $S^{-1}R$-submodule of $S^{-1}M$. Then $\left(\Delta^K_{S^{-1}M,S^{-1}R}(S^{-1}\widetilde{A})\right)^c = \Delta^{K^c}_{M,R}(\widetilde{A})$, where "$\cdot^c$" denotes contraction under the localization map. In particular, $\Delta^{K^c}_{M,R}(\widetilde{A}) = \Delta^{K^c}_{M,R}((S^{-1}\widetilde{A})^c)$.*

Proof. We have

$$\widetilde{m} \in \left(\Delta^{K}_{S^{-1}M,S^{-1}R}(S^{-1}\widetilde{A})\right)^{c} \Leftrightarrow \Delta_{S^{-1}M,S^{-1}R}(\frac{\widetilde{a}}{s},\frac{\widetilde{m}}{1}) = \frac{\Delta_{M,R}(\widetilde{a},\widetilde{m})}{s} \in K,$$

for each $\widetilde{a} \in \widetilde{A}$ and $s \in S$. This is equivalent to $\frac{\Delta_{M,R}(\widetilde{a},\widetilde{m})}{1} \in K$, that is, $\Delta_{M,R}(\widetilde{a},\widetilde{m}) \in K^c$ for all $\widetilde{a} \in \widetilde{A}$ or equivalently $\widetilde{m} \in \Delta^{K^c}_{M,R}(\widetilde{A})$. □

A proper submodule P of M is called *weakly prime*, when from $r_1r_2m \in P$ we can deduce that either either $r_1m \in$ P or $r_2m \in P$. It is easy to see that $\mathfrak{p} = (P : M)$ is a prime ideal, when P is weakly prime. In this case, we say that P is weakly $\mathfrak{p}$-prime. This concept was first introduced in [7] as another generalization of prime ideals. It should be mentioned that in some papers weakly prime submodules are called *classical prime*. The following shows that prime, weakly prime and primary submodules behave well under Δ.

Theorem 2.6. *Suppose that P is a $\mathfrak{p}$-primary (resp. $\mathfrak{p}$-prime, weakly $\mathfrak{p}$-prime) submodule of M and $\widetilde{N} \leq \widetilde{M}$. If $\widetilde{N} \nsubseteq P \oplus (P : M)$, then $\widetilde{D} = \Delta^P(\widetilde{N})$ is a $\mathfrak{p}$-primary (resp. $\mathfrak{p}$-prime, weakly prime) submodule of $\widetilde{M}$.*

Proof. First note that by 2.1d and also 2.2, $\widetilde{D} \neq \widetilde{M}$. Suppose that P is $\mathfrak{p}$-primary and $r\widetilde{m} \in \widetilde{D}$, where $r \in R \setminus \mathfrak{p}$ and $\widetilde{m} \in \widetilde{M}$. For each $\widetilde{n} \in \widetilde{N}$, we have $r\Delta(\widetilde{n},\widetilde{m}) = \Delta(\widetilde{n}, r\widetilde{m}) \in \Delta(\widetilde{N},\widetilde{D}) \subseteq P$, by definition of $\widetilde{D}$. So by P being $\mathfrak{p}$-primary and $r \notin \mathfrak{p}$, we deduce that $\Delta(\widetilde{n},\widetilde{m}) \in P$ for each $\widetilde{n} \in \widetilde{N}$, that is, $\widetilde{m} \in \widetilde{D}$. On the other hand, since $\widetilde{N} \nsubseteq P \oplus \mathfrak{p}$, we deduce that $\pi_1(\widetilde{N}) + \pi_2(\widetilde{N})M \nsubseteq P$. Hence by 2.3,

$$(P : M) \subseteq (\widetilde{D} : \widetilde{M}) = (P : \pi_1(\widetilde{N}) + \pi_2(\widetilde{N})M) \subseteq \sqrt{(P : M)},$$

for P is primary. So $\sqrt{(\widetilde{D} : \widetilde{M})} = \mathfrak{p}$, whence $\widetilde{D}$ is $\mathfrak{p}$-primary. The proof for primeness is similar.

Now assume that P is weakly $\mathfrak{p}$-prime and $r_1r_2\widetilde{m} \in \widetilde{D}$. Thus for each $\widetilde{n} \in \widetilde{N}$, we have $r_1r_2\Delta(\widetilde{m},\widetilde{n}) \in P$. By P being weakly prime, we deduce that either $r_1\Delta(\widetilde{m},\widetilde{n}) \in P$ or $r_2\Delta(\widetilde{m},\widetilde{n}) \in P$. Therefore, either $\Delta(\widetilde{m},\widetilde{N}) \subseteq (P :_M r_1)$ or $\Delta(\widetilde{m},\widetilde{N}) \subseteq (P :_M r_2)$. Consequently, either $r_1\widetilde{m} \in \Delta^P(\widetilde{N})$ or $r_2\widetilde{m} \in \Delta^P(\widetilde{N})$ and $\widetilde{D}$ is weakly prime. □

Under the conditions of the above result, in the case that P is weakly $\mathfrak{p}$-prime, it may happen that $(\widetilde{D} : \widetilde{M}) \neq \mathfrak{p}$, as the following example shows.

Example 2.7. Let $M = R \oplus R$ and $P = \mathfrak{p} \oplus \mathfrak{q}$ for prime ideals $\mathfrak{p} \subset \mathfrak{q}$ of R. One can readily check that P is weakly $\mathfrak{p}$-prime. Set $n = (0,1) \in M$ and $\widetilde{N} = Rn \leq M \leq \widetilde{M}$. Then $\widetilde{D} = \Delta^P(\widetilde{N}) = M \oplus \mathfrak{q}$ and $(\widetilde{D} : \widetilde{M}) = \mathfrak{q}$.

3. Prime submodules and radical of submodules

In this section, assuming that P is a prime submodule of M, we try to find exactly $\Delta^P(\widetilde{N})$ for an arbitrary submodule $\widetilde{N}$ of $\widetilde{M}$ and use it to present a formulation of $\mathrm{rad}(\widetilde{N})$. Throughout this section P is assumed to be a $\mathfrak{p}$-prime submodule of M. First we need the following well-known results (see for example [12]).

