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A FORMULAS FOR THE OPERATOR NORM AND THE EXTENSION OF A LINEAR FUNCTIONAL ON A LINEAR SUBSPACE OF A HILBERT SPACE

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Abstract

In this paper, formulas are given both for the operator norm and for the extension of a linear functional defined on a linear subspace of a Hilbert space and the results are illustrated with examples.

Keywords: Linear functional; Cauchy-Schwarz inequality; norm; operator norm; Hilbert spaces; Hahn-Banach theorem; Riesz representation theorem; orthogonal vectors; orthogonal decomposition.

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1 Introduction

Linear functionals occupy quite important place in mathematics in terms of both theory and application. The weak and weak-star topologies, which are fundamental and substantial subject in functional analysis, are generated by families of linear functionals. They are important in the theory of differential equations, potential theory, convexity and control theory [6]. Linear functionals play fundamental role in characterizing the topological closure of sets and therefore they are important for approximation theory. They play a very important role in defining vector valued analytic functions, generalizing Cauchy integral theorem and Liouville theorem. Therefore the need arises naturally to construct linear functionals with certain properties. The construction is usually achieved by defining the linear functional on a subspace of a normed linear space where it is easy to verify the desired properties and then extending it to the whole space with retaining the properties. This is not always easy in the case of general normed linear spaces. We specialize to linear functionals defined on the subspaces of a Hilbert space and provide formulas (Theorem 2) both for the operator norms and norm preserving linear extensions of linear functionals.

We start with basic definitions and results and fixed notations that will be used in the sequel. We denote the field of the real numbers $\mathbb{R}$ or the field of the complex numbers $\mathbb{C}$ by $\mathbb{F}$. We denote the absolute value function by $|.|$ defined on the field $\mathbb{F}$. So for $x \in \mathbb{R}$, if $x < 0$ then $|x| = -x$ and if $x \geq 0$ then $|x| = x$. For $z = x + iy \in \mathbb{C}$ we have $|z| = \sqrt{x^2 + y^2}$. The complex number $\overline{z} = x - iy$ is the complex conjugate of the number $z = x + iy$.

Definition 1.1. Let X be a linear space over the field $\mathbb{F}$ and $\|.\| : X \mapsto \mathbb{R}$ be a function. If the function $\|.\|$ satisfies the following properties

1. $\|0\| = 0$ and $\|x\| > 0$ for every $x \in X \setminus \{0\}$ (positivity definiteness);

2. $\|\alpha x\| = |\alpha| \, \|x\|$ for every $\alpha \in \mathbb{F}$ and $x \in X$ (homogeneity); and
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangular inequality);

then the function $\|.\|$ is called a norm on the space X and the pair $(X, \|.\|)$ is called a normed linear space.

Example 1.2. Let $p \in [1, \infty)$ and $\mathbb{R}^n = \{x = (x_1, \ldots, x_n) : x_j \in \mathbb{R} \text{ for } j = 1, \ldots, n\}$. $\mathbb{R}^n$ is a linear (vector) space over the field $\mathbb{R}$ with componentwise addition and scaler multiplication. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define $\|x\|_p = \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p}$ and $\|x\|_\infty = \max\{|x_j| : j = 1, \ldots, n\}$. Then for $1 \leq p \leq \infty$ the functions $\|.\|_p$ are norms on the space $\mathbb{R}^n$. Hence $(\mathbb{R}^n, \|.\|_p)$ is a normed linear spaces for each $p \in [1, \infty]$.

Inner products spaces are very important sources of normed linear spaces.

Definition 1.3. Let X be a linear space over the field $\mathbb{F}$. If the function $(.,.) : X \times X \mapsto \mathbb{F}$ satisfies the following properties

1. $(x, x) \geq 0$ for all $x \in X$ and $(x, x) = 0$ if and only if $x = 0$;
2. $(x, y) = \overline{(y, x)}$ for every $x, y \in X$;
3. $(x + y, z) = (x, z) + (y, z)$ for every $x, y, z \in X$; and
4. $(\alpha x, y) = \alpha(x, y)$ for every $\alpha \in \mathbb{F}$ and for every $x, y \in X$;

then the function $(.,.)$ is called an inner product on X and the pair $(X, (.,.))$ is called a inner product space over the field $\mathbb{F}$. The number $\|x\| = \sqrt{(x, x)}$ is called the norm of the vector $x \in X$. If $x, y \in X$ and $(x, y) = 0$ then the vectors x and y are called orthogonal vectors.

The inner product generates the most important inequality in mathematics, namely the Cauchy-Schwarz inequality.

Theorem 1.4. [Cauchy-Schwarz inequality][2, 4] Let $(X, (.,.))$ be an inner product space over the field $\mathbb{F}$. Then for every $x, y \in X$, $|(x, y)| \leq \sqrt{(x, x)}\sqrt{(y, y)} = \|x\| \, \|y\|$. The equality occurs if and only if the vectors x and y are linearly dependent.

