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Valid Inequalities for the Maximal Matching Polytope

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Abstract

Given a graph $G = (V, E)$, a subset M of E is called a matching if no two edges in M are adjacent. A matching is said to be maximal if it is not a proper subset of any other matching. The maximal matching polytope associated with graph G is the convex hull of the incidence vectors of maximal matchings in G. In this paper, we introduce new classes of valid inequalities for the maximal matching polytope.

Keywords: Matching, Maximal matching, Valid inequalities, Matching polytope, Maximal matching polytope

Maksimal Eşleme Politopu için Geçerli Eşitsizlikler

Öz

Verilen bir $G = (V, E)$ çizgesinde, uçları kesişmeyen kenarlardan oluşan $M \subseteq E$ kümesine eşleme denir. Herhangi başka bir eşlemenin öz altkümesi olmayan bir eşlemeye maksimal eşleme denir. G çizgesindeki maksimal eşlemelerin insidans vektörlerinin konveks örtüsüne G kümesi ile ilişkili maksimal eşleme politopu adı verilir. Bu çalışmada, maksimal eşleme politopu için yeni geçerli eşitsizlik sınıfları sunulmaktadır.

Anahtar Kelimeler: Eşleme, Maksimal eşleme, Geçerli eşitsizlikler, Eşleme politopu, Maksimal eşleme politopu

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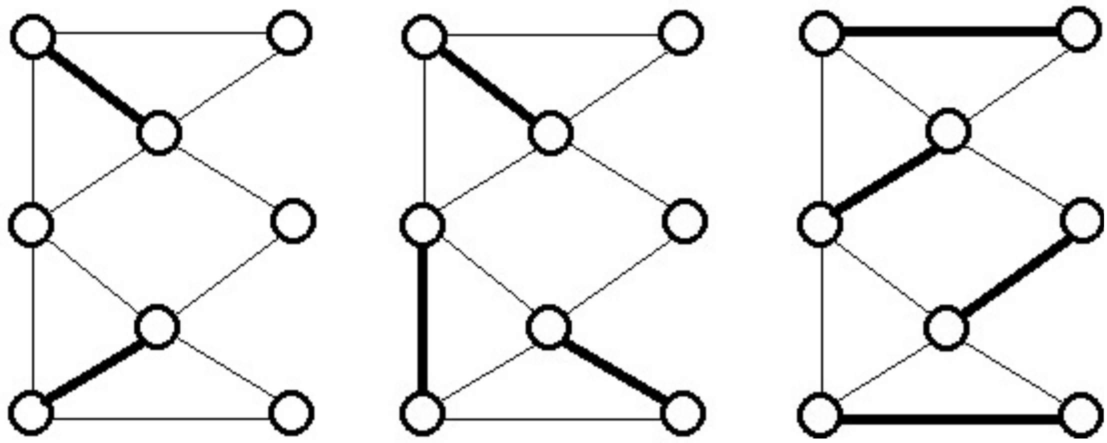
1. Introduction

Let $G = (V, E)$ be a simple undirected graph where V is the set of vertices and E is the set of edges of G. If $e = \{i, j\} \in E$, we say that the vertices i and j are adjacent. Two edges e, f $(e \neq f)$ are said to be adjacent if $e \cap f \neq \emptyset$. A vertex i is incident to an edge e if $i \in e$. The set of all edges incident to a vertex $i \in V$ is denoted by E_i. The cardinality of E_i is called the degree of vertex i. Given two graphs $G = (V, E)$ and $H = (\bar{V}, \bar{E})$, we say that H is a subgraph of G, denoted by $H \trianglelefteq G$, if $\bar{V} \subseteq V$ and $\bar{E} \subseteq E$. Following [1] and [2], we call a subgraph Q of G a clique if every pair of vertices in Q are adjacent in Q. A subgraph C of G is said to be a cycle if C is a connected graph and the degree of each vertex in C is two. A cycle in G is called an odd cycle if it has an odd number of vertices and an even cycle if has an even number of vertices. For a subgraph H of G, $V(H)$ and $E(H)$ represent the set of vertices and edges of H, respectively.