Lemma 3.1. *Suppose $K \leq L \leq M, M' \leq M$, $I \subseteq (K : M)$ and let A' be a submodule of an R-module A. Then*

- **(a)** *L is a $\mathfrak{p}$-prime submodule of M if and only if $\frac{L}{K}$ is a $\frac{\mathfrak{p}}{I}$-prime $\frac{R}{I}$-submodule of $\frac{M}{K}$;*
- **(b)** *the smallest $\mathfrak{p}$-prime submodule of M containing L, if any exists, is $(L + \mathfrak{p}M)_{\mathfrak{p}}{}^{c}$. If there is no such prime submodule, then $(L + \mathfrak{p}M)_{\mathfrak{p}}{}^{c} = M$;*
- **(c)** *L is a $\mathfrak{p}$-prime submodule of M if and only if $L = L_{\mathfrak{p}}{}^{c}$ and $L_{\mathfrak{p}}$ is a $\mathfrak{p}_{\mathfrak{p}}$-prime submodule of $M_{\mathfrak{p}}$;*

(d) *if L is $\mathfrak{p}$-prime in M, then $L \cap M'$ is either the whole M' or a $\mathfrak{p}$-prime submodule of M';*

(e) *every proper subspace of a vector space is 0-prime;*

(f) $rad_{M\oplus A}(M' \oplus A') = rad_M(M') \oplus rad_A(A')$.

Now we have all the stuff needed to characterize Δ^P of submodules of $\widetilde{M}$.

Theorem 3.2. *Assume that P is a $\mathfrak{p}$-prime submodule of M and $\widetilde{N} \leq \widetilde{M}$. Then*

(a) *if $\widetilde{N} \subseteq P \oplus \mathfrak{p}$, then $\Delta^P(\widetilde{N}) = \widetilde{M}$;*

(b) *if $\widetilde{N} \nsubseteq P \oplus \mathfrak{p}$ but $\widetilde{N} \subseteq M \oplus \mathfrak{p}$, then $\Delta^P(\widetilde{N}) = M \oplus \mathfrak{p}$;*

(c) *if $\widetilde{N} \nsubseteq M \oplus \mathfrak{p}$ and $\widetilde{N} \cap (M \oplus \mathfrak{p}) \nsubseteq P \oplus \mathfrak{p}$, then $\Delta^P(\widetilde{N}) = P \oplus \mathfrak{p}$;*

(d) *otherwise $\Delta^P(\widetilde{N}) = (\widetilde{N} + (P \oplus \mathfrak{p}))_{\mathfrak{p}}{}^c$ is the smallest $\mathfrak{p}$-prime submodule $\widetilde{P}$ of $\widetilde{M}$ containing $\widetilde{N}$ with $\widetilde{P} \cap M = P$.*

Proof. Case (a) follows from 2.1c and 2.1d. Now suppose that $\widetilde{N} \nsubseteq P \oplus \mathfrak{p}$ but $\widetilde{N} \subseteq M \oplus \mathfrak{p}$, then by 2.1c and 2.1d, we conclude that $M \oplus \mathfrak{p} \subseteq \widetilde{D} = \Delta^P(\widetilde{N})$. Also according to 2.6, $\widetilde{D}$ is a $\mathfrak{p}$-prime submodule of $\widetilde{M}$. But the only $\mathfrak{p}$-prime submodule of $\widetilde{M}$ containing $M \oplus \mathfrak{p}$ is $M \oplus \mathfrak{p}$.

Thus we can assume that $\widetilde{N} \nsubseteq M \oplus \mathfrak{p}$. Let $\widetilde{N}' = \widetilde{N} + P \oplus \mathfrak{p}$. Note that by regularity $\widetilde{N}' \cap (M \oplus \mathfrak{p}) = (\widetilde{N} \cap (M \oplus \mathfrak{p})) + P \oplus \mathfrak{p}$. Consequently, it follows from 2.4 and 3.1b that any of the conditions or the results of (c) or (d) holds for $\widetilde{N}'$ if and only if the same condition or result holds for $\widetilde{N}$. Therefore by replacing $\widetilde{N}$ with $\widetilde{N}'$, we can assume that $P \oplus \mathfrak{p} \subseteq \widetilde{N}$. Again by applying 2.4 and 3.1a, and by passing to $\frac{R}{\mathfrak{p}}$ and $\frac{M}{P}$, we assume that $P = 0 = \mathfrak{p}$. In particular, R is a domain and M is torsion-free.

If case (c) holds, that is, $\widetilde{N} \nsubseteq M$ and $\widetilde{N} \cap M \neq 0$, then there is a $0 \neq m \in \widetilde{N} \cap M$ and $\widetilde{n} \in \widetilde{N}$ with $0 \neq r = \pi_2(\widetilde{n})$. Let $\widetilde{m} \in \widetilde{D}$, then $\Delta(m, \widetilde{m}) \in P = 0$. Thus $\pi_2(\widetilde{m})m = 0$ and as $m \neq 0$ and M is torsion-free, we deduce that $\pi_2(\widetilde{m}) = 0$. Moreover, $0 = \Delta(\widetilde{n}, \widetilde{m}) = -r\pi_1(\widetilde{m})$, so $\pi_1(\widetilde{m}) = 0$ and hence $\widetilde{m} = 0$ as required.

Finally assume that $\widetilde{N} \nsubseteq M$ and $\widetilde{N} \cap M = 0$. As $\widetilde{N}_0 \cap M_0 = 0$ (here X_0 means localization of X on the zero ideal), $\widetilde{N}_0$ is a proper, and according to 3.1e, a 0-prime submodule of $\widetilde{M}_0$. Furthermore, by 3.1b $\widetilde{P} = \widetilde{N}_0{}^c$ is the smallest 0-prime submodule containing $\widetilde{N}$. If $x \in \widetilde{P} \cap M$, then there is a $0 \neq r \in R$ such that $rx \in \widetilde{N} \cap M = 0$. Since M is torsion-free, we get $x = 0$, that is, $\widetilde{P} \cap M = 0 = P$. This proves the second equality of d.