From the Cauchy-Schwarz inequality it follows that the function $\|x\| = \sqrt{(x, x)}$ is a norm on the space X. This norm is called the norm generated by the inner product function $(.,.)$. If the normed linear space $(X, \|.\|))$ is a Banach space, that is, if every Cauchy sequence in X converges to a point in X, the inner product space $(X, (.,.))$ is called a Hilbert space.

Example 1.5. For $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ the function $(.,.) : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ defined by $(x, y) = \sum_{j=1}^{n} x_j y_j$ is an inner product on the space $\mathbb{R}^n$. The norm generated by this inner product is the Euclidean norm $\|x\|_2 = \sqrt{(x, x)} = \left(\sum_{j=1}^{n} x_j^2\right)^{1/2}$. The inner product space $(\mathbb{R}^n, \|.\|)$ is a Hilbert space.

On normed linear spaces the primary objects of study are the linear operators and linear functionals which play central role in functional analysis.

Definition 1.6. Let $(X, \|.\|)$ and $(Y, \|.\|')$ be normed linear spaces over the same field $\mathbb{F}$ and $T : X \mapsto Y$ be a mapping. If $T(\alpha x + \beta y) = \alpha T(x) + \beta T(Y)$ for all $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$ then T is called a linear operator. A linear operator T is called bounded if there is a real constant $M > 0$ such that $\|T(x)\|' \leq M \|x\|$ for all $x \in X$. If T is a bounded linear operator the number $\|T\|_{op} = \|T\| =$

$\inf\{M : \|T(x)\|' \le M\|x\| \text{ for all } x \in X\}$ is called a operator norm of T. The equivalent definition of operator norm is given by the formulas

$$\begin{aligned} \|T\|_{op} = \|T\| &= \sup\left\{\frac{\|T(x)\|'}{\|x\|} : \|x\| \neq 0\right\} = \sup\left\{\|T(x)\|' : \|x\| \le 1\right\} \\ &= \sup\left\{\|T(x)\|' : \|x\| = 1\right\} \end{aligned}$$

The following is a very useful result for the computation of operator norms of linear operators.

Lemma 1.7. [Computation of operator norm] Let $(X, \|.\|)$ and $(Y, \|.\|')$ be normed linear spaces over the same field $\mathbb{F}$, $T : X \mapsto Y$ be a bounded linear operator and $M \ge 0$ be a real constant. If for every $x \in X$, $\|T(x)\|' \le M\|x\|$ and $\|T(x_0)\|' = M\|x_0\|$ for a vector $x_0 \in X \setminus \{0\}$ then the operator norm of T is $\|T\|_{op} = M$.

Proof. If for every $x \in X$, $\|T(x)\|' \le M\|x\|$ then by the definition of operator norm $\|T\|_{op} \le M$. On the other hand if for a vector $x_0 \in X \setminus \{0\}$, $\|T(x_0)\|' = M\|x_0\|$ then by the definition of operator norm we have $M\|x_0\| = \|T(x_0)\|' \le \|T\|_{op}\|x_0\|$ so that $M \le \|T\|_{op}$. Therefore $\|T\|_{op} = M$.

Remark 1.8. We note that in the finite dimensional case the operator norm can be computed by the method of Lagrange multipliers with constraints.

It is now a classical result that a linear operator is bounded if and only if it is continuous. The set $\mathcal{B}(X, Y)$ of bounded linear operators is a linear space over $\mathbb{F}$ with pointwise addition and scaler multiplication and $\|.\|_{op}$ is a norm on $\mathcal{B}(X, Y)$. If $(Y, \|.\|')$ is a Banach space then the space $(\mathcal{B}(X, Y), \|.\|_{op})$ is a Banach space.

A bounded linear operator $f : (X, \|.\|) \mapsto (\mathbb{F}, |.|)$ is called a **bounded linear functional**. The Banach space $(X^*, \|.\|_{op})$ of bounded linear functional is called dual or conjugate space of the normed linear space $(X, \|.\|)$.

Example 1.9. Let $(X, (.,.))$ be an inner product space over the field $\mathbb{F}$ and $a \in X$ be a fixed vector. Then $f : (X, \|.\|) \mapsto (\mathbb{F}, |.|), f(x) = (x, a)$ is a bounded linear functional and $\|f\|_{op} = \|a\|$.

Linear functionals are important in terms of generating and characterizing linear subspaces. If $(X, \|.\|)$ is a normed linear space over the field $\mathbb{F}$ and $\ell : X \mapsto \mathbb{F}$ is a linear functional then the the kernel or the null space $\ker(\ell) = \{x \in X : \ell(x) = 0\}$ of the linear functional ℓ is a linear subspace of X. It is well-known that a linear functional is continuous if and only if its null space is closed. On the other hand we have the following simple result which shows the relations between linear subspaces and linear functionals.