Given a graph $G = (V, E)$, a subset I of V is called an independent set if no two vertices in I are adjacent. A subset M of E is called a matching if no two edges in M are adjacent. Given a matching M in $G = (V, E)$, any vertex in $\cup_{e \in M} e$ is said to be matched and any vertex in $V - \cup_{e \in M} e$ is said to be unmatched by the matching M. If all the vertices in V are matched by the matching M, then M is called a perfect matching. A matching is said to be maximal if it is not a proper subset of any other matching. In Fig. 1, three maximal matchings are displayed for a given graph, where the bold edges represent the edges selected in each maximal matching. Note that the maximal matching in Fig. 1c is also a perfect matching.

Given an edge weight w_e for each $e \in E$, the minimum weighted maximal matching (MWMM) problem is the problem of finding a maximal matching in $G = (V, E)$ of minimum weight, where the weight of a maximal matching M is defined as $\cup_{e \in M} w_e$ [3]. When all the edge weights are 1, the MWMM problem becomes the problem of finding a maximal matching of minimum cardinality and is called the minimum maximal matching (MMM) problem. The MMM problem is in general NP-hard [4] and therefore the MWMM problem is in general NP-hard as well, whereas the problems of finding a minimum/maximum weighted matching or minimum/maximum weighted perfect matching are all solvable in polynomial time, see, e.g., [5]. In particular, the MMM problem has been shown to be NP-hard in bipartite graphs with maximum degree 3 [4], planar cubic graphs [6], and k-regular bipartite graphs for any $k \geq 3$ [7]. On the positive side, it has been shown that the MMM problem can be solved in polynomial time in some classes of graphs including trees [8], series-parallel graphs [9], and block graphs [10]. There are also some results in the literature that investigates the MMM problem from an approximation point of view, see e.g., [11, 12].



(a) (b) (c)

Figure 1. Examples of maximal matchings

The MWMM problem is used in applications where it is desirable to obtain a maximum weighted matching in a network (with positive edge weights) when the user has no control on the edges that are going to be selected in the matching. In such cases, the solution of the MWMM problem provides the worst-case performance of the network when the network is saturated (i.e., the minimum possible weight in a maximal matching). Yannakakis and Gavril provide such an application in a telephone switching network in [4]. For other applications of the MWMM, the reader is referred to [7].

Most of the studies in the literature on the MWMM problem explores the unweighted case, i.e., the MMM problem. These results, however, do not immediately generalize to the weighted case, i.e., to the MWMM problem [3]. Among the few studies on the MWMM problem with arbitrary edge weights, Taşkın and Ekim [3] propose an integer programming (IP) formulation for the MWMM problem. This is the first study on the MWMM problem from an optimization point of view. Moreover, they propose some valid inequalities for the problem. The same authors propose a decomposition algorithm for the MWMM problem in [13]. More recently, Tural [14] investigates the IP formulation proposed in [3] and shows that the linear programming (LP) relaxation of it always returns an integral solution for trees and therefore is able to solve the MWMM problem in trees. In other words, the LP relaxation of the IP formulation proposed in [3] characterizes the maximal matching polytope in trees.

In this paper, we propose several new classes of valid inequalities for the maximal matching polytope. The structure of the paper is as follows. In the next section, we describe the IP formulation for the MWMM problem proposed in [3] and discuss the known results about the maximal matching polytope including the valid inequalities. We then propose several new classes of valid inequalities for the maximal matching polytope.

2. An IP Formulation for the MWMM Problem and Known Valid Inequalities

Given a graph $G = (V, E)$ and an edge weight w_e for each $e \in E$, an IP formulation is introduced for the MWMM problem in [3]. We first define the decision variables of the formulation. For each edge $e \in E$, x_e is a binary variable that takes the value one if and only if e is selected in the maximal matching. Moreover, for each vertex $i \in V$, y_i is a binary variable that takes the value one if and only if vertex i is matched by the maximal matching. An IP formulation for the MWMM is given below [3].