Now note that by 2.5 and 3.1c the first equality of (d) is equivalent to $\Delta^0_{M_0,R_0}(\widetilde{N}_0) = \widetilde{N}_0$. Thus by changing R with R_0, we can assume that R is a field and we just need to prove $\widetilde{D} = \Delta^0(\widetilde{N}) = \widetilde{N}$. Assume that $\widetilde{m} = m + r \in \widetilde{M}$ and $\widetilde{n}_1 = m_1 + r_1 \in \widetilde{N} \setminus M$. Thus $r_1 \neq 0$. Now

$$\widetilde{m} \in \Delta^0(\widetilde{n}_1) \Leftrightarrow rm_1 - r_1 m = \Delta(\widetilde{n}_1, \widetilde{m}) = 0 \Leftrightarrow m = r\frac{m_1}{r_1} \Leftrightarrow m + r \in R(t_{\widetilde{n}_1} + 1),$$

where $t_{\widetilde{n}_1} = \frac{m_1}{r_1}$. Therefore as $\widetilde{N} \cap M = 0$, we have $\widetilde{D} = \Delta^0(\widetilde{N}) = \bigcap_{\widetilde{n} \in \widetilde{N} \setminus M} R(t_{\widetilde{n}} + 1)$. Assume that $\widetilde{n}_2 = m_2 + r_2 \in \widetilde{N} \setminus M$ such that $t_{\widetilde{n}_1} \neq t_{\widetilde{n}_2}$. Then $0 \neq r_2 m_1 - r_1 m_2 = r_2 \widetilde{n}_1 - r_1 \widetilde{n}_2 \in \widetilde{N} \cap M$, a contradiction. It follows that there is a $t \in M$ such that for each $0 \neq \widetilde{n} \in \widetilde{N}$, $t = t_{\widetilde{n}}$ and $\widetilde{n} = \pi_2(\widetilde{n})(t + 1)$. Thus in particular, $\widetilde{N} \subseteq \widetilde{D} = R(t + 1)$. On the other hand since $\widetilde{n}_1 \in \widetilde{N}$ and $r_1 \neq 0$, we see that $t + 1 = \frac{1}{r_1}\widetilde{n} \in \widetilde{N}$, that is, $\widetilde{N} = \widetilde{D}$ and the proof is concluded. □

As an application we get the following characterizations of prime submodules of $\widetilde{M}$.

Corollary 3.3. *For $\widetilde{N} \leq \widetilde{M}$ and a prime ideal $\mathfrak{p}$ of R the following are equivalent.*

(a) *$\widetilde{N}$ is $\mathfrak{p}$-prime.*

(b) *$\widetilde{N} = M \oplus \mathfrak{p}$ or $P = \widetilde{N} \cap M$ is $\mathfrak{p}$-prime in M and either $\widetilde{N} = P \oplus \mathfrak{p}$ or $\Delta^P(\widetilde{N}) = \widetilde{N}$.*

(c) *Either $\widetilde{N} = M \oplus \mathfrak{p}$ or $\widetilde{N} = P \oplus \mathfrak{p}$ or $\widetilde{N} = \Delta^P(\widetilde{m})$ for some $\mathfrak{p}$-prime submodule P of M and an $\widetilde{m} \in \widetilde{M} \setminus (M \oplus \mathfrak{p})$.*

Proof. (a) $\Rightarrow$ (b): Suppose $\widetilde{N} \neq M \oplus \mathfrak{p}$. Then by 3.1d, $P = \widetilde{N} \cap M$ is $\mathfrak{p}$-prime in M. Now since $(\widetilde{N} : \widetilde{M}) = \mathfrak{p}$, we see that $\mathfrak{p}M \oplus \mathfrak{p} \subseteq \widetilde{N}$ and whence $\widetilde{N} \cap (M \oplus \mathfrak{p}) = (\widetilde{N} \cap M) \oplus \mathfrak{p} = P \oplus \mathfrak{p}$. Thus cases (b) and (c) of 3.2 cannot occur. If case 3.2a holds, then $\widetilde{N} = P \oplus \mathfrak{p}$. Else according to 3.2d, $\Delta^P(\widetilde{N}) = {\widetilde{N}_{\mathfrak{p}}}^c = \widetilde{N}$ by 3.1c.

(b) $\Rightarrow$ (c): Assume that neither $\widetilde{N} = P \oplus \mathfrak{p}$ nor $\widetilde{N} = M \oplus \mathfrak{p}$. Then (b) says that $\widetilde{N} = \Delta^P(\widetilde{N})$ for some $\mathfrak{p}$-prime submodule P of M. Clearly cases (a)–(c) of 3.2 cannot happen. Thus $\widetilde{N} \nsubseteq M \oplus \mathfrak{p}$. Let $\widetilde{m} \in \widetilde{N} \setminus (M \oplus \mathfrak{p})$. If $r\widetilde{m} \in M \oplus \mathfrak{p}$, then $r\pi_2(\widetilde{m}) \in \mathfrak{p}$ and as $\pi_2(\widetilde{m}) \notin \mathfrak{p}$, we should have $r \in \mathfrak{p}$. So $r\pi_1(\widetilde{m}) \in \mathfrak{p}M \subseteq P$. Hence $R\widetilde{m} \cap (M \oplus \mathfrak{p}) \subseteq P \oplus \mathfrak{p}$ and $R\widetilde{m}$ satisfies the conditions of 3.2d and

$$\Delta^P(\widetilde{m}) = \Delta^P(R\widetilde{m}) = {(R\widetilde{m} + P \oplus \mathfrak{p})_{\mathfrak{p}}}^c \subseteq {(\widetilde{N} + P \oplus \mathfrak{p})_{\mathfrak{p}}}^c = \Delta^P(\widetilde{N}).$$

On the other hand, $R\widetilde{m} \subseteq \widetilde{N}$ and according to 2.1c, $\Delta^P(\widetilde{N}) \subseteq \Delta^P(\widetilde{m})$. Consequently, $\widetilde{N} = \Delta^P(\widetilde{N}) = \Delta^P(\widetilde{m})$.

(c) $\Rightarrow$(a): If $\widetilde{N} = P \oplus \mathfrak{p}$ or $\widetilde{N} = M \oplus \mathfrak{p}$, then the result is obvious. Assume $\widetilde{N} = \Delta^P(\widetilde{m})$ for a $\mathfrak{p}$-prime submodule P of M and an $\widetilde{m} \in \widetilde{M} \setminus (M \oplus \mathfrak{p})$. Clearly cases (a) and (b) of 3.2 do not occur for $R\widetilde{m}$ and in either of the cases (c) or (d) of the previous theorem, $\Delta^P(R\widetilde{m})$ is a $\mathfrak{p}$-prime submodule of $\widetilde{M}$, as required. □

Using this corollary we can inductively get a characterization of prime submodules of finitely generated free modules. For this first we need some notations. Note that the R-module $\widetilde{M}$ can also be considered as a commutative ring by defining $mm' = 0$ for all $m, m' \in M$ (this ring is usually called the *idealization* of M). Thus we can compute determinants of square matrices with entries in $\widetilde{M}$.