We recall that a linear subspace W of a linear space X is called a codimension one linear subspace if the dimension of the quotient space $X \setminus W$ is $\dim(X \setminus W) = 1$.

Lemma 1.10. Let $(X, \|.\|)$ be normed linear space over the field $\mathbb{F}$. Then W is a codimension one linear subspace of the space X if and only if there is a linear functional $\ell : X \mapsto \mathbb{F}$ such that $W = \ker(\ell)$.

Proof. Since the null space of a linear operator is a linear subspace if $W = \ker \ell$ for a linear functional $\ell : X \mapsto \mathbb{F}$, then W is a linear subspace of the space X. Conversely we assume that W is a codimension one linear subspace of the space X. Let $x_0 \in X \setminus W$ be arbitrary and $M = \{\alpha x_0 : \alpha \in \mathbb{F}\}$ be the linear subspace of X generated by the vector x_0. Then $X = W \oplus M$. Since each $x \in X$ has a unique representation of the form $x = w_x + \alpha_x x_0$ where $w_x \in W$ and $\alpha_x x_0 \in M$ the function $\ell : X \mapsto \mathbb{F}, \ell(x) = \ell(w_x + \alpha_x x_0) = \alpha_x$ is a linear functional with $\ker(\ell) = W$.

There are two fundamental results about bounded linear functionals, namely the Hahn-Banach theorem and the Riesz representation theorem. The Hahn-Banach theorem, one of the indispensable tools of modern analysis, play the central role in the investigation of geometric and analytic properties of bounded linear functionals. The Riesz representation theorem completely characterizes the bounded linear functionals on certain normed linear spaces. We state a version of each of these theorems that we need in what follows.

Theorem 1.11. [Hahn-Banach][2, 1, 7, 3] Let $(X, \|.\|)$ be normed linear space over the field $\mathbb{F}$ and W be a linear subspace of X. If $f : (W, \|.\|) \mapsto (\mathbb{F}, |.|)$ is a bounded linear functional then there is a bounded linear functional $F : (X, \|.\|) \mapsto (\mathbb{F}, |.|)$ such that $F \mid_W = f$, that is for all $x \in W, F(x) = f(x)$ and $\|F\|_{op} = \|f\|_{op}$.

Remark 1.12. The linear functional F is called a norm preserving linear functional extension of the linear functional f. The important and the difficult part of the theorem is to get the norm preserving linear extension. Otherwise it is well-known that there are many linear extensions of f easy to construct.

Theorem 1.13. [Riesz representation theorem] [2, 5, 3, 7] Let $(X, (.,.))$ be a Hilbert space over the field $\mathbb{F}$ and $\|.\|$ be the norm generated by the inner product. Then a function $f : (X, \|.\|) \mapsto (\mathbb{F}, |.|)$ is a bounded linear functional if and only if there is a unique vector $a \in X$ such that $f(x) = (x, a)$ for all $x \in X$. Furthermore, the operator norm of the linear functional f is $\|f\|_{op} = \|a\|$.

Remark 1.14. By Lemma 1.10 and the Riesz representation theorem in a Hilbert space $(X, (.,.))$ a codimension one linear subspace W of the space X is of the form $W = \{x \in X : \ell_a(x) = (x, a) = 0\}$ where $a \in X$ is a fixed vector.

On finite dimensional normed linear spaces, the Riesz representation theorem provides more concrete information about the structure of linear functionals. In this context, we state a version of the Riesz representation theorem for the finite dimensional spaces and give its proof for the sake of completeness.

Theorem 1.15. [Riesz representation theorem] Let $1 \leq p, q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then a function $f : (\mathbb{R}^n, \|.\|_p) \mapsto (\mathbb{R}, |.|)$ is a bounded linear functional if and only if there is constant vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ such that $f(x) = (x, a) = a_1x_1 + \cdots + a_nx_n$ for every $x \in \mathbb{R}^n$. Furthermore, the operator norm of f is $\|f\|_{op} = \|a\|_q$.