$$minimize \sum_{e \in E} w_e x_e$$

(IP-MWMM)

$$s.t. \ \textstyle\sum_{e \in E_i} x_e = y_i \leq 1, \forall i \in V \quad (1)$$

$$y_i + y_j - x_e \geq 1, \forall e = \{i, j\} \in E \quad (2)$$

$$x_e \in \{0,1\}, \forall e \in E \quad (3)$$

The objective function of (IP-MWMM) minimizes the weight of the maximal matching. Constraints in (1) make sure that the selected edges form a matching by enforcing that at most one edge that is incident with vertex i is selected. Moreover, they assure that vertex i is matched, i.e., $y_i = 1$, if and only if an edge that is incident with i is included in the maximal matching. Constraints in (2) imply that the set of the selected edges is a maximal matching by enforcing that if $x_e = 0$ for some edge $e = \{i, j\}$, then at least one of y_i and y_j should be one, i.e., at least one of i and j should be matched. Finally, binary restrictions on the x variables are defined in (3). Note that the y variables in (IP-MWMM) are used to make the exposition simpler. They are not necessary in the formulation and can be dropped. The LP relaxation of (IP-MWMM) is obtained by replacing the constraints in (3) by the constraints $0 \leq x_e \leq 1, \forall e \in E$.

Any feasible solution (x, y) of (IP-MWMM) corresponds to a maximal matching in $G = (V, E)$. More formally, the set $\{e \in E \mid x_e = 1\}$ is a maximal matching in G. On the reverse side, for any maximal matching M, if we define x_e as one if $e \in M$ and zero otherwise, then the vector $x = [x_e]_{e \in E}$ is a feasible solution of (IP-MWMM) (the values of the y variables are determined based on the values of the x variables). In this case, we say that x is the incidence vector of the maximal matching M.

Given a graph $G = (V, E)$, the maximal matching polytope associated with G, denoted by $MM(G)$, is the convex hull of the incidence vectors of maximal matchings in G. Denoting by $IP(G)$ and $LP(G)$ the feasible regions of (IP-MWMM) and its LP relaxation, respectively, we have that $MM(G) = conv(IP(G))$, where $conv(IP(G))$ is the convex hull of $IP(G)$. In general, we have that $MM(G) \subseteq LP(G)$. In [14], the author shows that if G is a tree, then the maximal matching polytope associated with G is characterized by the LP relaxation of (IP-MMWM), i.e., $MM(G) = LP(G)$. This implies that the MWMM problem in trees can be solved as an LP problem.

Taşkın and Ekim [3] propose two classes of valid inequalities for the maximal matching polytope, i.e., inequalities that are satisfied by the incidence vectors of all maximal matchings in G. For any clique Q in G, the authors show that the inequality

$$\sum_{i \in V(Q)} y_i \geq |V(Q)| - 1 \tag{4}$$

called the clique inequality is valid for the maximal matching polytope. To see that (4) is valid for the maximal matching polytope, observe that if less than $|V(Q)| - 1$ many vertices of the clique Q are included in a matching, then the matching cannot be maximal as there exist two adjacent vertices that are both unmatched by the matching. Therefore, any feasible solution (x, y) of (IP-MWMM) satisfies (4). Note that if there exist two cliques Q_1 and Q_2 in G such that $Q_1 \cup \{j\} = Q_2$, then the clique inequality written for Q_2

together with $y_j \leq 1$ imply the clique inequality written for Q_1. Therefore one does not need to consider clique inequalities for cliques that are not inclusion-wise maximal. Taşkın and Ekim perform a computational experiment in [3] by enumerating all maximal cliques and show that addition of the corresponding clique inequalities to formulation (IP-MWMM) results in lower computational times in solving the MWMM problem by a commercial solver.

Another valid inequality proposed in [3] for the maximal matching polytope is

$$\sum_{i \in V(C)} y_i \geq \left\lceil \frac{|V(C)|}{2} \right\rceil, \tag{5}$$

where C is an odd cycle in G. To see that (5) is valid for the maximal matching polytope, assume that the incidence vector of some maximal matching M in G does not satisfy (5) for some odd cycle C in G. This implies that among the vertices on this cycle, at most $\left\lceil \frac{|V(C)|}{2} \right\rceil - 1$ many of them are matched by M. This, however, again implies that there are two adjacent vertices in C and therefore in G that are both unmatched by the maximal matching resulting in a contradiction. Even though the inequality (5) when written for an even cycle is still valid for the maximal matching polytope, it is unnecessary as a stronger inequality is implied by the inequalities in (2).