Notation 3.4. Suppose that

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,n} \end{pmatrix}$$

is a $k \times n$ matrix with $k \leq n$ and entries in R. Let $M = R^{n-k+1}$. If in each row of $\mathbf{A}$ we consider the first $n - k + 1$ entries as an element of $M \leq \widetilde{M}$ and the other entries of the row as elements of $R \leq \widetilde{M}$, then we denote the determinant of $\mathbf{A}$ by $\det_{n-k+1}(\mathbf{A})$ which is an element of R^{n-k+1}.

Example 3.5. Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ -3 & 4 & 2 \end{pmatrix}$ over $\mathbb{Z}$. Then $\det_2(\mathbf{A}) = (1, 2)2 - 5(-3, 4) = (17, -16) \in \mathbb{Z}^2$.

In what follows, we consider two submodules A and B of R^n the same, *up to a permutation of coordinates*, if there exists a permutation $\sigma \in S_n$, such that $(x_1, \ldots, x_n) \in A \Leftrightarrow (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in B$. If I is an ideal of R, by I^n we mean the submodule $\underbrace{I \times I \times \cdots \times I}_{n \text{ times}}$ of R^n.

Theorem 3.6. *Assume that $\mathfrak{p}$ is a prime ideal of R and $\widetilde{P} < R^n$. Then $\widetilde{P}$ is a $\mathfrak{p}$-prime submodule of R^n if and only if there exist an integer $0 \leq k < n$ and $a_{i,j} \in R$ with $1 \leq i \leq k$ and $1 \leq j \leq n-i+1$, such that $a_{i,n-i+1} \notin \mathfrak{p}$ and up to a permutation of coordinates $\widetilde{P} = \{(x_1, \dots, x_n) | \det_{n-k}(\mathbf{A}(x_1, \dots, x_n)) \in \mathfrak{p}^{n-k}\}$, where*

$$\mathbf{A}(x_1, \dots, x_n) = \begin{pmatrix} x_1 & \dots & \dots & \dots & \dots & x_n \\ a_{1,1} & \dots & \dots & \dots & \dots & a_{1,n} \\ a_{2,1} & \dots & \dots & \dots & a_{2,n-1} & 0 \\ a_{3,1} & \dots & \dots & a_{3,n-2} & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & \dots & a_{k,n-k+1} & 0 & \dots & 0 \end{pmatrix}.$$

Before stating the proof, it should be noted that in the case $k = 0$, we have $\det_{n-k}(\mathbf{A}(x_1, \dots, x_n)) = (x_1, \dots, x_n)$ and hence the condition in this case holds if and only if $\widetilde{P} = \mathfrak{p}^n$.

Proof. ($\Rightarrow$): We prove the result by induction on n. If $n = 1$, then $\widetilde{P} = \mathfrak{p}$ and the result holds with $k = 0$, according to the above remark. Assume $n > 1$. If $\widetilde{P} = \mathfrak{p}^n$, then again the result holds by the above note. Thus we assume that an entry of an element of $\widetilde{P}$ is not in $\mathfrak{p}$. Since we are working up to a permutation of coordinates, we assume that this entry is on the last coordinate. Hence in part c of 3.3 (with $M = R^{n-1}$ and $\widetilde{N} = \widetilde{P}$) the first two cases cannot happen. Therefore, we have $\widetilde{P} = \Delta^P(\widetilde{m})$ for some $\mathfrak{p}$-prime submodule P of R^{n-1} and an $\widetilde{m} \in R^n \setminus (R^{n-1} \oplus \mathfrak{p})$. Suppose that $\widetilde{m} = (a_{1,1}, \dots, a_{1,n})$. Then $a_{1,n} \notin \mathfrak{p}$.

By induction hypothesis, up to a permutation of coordinates $P = \{(x_1, \dots, x_{n-1}) | \det_{n-k}(\mathbf{B}(x_1, \dots, x_{n-1})) \in \mathfrak{p}^{n-k}\}$, where

$$\mathbf{B}(x_1, \dots, x_{n-1}) = \begin{pmatrix} x_1 & \dots & \dots & \dots & \dots & x_{n-1} \\ a_{2,1} & \dots & \dots & \dots & \dots & a_{2,n-1} \\ a_{3,1} & \dots & \dots & \dots & a_{3,n-2} & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & \dots & a_{k,n-k+1} & 0 & \dots & 0 \end{pmatrix}, \qquad (*)$$

for suitable k and $a_{i,j}$ (note that we have started the first indices of $a_{i,j}$'s in $\mathbf{B}$ from 2 instead of 1). Now:

$$\begin{aligned} & (x_1, \dots, x_n) \in \widetilde{P} = \Delta^P(\widetilde{m}) \\ \Leftrightarrow\ & m := x_n(a_{1,1}, \dots, a_{1,n-1}) - a_{1,n}(x_1, \dots, x_{n-1}) \in P \\ \Leftrightarrow\ & \det_{n-k}(\mathbf{B}(m)) \in \mathfrak{p}^{n-k} \qquad (**) \\ \Leftrightarrow\ & x_n \det_{n-k}(\mathbf{B}(a_{1,1}, \dots, a_{1,n-1})) - a_{1,n} \det_{n-k}(\mathbf{B}(x_1, \dots, x_{n-1})) \in \mathfrak{p}^{n-k} \\ \Leftrightarrow\ & \det_{n-k}(\mathbf{A}(x_1, \dots, x_n)) \in \mathfrak{p}^{n-k}, \end{aligned}$$

where the last equivalency holds by expanding $\det_{n-k}(\mathbf{A}(x_1, \dots, x_n))$.