Proof. We first assume that for a constant vector $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ $f(x) = (x, a) = a_1x_1 + \cdots + a_nx_n$. Since the inner product is a linear functional with respect to the first variable it follows that f is a linear functional. On the other hand we assume that $f : (\mathbb{R}^n, \|.\|_p) \mapsto (\mathbb{R}, |.|)$ is a linear functional. If $f \equiv 0$ then for the vector $a = 0$, f is the required form $f(x) = a_1x_1 + \cdots + a_nx_n = (x, a)$. Therefore we may assume that $f \neq 0$. For $j = 1, 2, \ldots, n$ let $e_j = (0, \ldots, 0, \underset{j}{1}, 0 \ldots, 0)$. The set $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a standard (Hamel) basis of the space $\mathbb{R}^n$. Hence every vector $x \in \mathbb{R}^n$ has a unique representation of the form $x = \sum_{j=1}^n x_je_j$. For $j = 1, 2, \ldots, n$ let $a_j = f(e_j)$. $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. Since f is a linear functional we have $f(x) = \sum_{j=1}^n x_jf(e_j) = \sum_{j=1}^n x_ja_j = a_1x_1 + \cdots + a_nx_n = (x, a)$. So f is the required form.

For $1 \leq p < \infty$ and for each $x \in \mathbb{R}^n$ by the Cauchy-Schwarz inequality we have $|f(x)| \leq \|a\|_q \|x\|_p$. If $p = \infty$ then $q = 1$ and $|f(x)| \leq \|a\|_1 \|x\|_\infty = \|a\|_q \|x\|_p$. Hence by the definition of operator norm $\|f\|_{op} \leq \|a\|_q$.

For $1 \leq p < \infty$, if $a_j = 0$ we define $x_j(0) = 0$, and if $a_j \neq 0$ we define $x_j(0) = \frac{|a_j|^q}{a_j}$ and let $x(0) = (x_1(0), \ldots, x_n(0))$. Since $\|x(0)\|_p = \|a\|_q^{q/p}$ and $|f(x(0))| = \|a\|_q^q = \|a\|_q^{q(\frac{1}{p}+\frac{1}{q})} = \|a\|_q \|a\|_q^{\frac{q}{p}} = \|a\|_q \|x(0)\|_p$ from the Lemma1.7 it follows that $\|f\|_{op} = \|a\|_q$.

For $p = \infty$, if $a_j \geq 0$ we define $x_j(0) = 1$ and if $a_j < 0$ we define $x_j(0) = -1$ and let $x(0) = (x_1(0), \ldots, x_n(0))$. Since $\|x(0)\|_\infty = 1$ and $|f(x(0))| = \|a\|_1 = \|a\|_1 \|x(0)\|_\infty$ from the Lemma 1.7 it follows that $\|f\|_{op} = \|a\|_1$. Therefore we have $\|f\|_{op} = \|a\|_q$ for all $1 \leq p \leq \infty$.

2 Operator Norms and Extension of Linear Functionals

The Hahn-Banach theorem states that a bounded linear functional on a linear subspace of a normed linear space can be extended to the whole space without changing its operator norm. On the other hand, the Riesz representation theorem provides formulas both for the linear functional and its operator norm on a Hilbert space. But, as far as I know there is no such a formula for the operator norm of a linear functional defined on a linear subspace of a normed linear space.

By analyzing the orthogonal decomposition theorem and the Riesz representation theorem [5], [7](4.11 Theorem, 4.12 Theorem) we get two methods of the unique norm preserving linear extension of a linear functional defined on a closed linear subspace a Hilbert space. We note and state these methods without proofs.

Lemma 2.1. Let $(X,(.,.))$ be a Hilbert space over the field $\mathbb{F}$, $\|.\|$ be the norm generated by the inner product, W be a closed linear subspace of the space X and $f:(W,\|.\|)\mapsto(\mathbb{F},|.|)$ be a nontrivial bounded linear functional. Let $M=\ker(f)$ be the null space of f and $M^{\perp}=\{x\in W:(x,y)=0 \text{ for all } y\in M\}$ be the orthogonal complement of the space M in W. Choose any vector $x_0\in M^{\perp}\setminus\{0\}$ and let $a=\frac{\overline{f(x_0)}}{\|x_0\|^2}x_0$. Then $f(x)=(x,a)$ for all $x\in W$, $\|f\|_{op}=\|a\|$ and the norm preserving linear extension of the functional f is the linear functional $F:(X,\|.\|)\mapsto(\mathbb{F},|.|), F(x)=(x,a)$.

Lemma 2.2. Let $(X,(.,.))$ be a Hilbert space over the field $\mathbb{F}$, $\|.\|$ be the norm generated by the inner product, W be a closed linear subspace of the space X and $f:(W,\|.\|)\mapsto(\mathbb{F},|.|)$ be a bounded linear functional. Let $p:X\mapsto W$ be the orthogonal projection of the space X onto the space W. Then $F:(X,\|.\|)\mapsto(\mathbb{F},|.|), F(x)=f\circ p(x)=f(p(x))$ is the norm preserving linear functional extension of the functional f.

The applications of these methods, without doubt, requires certain amount of work. In the case of a bounded linear functional defined on a codimension one subspace of a Hilbert space we provide simple formula both for the operator norm and for the norm preserving linear functional extension.