As any maximal matching is a matching, valid inequalities for the matching polytope are also valid for the maximal matching polytope. It is widely known that the matching polytope associated with a graph G is characterized by the inequalities given in (1) without the y variables and the following inequalities, see e.g., [5].

$$\sum_{e \in E(H)} x_e \leq \left\lfloor \frac{|V(H)|}{2} \right\rfloor, \forall H \trianglelefteq G \; with \; |V(H)| \; odd \tag{6}$$

$$0 \leq x_e \leq 1, \forall e \in E \tag{7}$$

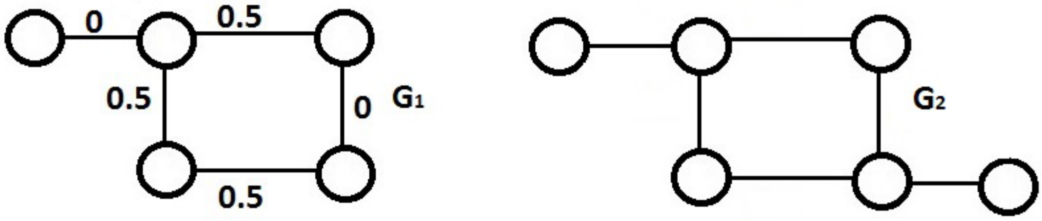
The inequalities in (6) therefore form another class of valid inequalities for the maximal matching polytope. Note that for bipartite graphs, the matching polytope is characterized by the inequalities given in (1) and (7) only.

Adding valid inequalities to an IP formulation makes it stronger. Branch-and-bound algorithms usually enumerate a smaller number of nodes for a strengthened formulation. Therefore, if the solution times of the LP relaxations are not increased too much by the addition of the valid inequalities, this may result in lower computational times for the strengthened formulation. Moreover, addition of some classes of valid inequalities may lead to solving the problem as an LP in certain cases. In this paper, we investigate the maximal matching polytope and propose new classes of valid inequalities for it. One of these inequalities generalizes both of the valid inequalities proposed in [3]. We hope that clever use of the valid inequalities we propose may expedite the solution times of the MWMM problem. Moreover, the new inequalities may help researchers in characterizing the maximal matching polytope in some classes of graphs and therefore may allow them to solve the MWMM problem as an LP in those graph classes.

3. Maximal Matching Polytope and New Valid Inequalities

It is currently not known for which graphs G, we have that $MM(G) = LP(G)$. It is shown in [14] that if G is a tree, then $MM(G) = LP(G)$. Is there any connected graph G which is not a tree for which $MM(G) = LP(G)$? If G contains an odd cycle, then one can easily show that $MM(G) \neq LP(G)$. To see this, let C be an odd cycle in G. Then for each edge $e \in E(C)$, we can take x_e as 0.5. For each edge f not in C that is incident with at least one vertex of C, we can take x_f as 0. We then remove C and all the edges that are incident with at least one vertex of C from G to obtain a new graph say H. Afterwards, we find a maximal matching in H, i.e., give 0 or 1 value to x_g for each $g \in E(H)$. Now, the vector x constructed in this manner is in $LP(G)$. However, x is not in $MM(G)$ as x is not in the matching polytope associated with G. This can be seen by observing that x violates the constraint (6) that is written for the cycle C.

Considering now bipartite graphs, i.e., graphs that do not contain any odd cycle, is $MM(G) = LP(G)$ for every bipartite graph G? Two bipartite graphs G_1 and G_2 are displayed in Fig. 2. For G_1, a feasible solution, say $\bar{x}$, of the LP relaxation of (IP-MWMM) is also given in Fig. 2a. This solution is in the matching polytope associated with G_1 and also in $LP(G_1)$. One can show that $\bar{x}$ is not in $MM(G_1)$ (we will show this later!). Note that $\bar{x}$ does not violate any of the proposed valid inequalities (4), (5), or (6) for the maximal matching polytope. On the other hand, for the second graph G_2 displayed in Fig. 2b, we have that $MM(G) = LP(G)$. We leave the characterization of the graphs for which the feasible region of the LP relaxation of (IP-MWMM) is equal to the maximal matching polytope as a future research direction.



(a) Graph G_1 (b) Graph G_2

Figure 2. Two bipartite graphs

We now propose new classes of valid inequalities for the maximal matching polytope.