($\Leftarrow$): We use induction on n. If $n = 1$, then $k = 0$ and $\widetilde{P} = \mathfrak{p}$ is a prime submodule of R. Assume $n > 1$. For $(x_1, \dots, x_{n-1}) \in R^{n-1}$, let $B(x_1, \dots, x_{n-1})$ be defined by $*$. Set $P = \{(x_1, \dots, x_{n-1}) | \det_{n-k}(\mathbf{B}(x_1, \dots, x_{n-1})) \in \mathfrak{p}^{n-k}\}$ which is a prime submodule of $M = R^{n-1}$ by induction hypothesis. Now $**$ shows that $\widetilde{P} = \Delta^P(\widetilde{m})$ where $\widetilde{m} = (a_{1,1}, \dots, a_{1,n})$. Note that since $a_{1,n} \notin \mathfrak{p}$, we have $\widetilde{m} \in \widetilde{M} \setminus (M \oplus \mathfrak{p})$. Thus by 3.3c, $\widetilde{P}$ is a prime submodule of $\widetilde{M} = R^n$. □

The above theorem should be compared with [15, Theorem 1.6], which presents a characterization of prime submodules of R^n using determinants of some matrices. In Theorem 1.6 of [15], for checking if a submodule of R^n is prime, one should consider all k-minors of a certain matrix. But in Theorem 3.6, we just need to find one determinant. Also the matrix in 3.6, has a simpler form (it has many zeros) than the matrix in [15].

Next we are going to present a formulation for $\mathrm{rad}(\widetilde{N})$ where $\widetilde{N}$ is an arbitrary submodule of $\widetilde{M}$. For this we need the following lemma.

Lemma 3.7. *Assume that $\mathfrak{p}$ is a prime ideal of R and $\widetilde{N} \leq \widetilde{M}$ is such that $\widetilde{N} \nsubseteq M \oplus \mathfrak{p}$ and set $N_1 = \widetilde{N} \cap M$, and $N_2 = \pi_1(\widetilde{N} \cap (M \oplus \mathfrak{p}))$. Then $(N_1 + \mathfrak{p}M)_{\mathfrak{p}}{}^c = (N_2 + \mathfrak{p}M)_{\mathfrak{p}}{}^c$.*

Proof. ($\subseteq$): It is satisfied since $N_1 \subseteq N_2$.
($\supseteq$): We just need to show that $N_2 \subseteq (N_1 + \mathfrak{p}M)_{\mathfrak{p}}{}^c$. Let $n_2 \in N_2$. Then by definition of N_2, there should exist an $\tilde{n} \in \widetilde{N} \cap (M \oplus \mathfrak{p})$ such that $\tilde{n} = n_2 + r$ for some $r \in R$. Thus $r \in \mathfrak{p}$. By assumption there are $s \in R \setminus \mathfrak{p}$ and $m \in M$ with $m + s \in \widetilde{N}$. Now $x = s(n_2 + r) - r(m + s) = sn_2 - rm \in \widetilde{N} \cap M = N_1$ and $sn_2 = x + rm \in N_1 + \mathfrak{p}M$. Consequently, $n_2 \in (N_1 + \mathfrak{p}M)_{\mathfrak{p}}{}^c$, and the result follows. □

Theorem 3.8. *Suppose that $\widetilde{N} \leq \widetilde{M}$ and set $N_1 = \widetilde{N} \cap M$. Then*

$$rad_{\widetilde{M}}(\widetilde{N}) = \left(rad_M(\pi_1(\widetilde{N})) \oplus \sqrt{\pi_2(\widetilde{N})} \right) \cap \Delta^{rad_M(N_1)}(\widetilde{N}).$$

Proof. ($\subseteq$): By 3.1f, we have $\mathrm{rad}_{\widetilde{M}}(\widetilde{N}) \subseteq \mathrm{rad}_M(\pi_1(\widetilde{N})) \oplus \sqrt{\pi_2(\widetilde{N})}$, because $\widetilde{N} \subseteq \pi_1(\widetilde{N}) \oplus \pi_2(\widetilde{N})$. According to 3.1b, $\mathrm{rad}_M(N_1) = \bigcap_{\mathfrak{p} \in \mathrm{spec}(R)} (N_1 + \mathfrak{p}M)_{\mathfrak{p}}{}^c$, where $\mathrm{spec}(R)$ denotes the set of prime ideals of R. Therefore by 2.1f,

$$\Delta^{\mathrm{rad}_M(N_1)}(\widetilde{N}) = \bigcap_{\mathfrak{p} \in \mathrm{spec}(R)} \Delta^{(N_1 + \mathfrak{p}M)_{\mathfrak{p}}{}^c}(\widetilde{N}).$$

Thus if $\mathfrak{p} \in \mathrm{spec}(R)$ and $P = (N_1 + \mathfrak{p}M)_{\mathfrak{p}}{}^c$, we just need to show that $\mathrm{rad}_{\widetilde{M}}(\widetilde{N}) \subseteq \Delta^P(\widetilde{N})$. If $P = M$ we get $\Delta^P(\widetilde{N}) = \widetilde{M} \supseteq \mathrm{rad}_{\widetilde{M}}(\widetilde{N})$. So assume $P \neq M$, hence P is a $\mathfrak{p}$-prime submodule of M, by 3.1b. If $\widetilde{N} \subseteq M \oplus \mathfrak{p}$, then by 3.2, $\Delta^P(\widetilde{N})$ is either $M \oplus \mathfrak{p}$ or $\widetilde{M}$. In both cases clearly $\mathrm{rad}_{\widetilde{M}}(\widetilde{N}) \subseteq \Delta^P(\widetilde{N})$.

Thus we can assume that $\widetilde{N} \nsubseteq M \oplus \mathfrak{p}$. According to previous lemma, if $N_2 = \pi_1(\widetilde{N} \cap (M \oplus \mathfrak{p}))$, then $P = (N_2 + \mathfrak{p}M)_{\mathfrak{p}}{}^c$, in particular, $N_2 \subseteq P$. Hence $\widetilde{N} \cap (M \oplus \mathfrak{p}) \subseteq N_2 \oplus \mathfrak{p} \subseteq P \oplus \mathfrak{p}$. So by 3.2, $\Delta^P(\widetilde{N})$ is a $\mathfrak{p}$-prime submodule containing $\widetilde{N}$ and hence $\mathrm{rad}_{\widetilde{M}}(\widetilde{N})$, as required.