Theorem 2.3. [Formula for the operator norm and linear extension] Let $(X,(.,.))$ be a Hilbert space over the field $\mathbb{F}$, $\|.\|$ be the norm generated by the inner product, $a,b\in X\setminus\{0\}$ be fixed vectors, $W=\{x\in X:\ell_b(x)=(x,b)=0\}$ be a linear subspace of the space X and $f_a:(W,\|.\|)\mapsto(\mathbb{F},|.|)$, $f_a(x)=(x,a)$ be a linear functional. Then the operator norm of the linear functional f_a is $\|f_a\|_{op}=\left\|a-\frac{(a,b)}{\|b\|^2}b\right\|=\frac{1}{\|b\|}\sqrt{\|a\|^2\|b\|^2-|(a,b)|^2}$ and the norm preserving extension of the linear functional f_a is the linear functional $F:(X,\|.\|)\mapsto(\mathbb{F},|.|), F(x)=f_a(x)-\frac{(a,b)}{\|b\|^2}\ell_b(x)=\left(x,a-\frac{(a,b)}{\|b\|^2}b\right)$.

Proof. Since for each $x\in W$, $\ell_b(x)=0$ we have $F(x)=f_a(x)$. So the function F is an extension of the function f_a. By the Riesz representation theorems F is a linear functional on the space X and its operator norm is $\|F\|_{op}=\left\|a-\frac{(a,b)}{\|b\|^2}b\right\|$. Since by the properties of the inner product

$$\begin{aligned}\|F\|_{op}=\left\|a-\frac{(a,b)}{\|b\|^2}b\right\|&=\sqrt{\left(a-\frac{(a,b)}{\|b\|^2}b,a-\frac{(a,b)}{\|b\|^2}b\right)}\\&=\sqrt{\|a\|^2-\frac{2|(a,b)|^2}{\|b\|^2}+\frac{|(a,b)|^2\|b\|^2}{\|b\|^4}}\\&=\sqrt{\|a\|^2-\frac{|(a,b)|^2}{\|b\|^2}}=\frac{1}{\|b\|}\sqrt{\|a\|^2\|b\|^2-|(a,b)|^2}\end{aligned}$$

it suffices to show that $\|f_a\|_{op} = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\|$. Since for $x \in W$, $\ell_b(x) = 0$ by the Cauchy-Schwarz inequality we have

$$|f_a(x)| = \left| f_a(x) - \frac{(a,b)}{\|b\|^2} \ell_b(x) \right| = \left| (x,a) - \frac{(a,b)}{\|b\|^2} (x,b) \right|$$
$$= \left| \left(x, a - \frac{(a,b)}{\|b\|^2} b \right) \right| \leq \left\| a - \frac{(a,b)}{\|b\|^2} b \right\| \|x\| .$$

By the definition of operator norm $\|f_a\|_{op} \leq \left\| a - \frac{(a,b)}{\|b\|^2} b \right\|$. Since the equality holds in Cauchy-Schwarz inequality when $x = a - \frac{(a,b)}{\|b\|^2} b \in W$ and $\left| f_a \left(a - \frac{(a,b)}{\|b\|^2} b \right) \right| = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\|^2 = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\| \|x\|$ it follows from the Lemma 1.7 that

$$\|f_a\|_{op} = \left\| a - \frac{(a,b)}{\|b\|^2} b \right\| = \frac{1}{\|b\|} \sqrt{\|a\|^2 \|b\|^2 - |(a,b)|^2}.$$

Remark 2.4. Since $a = \frac{(a,b)}{\|b\|^2} b + a - \frac{(a,b)}{\|b\|^2} b$ and $\left(a - \frac{(a,b)}{\|b\|^2} b, b \right) = 0$, the vector $a - \frac{(a,b)}{\|b\|^2} b$ is the component of the vector a orthogonal to the vector b. This observation gives the following results.

Corollary 2.5. In Theorem 2.3, if the vectors a and b are orthogonal, that is $(a,b) = 0$ then the operator norm of the linear functional f_a is $\|f_a\|_{op} = \|a\|$ and its norm preserving extension is the linear functional $F : (X, \|.\|) \mapsto (\mathbb{F}, |.|), F(x) = f_a(x)$.

Corollary 2.6. In Theorem 2.3., if the vectors a and b are collinear, that is $b = ta$ for a scaler $t \in \mathbb{F}$ then $f_a \equiv 0$, its operator norm $\|f_a\|_{op} = 0$ and its norm preserving extension is the linear functional $F : (X, \|.\|) \mapsto (\mathbb{F}, |.|), F(x) = 0$.

In the following result we assume that $\dim(\mathbb{R}^n) = n \geq 2$.