Theorem 1. *Let* $H \trianglelefteq G$ *and* I *be an independent set in* H *of maximum cardinality. Then the inequality*

$$\sum_{i \in V(H)} y_i \geq |V(H)| - |I| \tag{8}$$

is valid for $MM(G)$.

Proof. Proof is by contradiction. Assume that the incidence vector of some maximal matching M in G does not satisfy (8). This means that less than $|V(H)| - |I|$ many vertices of H are matched by M. In other words, more than $|I|$ many vertices of H are unmatched by M. As I is an independent set in H of maximum cardinality, any set of more than $|I|$ many vertices include two adjacent vertices. This means that there are two adjacent vertices in H and hence in G that are unmatched by M and hence M is not a maximal matching.

The valid inequality (8) generalizes both of the valid inequalities (4) and (5) proposed by Taşkın and Ekim [3]. Taking H as a clique in G, we obtain from (8) the inequality (4) and taking H as an odd cycle in G, we obtain from (8) the inequality (5). One can consider interesting substructures in a given graph and obtain valid inequalities for the maximal matching polytope utilizing the inequality given in (8).

We now propose two classes of valid inequalities for the maximal matching polytope in Theorems 2 and 3. The proofs of these theorems will follow when another theorem, namely, Theorem 4, is proved. Therefore, we do not give the proofs of Theorems 2 and 3.

Theorem 2. *For any clique Q in G, the inequality*

$$\sum_{i \in V(Q)} y_i - \sum_{e \in E(Q)} x_e \geq \left\lfloor \frac{|V(Q)|}{2} \right\rfloor \tag{9}$$

is valid for $MM(G)$.

Note that when the cardinality of $V(Q)$ is odd, inequality (9) is implied by (4) and (6) written for $H = Q$. Therefore, inequality (9) is only useful for cliques having an even number of vertices.

Theorem 3. *For any cycle C in G, the inequality*

$$\sum_{i \in V(C)} y_i - \sum_{e \in E(C)} x_e \geq \left\lceil \frac{|V(C)|}{3} \right\rceil \tag{10}$$

is valid for $MM(G)$.

It should be noted that when C is an odd cycle, an inequality that is at least as strong as inequality (10) is implied by (4) and (6) written for $H = C$. Henceforth, inequality (10) is interesting only when C is an even cycle. Consider the graph G_1 displayed in Figure 2a and the solution $\bar{x}$ which we claim to be outside of $MM(G_1)$. When we write inequality (10) for the even cycle in G_1, we can see that $\bar{x}$ does not satisfy the inequality (1.5 is not greater than or equal to 2!) proving that $\bar{x} \notin MM(G_1)$.

Next we state Theorem 4 which generalizes Theorems 2 and 3. Correctness of Theorems 2 and 3 will follow from the proof of Theorem 4.

Theorem 4. *Let $H \trianglelefteq G$ and N be a maximal matching in H of minimum cardinality. Then the inequality*

$$\sum_{i \in V(H)} y_i - \sum_{e \in E(H)} x_e \geq |N| \tag{11}$$

is valid for $MM(G)$.

Proof. Let M be a maximal matching in G and let $\widetilde{M} = M \cap E(H)$. $\widetilde{M}$ is a matching in H. Assume that $\widetilde{M} \cup \tilde{O}$ is a maximal matching in H for some $\tilde{O} \subseteq E(H)$ where $\tilde{O} \cap \widetilde{M} = \emptyset$. Note that $\tilde{O}$ can be the empty set. Let x be the incidence vector of M (corresponding to x, we have y). As $\sum_{i \in V(H)} y_i \geq 2 \sum_{e \in E(H)} x_e + |\tilde{O}|$ and $\sum_{e \in E(H)} x_e = |\widetilde{M}|$, we have that $\sum_{i \in V(H)} y_i - \sum_{e \in E(H)} x_e \geq \sum_{e \in E(H)} x_e + |\tilde{O}| = |\widetilde{M}| + |\tilde{O}|$. Now, using the fact that N is a maximal matching in H of minimum cardinality, we have that $|\widetilde{M}| + |\tilde{O}| \geq |N|$ proving inequality (10).

Note that when H is taken as a clique in Theorem 4, we obtain Theorem 2. On the other hand, when H is taken as a cycle in Theorem 4, we get Theorem 3.