($\supseteq$): Let $\mathfrak{p}$ be an arbitrary prime ideal of R and set $\widetilde{P} = (\widetilde{N} + \mathfrak{p}\widetilde{M})_{\mathfrak{p}}{}^c$. We just need to show that the right hand side of the claimed equality is contained in $\widetilde{P}$. If $\widetilde{P} = P \oplus I$ where I is either R or $\mathfrak{p}$ and P is either M or a $\mathfrak{p}$-prime submodule of M, then as $\widetilde{N} \subseteq \widetilde{P}$ we get $\pi_1(\widetilde{N}) \subseteq P$ and $\pi_2(\widetilde{N}) \subseteq I$. Therefore $\mathrm{rad}_M(\pi_1(\widetilde{N})) \oplus \sqrt{\pi_2(\widetilde{N})} \subseteq \widetilde{P}$, as required. Thus we assume that $\widetilde{P}$ is not in the form mentioned above.

Set $N_2 = \pi_1(\widetilde{N} \cap (M \oplus \mathfrak{p}))$ and $P = (N_2 + \mathfrak{p}M)_{\mathfrak{p}}{}^c$. If $n_2 \in N_2$, then for some $r \in \mathfrak{p}$ we have $n_2 + r \in \widetilde{N}$. So $n_2 \in \widetilde{N} + \mathfrak{p}\widetilde{M}$. Therefore, $(N_2 + \mathfrak{p}M) \oplus \mathfrak{p} \subseteq \widetilde{N} + \mathfrak{p}\widetilde{M}$, whence $P \oplus \mathfrak{p} = ((N_2 + \mathfrak{p}M) \oplus \mathfrak{p})_{\mathfrak{p}}{}^c \subseteq (\widetilde{N} + \mathfrak{p}\widetilde{M})_{\mathfrak{p}}{}^c = \widetilde{P}$. Thus if $P = M$, then $\widetilde{P}$ should be either $M \oplus R$ or $M \oplus \mathfrak{p}$, against what we assumed above. Hence P is a $\mathfrak{p}$-prime submodule of M. Also if $\widetilde{N} \subseteq P \oplus \mathfrak{p}$, then the $\mathfrak{p}$-prime submodule $P \oplus \mathfrak{p}$ should contain $\widetilde{P}$ which is the smallest $\mathfrak{p}$-prime submodule of $\widetilde{M}$ containing $\widetilde{N}$. Therefore, $\widetilde{P} = P \oplus \mathfrak{p}$, again contradicting our assumption. Moreover, $\widetilde{N} \cap (M \oplus \mathfrak{p}) \subseteq N_2 \oplus \mathfrak{p} \subseteq P \oplus \mathfrak{p}$. Consequently, we see that P and $\widetilde{N}$ satisfy the conditions of case d of 3.2. It follows that

$$\widetilde{P} = (\widetilde{N} + \mathfrak{p}\widetilde{M})_{\mathfrak{p}}{}^c \subseteq \Delta^P(N) = (\widetilde{N} + (P \oplus \mathfrak{p}))_{\mathfrak{p}}{}^c \subseteq \widetilde{P}_{\mathfrak{p}}{}^c = \widetilde{P},$$

that is, $\widetilde{P} = \Delta^P(\widetilde{N})$. Because $\widetilde{N}$ satisfies case d of 3.2, we deduce that $\widetilde{N} \nsubseteq M \oplus \mathfrak{p}$. Therefore according to 3.7, $P = (N_1 + \mathfrak{p}M)_{\mathfrak{p}}{}^c \supseteq \mathrm{rad}(N_1)$ and $\widetilde{P} = \Delta^P(\widetilde{N}) \supseteq \Delta^{\mathrm{rad}(N_1)}(\widetilde{N})$, as claimed. □

The following example shows how we can apply 3.8.

Example 3.9. Assume that $M = R = \mathbb{Z}$ and $\widetilde{N} = \mathbb{Z}(2,2) + \mathbb{Z}(3,0)$. Then $\pi_1(\widetilde{N}) = \mathbb{Z}$, $\pi_2(\widetilde{N}) = 2\mathbb{Z}$ and $N_1 = \widetilde{N} \cap M = \widetilde{N} \cap (\mathbb{Z} \oplus 0) = 3\mathbb{Z}$. By definition

$$\begin{aligned}\Delta^{3\mathbb{Z}}(\widetilde{N}) &= \Delta^{3\mathbb{Z}}(2,2) \cap \Delta^{3\mathbb{Z}}(3,0)\\ &= \{(n_1,n_2)|2n_1 - 2n_2 \in 3\mathbb{Z}\} \cap \{(n_1,n_2)|3n_2 \in 3\mathbb{Z}\}\\ &= \{(n_1,n_2)|n_1 - n_2 \in 3\mathbb{Z}\}.\end{aligned}$$

Therefore according to 3.8,

$$\begin{aligned}\mathrm{rad}(\widetilde{N}) &= (\sqrt{\mathbb{Z}} \oplus \sqrt{2\mathbb{Z}}) \cap \Delta^{3\mathbb{Z}}(\widetilde{N}) = (\mathbb{Z} \oplus 2\mathbb{Z}) \cap \{(n_1,n_2)|n_1 - n_2 \in 3\mathbb{Z}\}\\ &= \{(3t+2k, 2k)|t,k \in \mathbb{Z}\} = \widetilde{N},\end{aligned}$$

that is, $\widetilde{N}$ is a radical submodule of $\widetilde{M}$.

4. Delta operation and primary decompositions

At this final section we pay some attention to primary decompositions of submodules and their behavior under the delta operation. Recall that if $A = \bigcap_{i=1}^{n} Q_i$ is a minimal primary decomposition of a submodule A of M, then $\mathrm{Ass(A)} = \{\sqrt{(\mathrm{Q_i : M})}\}_{\mathrm{i=1}}^{\mathrm{n}}$ and $\min(A)$ is the set of minimal elements of Ass(A).