Corollary 2.7. Let $a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \setminus \{0\}$ be fixed vectors, $W = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \ell_b(x) = (x,b) = b_1x_1 + b_2x_2 + \cdots + b_nx_n = 0\}$ be a linear subspace and $f_a : (W, \|.\|_2) \mapsto (\mathbb{R}, |.|)$, $f_a(x) = (x,a) = a_1x_1 + a_2x_2 + \cdots + a_nx_n$ be a linear functional. Then the operator norm of the linear functional f_a is $\|f_a\|_{op} = \sqrt{\|a\|_2^2 - \frac{(a,b)^2}{\|b\|_2^2}} = \frac{1}{\|b\|_2} \sqrt{\|a\|_2^2 \|b\|_2^2 - (a,b)^2}$ and its norm preserving extension is the linear functional $F : (\mathbb{R}^n, \|.\|_2) \mapsto (\mathbb{R}, |.|), F(x) = f_a(x) - \frac{(a,b)}{\|b\|_2^2} \ell_b(x) = \left(x, a - \frac{(a,b)}{\|b\|_2^2} b \right)$.

Remark 2.8. Since the computation of an operator norm is an extremum value problem we note that Corollary 2.7 may be used to solve certain type of extremum value problems.

Example 2.9. Let $W = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0 \right\}$ be a linear subspace of the space $\mathbb{R}^3$. Find the operator norm of the linear functional $f : (W, \|.\|_2) \mapsto (\mathbb{R}, |.|), f(x) = 2x_1 + 3x_3$ and its norm preserving linear functional extension $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$.

Solution. For the vectors $b = (b_1, b_2, b_3) = (1,1,1)$, $a = (a_1, a_2, a_3) = (2,0,3) \in \mathbb{R}^3$ we have $W = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ell_b(x) = (x,b) = x_1 + x_2 + x_3 = 0\}$ and $f(x) = f_a(x) = (x,a) = 2x_1 + 3x_3$. Since $(a,b) = 5$, $\|a\|_2 = \sqrt{13}$ and $\|b\|_2 = \sqrt{3}$ by the Corollary 2.7 the operator norm of the linear functional f is $\|f\|_{op} = \frac{1}{\sqrt{3}} \sqrt{39 - 25} = \sqrt{\frac{14}{3}} = \frac{\sqrt{42}}{3}$ and its norm preserving linear functional extension is $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$ $F(x) = f_a(x) - \frac{(a,b)}{\|b\|_2^2} \ell_b(x) = 2x_1 + 3x_3 - \frac{5}{3}(x_1 + x_2 + x_3) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$.

We use Lemma 2.1 and Lemma 2.2 to give alternative solutions of this example.

Alternative solution. $(W, \|.\|_2)$ is a Hilbert space. The kernel or the null space of the linear functional f is the linear subspace $M = \ker(f) = \left\{\alpha\left(1, -\frac{1}{3}, -\frac{2}{3}\right) : \alpha \in \mathbb{R}\right\}$ and its orthogonal complement in W is the linear subspace $M^{\perp} = \{\alpha(1, -5, 4) : \alpha \in \mathbb{R}\}$. Choose $x_0 = (1, -5, 4)$. and let $a = \frac{\overline{f(x_0)}}{\|x_0\|_2^2} x_0 = \frac{1}{3}(1, -5, 4)$. Then by Lemma 2.1 we have $f(x) = (x, a) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$ for all $x \in W$. By the Riesz representation theorem $\|f\|_{op} = \|a\|_2 = \frac{\sqrt{42}}{3}$ and by the uniqueness of extension the norm preserving linear extension of the linear functional f is the linear functional $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$, $F(x) = (x, a) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$.

Alternative solution 2. Since the space W is the kernel of the linear functional $\ell : \mathbb{R}^3 \mapsto \mathbb{R}, \ell(x) = x_1 + x_2 + x_3$ it is a closed codimension one linear subspace of $\mathbb{R}^3$ and hence $\dim W = 2$. The orthogonal projection of the space $\mathbb{R}^3$ onto the space W is the bounded linear operator $p : \mathbb{R}^3 \mapsto W$, $p(x) = \left(\frac{2x_1 - x_2 - x_3}{3}, \frac{2x_2 - x_1 - x_3}{3}, \frac{2x_3 - x_2 - x_1}{3}\right)$. So by Lemma 2.2 the norm preserving linear extension of the linear functional f is the linear functional $F : (\mathbb{R}^3, \|.\|_2) \mapsto (\mathbb{R}, |.|)$,

$$
\begin{aligned}
F(x) =& f(p(x)) = f\left(\frac{2x_1 - x_2 - x_3}{3}, \frac{2x_2 - x_1 - x_3}{3}, \frac{2x_3 - x_2 - x_1}{3}\right) \\
=& 2\left(\frac{2x_1 - x_2 - x_3}{3}\right) + 3\left(\frac{2x_3 - x_2 - x_1}{3}\right) \\
=& \frac{1}{3}(x_1 - 5x_2 + 4x_3) = ((x_1, x_2, x_3), \frac{1}{3}(1, -5, 4)).
\end{aligned}
$$

By the Riesz representation theorem $\|f\|_{op} = \|F\|_{op} = \left\|\frac{1}{3}(1, -5, 4)\right\|_2 = \frac{\sqrt{42}}{3}$.