Consider the cycle C and a solution $\hat{x}$ displayed in Fig. 3. Letting $e_1 = \{1,2\}$, $e_2 = \{2,3\}$, $e_3 = \{3,4\}$, $e_4 = \{4,5\}$, $e_5 = \{5,6\}$, and $e_6 = \{6,1\}$, we have that $\hat{x}_{e_1} = \hat{x}_{e_2} = \hat{x}_{e_3} = \hat{x}_{e_6} = 0.5$ and $\hat{x}_{e_4} = \hat{x}_{e_5} = 0.4$. The vector $\hat{x}$ is in $LP(C)$ and moreover it satisfies the valid inequalities (5) and (10). However, we claim that $\hat{x} \notin MM(C)$. To see this, note that the incidence vector of any maximal matching in C satisfies $x_{e_1} + x_{e_2} = x_{e_4} + x_{e_5}$ and $\hat{x}$ violates this equality. Now, using this observation, we propose another class of valid inequalities in Theorem 5.

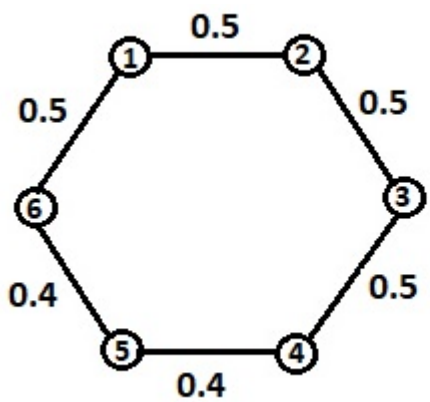


Figure 3. A cycle C with 6 vertices and a solution in $LP(C)$

Theorem 5. *Let C be a cycle in G having 6 vertices and e_1 and e_2 be two adjacent edges in C. Moreover, let e_4 and e_5 be the edges in C that are not adjacent to e_1 or e_2. Then the inequalities*

$$\sum_{i\in V(C)} y_i - 2\sum_{e\in E(C)} x_e \geq x_{e_1} + x_{e_2} - x_{e_4} - x_{e_5} \tag{12}$$

and

$$\sum_{i\in V(C)} y_i - 2\sum_{e\in E(C)} x_e \geq x_{e_4} + x_{e_5} - x_{e_1} - x_{e_2} \tag{13}$$

are both valid for $MM(G)$.

Proof. Let x be the incidence vector of some maximal matching M in G. $\sum_{i\in V(C)} y_i - 2\sum_{e\in E(C)} x_e$ represents the number of vertices in C that are matched using edges not in $E(C)$. It is clear that $\sum_{i\in V(C)} y_i - 2\sum_{e\in E(C)} x_e \geq 0$. On the other hand, if $x_{e_1} + x_{e_2} > x_{e_4} + x_{e_5}$, i.e., $x_{e_1} + x_{e_2} = 1$ and $x_{e_4} + x_{e_5} = 0$, then at least one vertex of C has to be matched using an edge not in $E(C)$. In other words, the inequality in (12) is a valid inequality for $MM(G)$. The fact that (13) is valid for $MM(G)$ follows from symmetry.

For the graph given in Fig. 3, we have that $G = C$. The solution $\hat{x}$ clearly violates (12) as $\sum_{i\in V(C)} \hat{y}_i - 2\sum_{e\in E(C)} \hat{x}_e = 0$, but $\hat{x}_{e_1} + \hat{x}_{e_2} - \hat{x}_{e_4} - \hat{x}_{e_5} = 0.2$.

4. Conclusion

In this paper, we investigate the maximal matching polytope and propose new classes of valid inequalities for it. The strengths of these valid inequalities should be investigated by future studies. The inequality we prove in Theorem 1 generalizes the existing inequalities in the literature, so the known inequalities become special cases of the inequality given in Theorem 1. As the MMM problem is solvable in polynomial time for some classes of graphs, it may be possible to characterize the maximal matching polytope for these graphs. The proposed valid inequalities may be useful in this respect. Moreover, the proposed valid inequalities can be used to strengthen the formulation (IP-MWMM) and may help solve the MWMM problem faster using commercial solvers. A computational study investigating the effects of the proposed valid inequalities is left as a future research task. Finally, characterizing graphs G for which $MM(G) = LP(G)$ is an open question for future studies.

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