Theorem 4.1. *Suppose that $A = \bigcap_{i=1}^{n} Q_i$ is a minimal primary decomposition and $\widetilde{A} = \Delta^A(\widetilde{N})$ for some submodule $\widetilde{N}$ of $\widetilde{M}$ such that $\widetilde{N} \not\subseteq Q_i \oplus (Q_i : M)$ for each i. Then $\widetilde{A} = \bigcap_{i=1}^{n} \Delta^{Q_i}(\widetilde{N})$ is a primary decomposition of $\widetilde{A}$ and $\mathrm{Ass}(\widetilde{\mathrm{A}}) \subseteq \mathrm{Ass(A)}$. Moreover $\min(\widetilde{A}) = \min(A)$ and if $\widetilde{N} \not\subseteq M \oplus \mathfrak{p}$ for each embedded prime $\mathfrak{p}$ of A, then this primary decomposition of $\widetilde{A}$ is minimal.*

Proof. The first statement follows from 2.1f and 2.6. Suppose that this primary decomposition of $\widetilde{A}$ is not minimal and let $\mathfrak{p}_i = \sqrt{(Q_i : M)}$. Then for some $1 \leq i \leq n$, we should have $\bigcap_{i\neq j=1}^{n} \Delta^{Q_j}(\widetilde{N}) \subseteq \Delta^{Q_i}(\widetilde{N})$. Then $\left(\bigcap_{i\neq j=1}^{n} Q_j\right) \oplus 0 \subseteq \bigcap_{i\neq j=1}^{n}(Q_j \oplus (Q_j : M)) \subseteq \Delta^{Q_i}(\widetilde{N})$ by 2.4. Hence $\pi_2(\widetilde{N})\left(\bigcap_{i\neq j=1}^{n} Q_j\right) = \Delta\left(\left(\bigcap_{i\neq j=1}^{n} Q_j\right) \oplus 0, \widetilde{N}\right) \subseteq Q_i$. Because of the minimality of the primary decomposition of A and the fact that Q_i is $\mathfrak{p}_i$-primary, we deduce that $\pi_2(\widetilde{N}) \subseteq \mathfrak{p}_i$, that is, $\widetilde{N} \subseteq M \oplus \mathfrak{p}_i$. On the other hand,

$$\bigcap_{i\neq j=1}^{n} \mathfrak{p}_j = \bigcap_{i\neq j=1}^{n} \sqrt{(\Delta^{Q_j}(\widetilde{N}) : \widetilde{M})} \subseteq \sqrt{(\Delta^{Q_i}(\widetilde{N}) : \widetilde{M})} = \mathfrak{p}_i,$$

where the flanking equalities follow from 2.6. Consequently $\mathfrak{p}_j \subseteq \mathfrak{p}_i$ for some $j \neq i$ and $\mathfrak{p}_i$ is an embedded prime of A, and the second statement is established. □

The above theorem proposes the question "when a submodule $\widetilde{A}$ of $\widetilde{M}$ is of the form $\Delta^B(\widetilde{N})$ for some submodules B and $\widetilde{N}$ of M and $\widetilde{M}$, respectively?" Regarding this, we have:

Proposition 4.2. *Suppose that $\widetilde{N} = \Delta^A(\widetilde{K})$ for some $\widetilde{K} \leq \widetilde{M}$ and $(A :_M \pi_2(\widetilde{N})) = A$ (for example, if $\pi_2(\widetilde{N}) \not\subseteq Z\left(\frac{M}{A}\right) = \{r \in R | \exists m \in M \setminus A :\ rm \in A\}$). Then $\widetilde{N} = \Delta^B(\widetilde{N})$ where $B = \Delta(\widetilde{N}, \widetilde{N})$.*

Proof. Let $B' = \left(A :_M \pi_2(\Delta^A(\widetilde{N}))\right)$. First we show that $\Delta^A(\Delta^A(\widetilde{N})) = \Delta^{B'}(\widetilde{N})$. In the following for any element $\widetilde{m} \in \widetilde{M}$ we denote $\pi_i(\widetilde{m})$ by m_i $(i = 1, 2)$.

($\subseteq$): Assume that $\widetilde{x} \in \Delta^A(\Delta^A(\widetilde{N})), \widetilde{n} \in \widetilde{N}$ and $\widetilde{a} \in \Delta^A(\widetilde{N})$. Then $z = a_1n_2 - a_2n_1 = \Delta(\widetilde{a}, \widetilde{n}) \in A$ and $\Delta(\widetilde{a}, \widetilde{x}) \in A$. Thus

$$\begin{aligned} a_2\Delta(\widetilde{n}, \widetilde{x}) &= a_2(n_1x_2 - n_2x_1) = (a_2n_1)x_2 - a_2n_2x_1 \\ &= (a_1n_2 - z)x_2 - a_2n_2x_1 = n_2(a_1x_2 - a_2x_1) - zx_2 \\ &= n_2\Delta(\widetilde{a}, \widetilde{x}) - zx_2 \in A, \end{aligned}$$

that is, $\Delta(\widetilde{n}, \widetilde{x}) \in (A :_M a_2)$. Since $\widetilde{a} \in \Delta^A(\widetilde{N})$ was arbitrary, we conclude that $\Delta(\widetilde{n}, \widetilde{x}) \in (A :_M \pi_2(\Delta^A(\widetilde{N}))) = B'$, and because $\widetilde{n}$ was arbitrary we deduce $\widetilde{x} \in \Delta^{B'}(\widetilde{N})$ and hence $\Delta^A(\Delta^A(\widetilde{N})) \subseteq \Delta^{B'}(\widetilde{N})$.

($\supseteq$): Let $\widetilde{x} \in \Delta^{B'}(\widetilde{N}), \widetilde{n} \in \widetilde{N}$ and $\widetilde{a} \in \Delta^A(\widetilde{N})$. Then $z = a_1n_2 - n_1a_2 \in A$ and $\Delta(\widetilde{n}, \widetilde{x}) \in A$

$$\begin{aligned} n_2\Delta(\widetilde{a}, \widetilde{x}) &= (n_2a_1)x_2 - n_2a_2x_1 = (a_2n_1 + z)x_2 - n_2a_2x_1 \\ &= a_2(n_1x_2 - n_2x_1) + zx_2 = a_2\Delta(\widetilde{n}, \widetilde{x}) + zx_2 \in A, \end{aligned}$$

which similarly to the ($\subseteq$) case, gives $\widetilde{x} \in \Delta^{(A:_M\pi_2(\widetilde{N}))}(\Delta^A(\widetilde{N})) = \Delta^A(\Delta^A(\widetilde{N}))$ according to the assumption of the theorem. So $\Delta^{B'}(\widetilde{N}) \subseteq \Delta^A(\Delta^A(\widetilde{N}))$.