We give a different solution of this example which is also important in terms of the method used.

Alternative solution 3. By the definition of operator norm combined probably with the method of Lagrange multipliers we have

$$
\begin{aligned}
\|f\|_{op} =& \sup\{|f(x)| : x = (x_1, x_2, x_3) \in W, \|x\|_2 = 1\} \\
=& \sup\left\{|2x_1 + 3x_3| : x = (x_1, x_2, x_3) \in \mathbb{R}^3, x_1 + x_2 + x_3 = 0, \|x\|_2 = 1\right\} \\
=& \sup\{2x_1 + 3x_3 : x_1, x_3 \geq 0, x_1 + x_2 + x_3 = 0, \|x\|_2 = 1\} = \frac{\sqrt{42}}{3}.
\end{aligned}
$$

By the Hahn-Banach theorem there is at least one norm preserving linear functional extension F of f to the space $(\mathbb{R}^3, \|.\|_2)$. By the Riesz representation theorem this extension is of the form $F(x) = (x, a) = a_1x_1 + a_2x_2 + a_3x_3$ where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ is a constant vector and $\|F\|_{op} = \|a\|_2$. For $x \in W$ by solving the linear extension equality $2x_1 + 3x_3 = f(x) = F(x) = a_1x_1 + a_2x_2 + a_3x_3$ and the operator norm equality $\frac{\sqrt{42}}{3} = \|f\|_{op} = \|F\|_{op} = \|a\|_2 = \sqrt{a_1^2 + a_2^2 + a_3^2}$ simultaneously we get $a_1 = \frac{1}{3}, a_2 = \frac{-5}{3}$ and $a_3 = \frac{4}{3}$. Therefore the unique norm preserving linear extension of the linear functional f is the linear functional $F(x) = \frac{1}{3}(x_1 - 5x_2 + 4x_3)$.

Example 2.10. Let $X = \mathcal{P}_3(\mathbb{R})$ be the linear space of all real polynomial functions of degree at most 3. The function $(.,.) : X \times X \mapsto \mathbb{R}$ defined by $(p, q) = \int_{-1}^{1} p(x)q(x)dx$ is an inner product on X and it generates the norm $\|p\| = \sqrt{(p, p)} = \left(\int_{-1}^{1}(p(x))^2 dx\right)^{1/2}$. The inner product space $(X, (.,.))$ is a Hilbert space. Let $W = \{p \in X : \int_{-1}^{1}(p(x) + xp(x))dx = 0\}$ and $\ell : (W, \|.\|) \mapsto (\mathbb{R}, |.|), \ell(p) = \int_{-1}^{1}(p(x) + x^2 p(x))dx$. Show that W is a linear subspace of X and ℓ is bounded linear functional. Find the operator norm of the functional ℓ and its norm preserving linear extension to the space X.

Solution. For the polynomial functions $a : \mathbb{R} \mapsto \mathbb{R}, a(x) = 1 + x^2$ and $b : \mathbb{R} \mapsto \mathbb{R}, b(x) = 1 + x$ we have $W = \{p \in X : \int_{-1}^{1}(p(x) + xp(x))dx = 0\} = \{p \in X : (p, b) = 0\}$ and $\ell(p) = \int_{-1}^{1}(p(x) + x^2 p(x))dx = \int_{-1}^{1}(1 + x^2)p(x)dx = (p, a)$. Since the inner product is a bounded linear functional with respect to the first variable it follows that W is a closed codimension one linear subspace of the space X and ℓ is a bounded linear functional. Since

$$\begin{aligned}\|b\| &= \left(\int_{-1}^{1}(b(x))^2dx\right)^{1/2} = \left(\int_{-1}^{1}(1+x)^2dx\right)^{1/2}\\ &= \left(\int_{0}^{2}t^2dt\right)^{1/2} = \sqrt{\frac{t^3}{3}\mid_0^2} = \frac{2\sqrt{6}}{3} \quad \text{and}\\ (a,b) &= \int_{-1}^{1}a(x)b(x)dx = \int_{-1}^{1}(1+x+x^2+x^3)dx\\ &= \left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}\right)\mid_{-1}^{1} = 2+\frac{2}{3} = \frac{8}{3},\end{aligned}$$

by the Theorem 2.3 the operator norm of the linear functional ℓ is

$$\begin{aligned}\|\ell\|_{op} &= \left\|a-\frac{(a,b)}{\|b\|^2}b\right\| = \|a-b\| = \left(\int_{-1}^{1}(a(x)-b(x))^2dx\right)^{1/2}\\ &= \left(\int_{-1}^{1}(x^2-x)^2dx\right)^{1/2} = \left(\int_{-1}^{1}(x^4-2x^3+x^2)dx\right)^{1/2}\\ &= \sqrt{\left(\frac{x^5}{5}-\frac{2x^4}{4}+\frac{x^3}{3}\right)\mid_{-1}^{1}} = \sqrt{\frac{2}{5}+\frac{2}{3}} = \frac{4\sqrt{15}}{15}\end{aligned}$$

and its norm preserving linear functional extension is the linear functional $L : (X, \|.\|) \mapsto (\mathbb{R}, |.|)$ defined by

$$\begin{aligned}L(p) &= \left(p, a-\frac{(a,b)}{\|b\|^2}b\right) = (p, a-b)\\ &= \int_{-1}^{1}(a(x)-b(x))p(x)dx = \int_{-1}^{1}(x^2-x)p(x)dx.\end{aligned}$$

The following example shows that our formula works not just for finite dimensional Hilbert spaces but also works for infinite dimensional Hilbert spaces.

Example 2.11. Let $X = L_2([0,1]) = \{f : f : [0,1] \mapsto \mathbb{R}$, Lebesque measurable and $\|f\|_2 = \left(\int_0^1(f(x))^2dx\right)^{1/2}\}$ be the linear space of Lebesque square integrable functions. The function $(.,.) : X \times X \mapsto \mathbb{R}$ defined by $(f,g) = \int_0^1 f(x)g(x)dx$ is an inner product on X and it generates the norm $\|f\| = \|f\|_2$. Let $W = \{f \in X : \int_0^1 f(x)dx = 0\}$ and $\ell : (W, \|.\|) \mapsto (\mathbb{R}, |.|), \ell(f) = \int_0^1 x^2 f(x)dx$. Show that W is a linear subspace of X and ℓ is bounded linear functional on W. Find the operator norm of the functional ℓ and its norm preserving linear extension to the space X.

Solution. For the functions $a : [0,1] \mapsto \mathbb{R}, a(x) = x^2$ and $b : [0,1] \mapsto \mathbb{R}, b(x) = 1$ we have $W = \{f \in X : \int_0^1 f(x)dx = 0\} = \{f \in X : (f,b) = 0\}$ and $\ell(f) = \int_0^1 x^2 f(x)dx = (f,a)$. Since the inner product is a bounded linear functional with respect to the first variable it follows that W is a codimension one linear subspace of the space X and ℓ is a bounded linear functional. Since $\|b\| = \left(\int_0^1(b(x))^2dx\right)^{1/2} = \left(\int_0^1 1^2dx\right)^{1/2} = \sqrt{x\mid_0^1} = 1$ and $(a,b) = \int_0^1 a(x)b(x)dx = \int_0^1 x^2dx = \frac{x^3}{3}\mid_0^1 = \frac{1}{3}$ by the Theorem 2.3 the operator norm of the linear functional ℓ is

$$\begin{aligned}\|\ell\|_{op} &= \left\|a-\frac{(a,b)}{\|b\|^2}b\right\| = \left\|a-\frac{b}{3}\right\| = \left(\int_0^1\left(a(x)-\frac{b(x)}{3}\right)^2dx\right)^{1/2}\\ &= \left(\int_0^1\left(x^2-\frac{1}{3}\right)^2dx\right)^{1/2} = \left(\int_0^1\left(x^4-\frac{2}{3}x^2+\frac{1}{9}\right)dx\right)^{1/2}\\ &=\sqrt{\left(\frac{x^5}{5}-\frac{2}{9}x^3+\frac{1}{9}x\right)|_0^1} = \sqrt{\left(\frac{1}{5}-\frac{2}{9}+\frac{1}{9}\right)-0}\\ &=\sqrt{\frac{9-5}{9\cdot 5}} = \sqrt{\frac{4}{9\cdot 5}} = \frac{2}{3\sqrt{5}} = \frac{2\sqrt{5}}{15}\end{aligned}$$

and its norm preserving linear functional extension is the linear functional $L:(X,\|.\|)\mapsto(\mathbb{R},|.|)$, $L(p)=\left(f,a-\frac{(a,b)}{\|b\|^2}b\right)=\left(p,a-\frac{b}{3}\right)=\int_0^1(a(x)-b(x))f(x)dx=\int_0^1\left(x^2-\frac{1}{3}\right)f(x)dx.$

We end the paper with the following question.

Question. Can we remove the codimension one hypothesis in Theorem 2.3? Is it possible to generalize these results to normed linear spaces under some smoothness conditions.

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