Now note that since $\widetilde{N} = \Delta^A(\widetilde{K})$ for some $\widetilde{K} \leq \widetilde{M}$, $\widetilde{N} = \Delta^A(\Delta^A(\widetilde{N})) = \Delta^{B'}(\widetilde{N})$ according to 2.2. In particular, $B = \Delta(\widetilde{N}, \widetilde{N}) \subseteq B'$. Consequently, $\widetilde{N} \subseteq \Delta^B(\widetilde{N}) \subseteq \Delta^{B'}(\widetilde{N}) = \widetilde{N}$, and hence $\widetilde{N} = \Delta^B(\widetilde{N})$. □

This suggests to search for submodules $\widetilde{N}$ of $\widetilde{M}$ with $\widetilde{N} = \Delta^B(\widetilde{N})$ with $B = \Delta(\widetilde{N}, \widetilde{N})$.

Theorem 4.3. *Suppose that M is torsion-free and $B = \Delta(\widetilde{N}, \widetilde{N}) \neq 0$ and is cyclic. Then $\widetilde{N} = \Delta^B(\widetilde{N})$. In particular, if B is proper and has a minimal primary decomposition $B = \bigcap_{i=1}^n Q_i$, then $\widetilde{N} = \bigcap_{i=1}^n \Delta^{Q_i}(\widetilde{N})$ is a primary decomposition of $\widetilde{N}$.*

Proof. By assumption $B = Rd$, where $0 \neq d = \sum_{i=1}^k a_i\alpha_i$ with $a_i \in R$, $0 \neq \alpha_i = \Delta(\widetilde{m}_i, \widetilde{n}_i)$ and $\widetilde{m}_i, \widetilde{n}_i \in \widetilde{N}$. In particular as $\alpha_i \in B$, there exists $s_i \in R$, with $\alpha_i = s_id$. Thus $d = \left(\sum_{i=1}^k a_is_i\right) d$ and since M is torsion-free we get $\sum_{i=1}^k a_is_i = 1 \quad (*)$. By definition of B, it is obvious that $\widetilde{N} \subseteq \Delta^B(\widetilde{N})$. For the converse inclusion, let $\widetilde{x} \in \Delta^B(\widetilde{N})$ be arbitrary. Then $\Delta(\widetilde{x}, \widetilde{m}_i), \Delta(\widetilde{x}, \widetilde{n}_i) \in B = Rd$, that is, there are $r_i, r'_i \in R$ such that for each i:

$$\begin{cases} m_{i2}x_1 - m_{i1}x_2 = r_id & (1) \\ n_{i2}x_1 - n_{i1}x_2 = r'_id & (2) \end{cases},$$

where for any $j = 1, 2$ and $\widetilde{z} \in \widetilde{M}$ we have set $z_j = \pi_j(\widetilde{z})$. Now if we set $c_i = n_{i2}r_i - m_{i2}r'_i$, then by subtracting m_{i2} times Eq. (2) from n_{i2} times Eq. (1) it follows that $-\alpha_ix_2 = c_id$. Hence $c_i\alpha_i = c_is_id = -s_ix_2\alpha_i$ and so by torsion-freeness, $c_i = -s_ix_2 \quad (**)$. Also $dx_2 = \sum_{i=1}^k a_i\alpha_ix_2 = -\sum_{i=1}^k a_ic_id$. Therefore,

$$x_2 = -\sum_{i=1}^k a_ic_i = \sum_{i=1}^k (a_ir'_im_{i2} - a_ir_in_{i2}) \quad (3).$$

Let $1 \leq i \leq k$ be such that $m_{i2} \neq 0$. Then from (1) we deduce that $s_im_{i2}x_1 = s_ir_id + s_im_{i1}x_2 = r_i\alpha_i - c_im_{i1}$ (by $(**)$). Replacing c_i and α_i with their definitions, we get that $s_im_{i2}x_1 = m_{i2}m_{i1}r'_i - r_im_{i2}n_{i1}$. Cancelling out m_{i2}'s, we conclude $s_ix_1 = r'_im_{i1} - r_in_{i1}$. If i is such that $m_{i2} = 0$, then $n_{i2} \neq 0$ (else $\alpha_i = 0$), so again we deduce the same equation for s_ix_1, using Eq. (2) instead of (1). Now summing up over all i's and using $(*)$ we see that

$$x_1 = \sum_{i=1}^k a_is_ix_1 = \sum_{i=1}^k (a_ir'_im_{i1} - a_ir_in_{i1}) \quad (4).$$

Adding Eq. (3) and (4) we finally get $\widetilde{x} = x_1 + x_2 = \sum_{i=1}^{k}(a_i r_i' \widetilde{m}_i - a_i r_i \widetilde{n}_i) \in \widetilde{N}$, as required.

For the "in particular" statement, by 4.1, we just need to show that $\widetilde{N} \not\subseteq Q_i \oplus (Q_i : M)$ for each $1 \leq i \leq n$. Suppose that for $i \leq t$ we have $\widetilde{N} \not\subseteq Q_i \oplus (Q_i : M)$ and for $t < i \leq n$ we have $\widetilde{N} \subseteq Q_i \oplus (Q_i : M)$. If $t = 0$, then $\widetilde{N} = \cap_{i=1}^{n} \Delta^{Q_i}(\widetilde{N}) = \cap_{i=1}^{n} \widetilde{M} = \widetilde{M}$ by 2.1. Consequently, for each $m \in M$, $m = \Delta(m + 0, 0 + 1) \in \Delta(\widetilde{N}, \widetilde{N}) = B$, that is, $B = M$ contradicting the properness assumption on B.

Thus $t > 0$. Note that since $\Delta^{Q_i}(\widetilde{N}) = \widetilde{M}$ for each $t < i \leq n$, we get $\widetilde{N} = \bigcap_{i=1}^{t} \Delta^{Q_i}(\widetilde{N})$ which contains $\bigcap_{i=1}^{t}(Q_i \oplus (Q_i : M))$ by 2.4. On the other hand, if $t < n$ for any $t < j \leq n$ and by 2.2, 2.4 and 2.1, $\widetilde{N} \subseteq \Delta^{Q_j}(\Delta^{Q_j}(\widetilde{N})) = \Delta^{Q_j}(\widetilde{M}) = Q_j \oplus (Q_j : M)$. Therefore, $\bigcap_{i=1}^{t}(Q_i \oplus (Q_i : M)) \subseteq Q_j \oplus (Q_j : M)$, hence $\bigcap_{i=1}^{t} Q_i \subseteq Q_j$ which contradicts the minimality of the decomposition of B. So $t = n$ and the result is established. □

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