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Numerical Assessment of Symmetric and Non-Symmetric Kernel Functions on Second Order Non-Homogenous Volterra Integro-Differential Equations

Kazeem Iyanda FALADE[*1], Ismail Gboyeka BAOKU[2], Abdulgafar Tunde TIAMIYU[3]

Abstract

In this paper, we present numerical assessment of symmetric and non-symmetric kernel functions on non-homogenous Volterra integro-differential equations. Simple MAPLE 18 software commands codes procedures are employ based on newly introduced techniques: exponentially fitted collocation approximation method and Adomian decomposition method for the numerical solutions of the non-homogenous Volterra integro-differential equations. The procedures are sought to obtain convergent point of the problems. Considering the property of symmetric and non-symmetric kernel ($K(t,\ s) = K(s,\ t)\ and\ K(t,\ s) \neq K(s,\ t)$), the computational lengths are considered to archive the best numerical solutions for the four examples considered. The reliability and efficiency of the proposed techniques are demonstrated using some examples available in literature.

Keywords: symmetric and non-symmetric kernel functions, non-homogenous Volterra integro-differential equations, exponentially fitted collocation approximate method, Adomian decomposition method.

1. INTRODUCTION

The Volterra integro-differential equation appeared after its establishment by Volterra. It then appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer and diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating and wind ripple in the desert. It gained a lot of interest in many applied mathematical sciences, such as modelling of physical phenomenon: thermodynamics, solid mechanics, rocketing sciences, biological models and chemical kinetics**.** As such, the solution of integro-differential equations has a major role in the fields of applied sciences. Therefore, they received special attention of scientists and researchers [1-5]. When a physical system is modelled under the differential sense, it finally gives a differential equation, an integral equation or an integro-differential equation. There are various techniques for solving an integral or integro differential equation, e.g, wavelet Galerkin, Taylor expansion method, Haar functions method, variational iteration method and differential transform method [6-10]. Homotopy perturbation method (HPM) [11], linear multi step methods [12] and just to mention a few.

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The purpose of this article is to extend the numerical method proposed by [13] and compare the results obtain with analytical solutions and the Adomian decomposition method discussed by [14] for the assessment of symmetric and non-symmetric kernel functions on second order non-homogenous Volterra intgro-differential equations of the form:

$$\frac{d^2u(t)}{dt^2} = g(t) + \int_a^t K(t,s)u(s)ds, t\epsilon[a,b] \quad (1)$$

subject to initial conditions:

$$u(a) = \beta \qquad u'(a) = \gamma \qquad (2)$$

where β and γ are constants, $g(t)$ is the source inhomogeneous term, a and b are limit of function $u(t)$, $K(t,s) = K(s,t)$ symmetric kernel, $K(t,s) \neq K(s,t)$ non-symmetric kernel.

2. DEFINITION AND THEOREM [15]

Definition 2.1. A kernel $K(t,s)$ is said to be symmetric (real) if

$$K(t,s) = K(s,t) \qquad (3)$$

otherwise is non-symmetric i.e

$$K(t,s) \neq K(s,t) \qquad (4)$$

Theorem 2.2. If the kernel $K(t,s) = K(s,t)$ is symmetric and real then

$$(Ku,v) = (u,Kv) \qquad (5)$$

Proof. We take the general case where

$$(u,v) = \int_a^b \int_a^b \left(u(t)\underline{v}(t)dt\right) \qquad Now$$

$$(Ku,v) = \int_a^b \left(\int_a^b (k(t,s)u(s)ds)\right)\underline{v}(t)dt$$

$$(Ku,v) = \int_a^b \int_a^b k(t,s)u(s)\underline{v}(t)dsdt$$

$$re - labelling\ of\ variables$$

$$(Ku,v) = \int_a^b \int_a^b k(s,t)u(s)\underline{v}(t)dtds$$

$$symmetric\ kernel$$

$$(Ku,v) = \int_a^b u(s)\left(\underline{\int_a^b k(s,t)v(t)dt}\right)ds$$

$$real\ kernel$$

$$(Ku,v) = (u,Kv) \qquad (6)$$

3. DESCRIPTION OF THE METHODS

3.1. Adomian decomposition method (ADM)

The Adomian decomposition method [14,16, 17] gives the solution in an infinite series of components that can be recurrently determined. The obtained series may give the exact solution if such a solution exists. Otherwise, the series gives an approximation for the solution that gives high accuracy level.

Consider equation (1) and integrating both sides of from a to t twice lead to:

$$u(t) = a_0 + a_1 t + L^{-1}(g(t)) + L^{-1}\left(\int_a^t K(t,s)u(s)ds\right) \qquad (7)$$

where the initial conditions $u(a)$ and $u'(a)$ are used and L^{-1} is a two-fold integral operator.

We then use the decomposition series:

$$u(t) = \sum_{p=0}^{\infty} u_p(t) \qquad (8)$$

Integrate both sides of (7) to obtain:

$$\sum_{p=0}^{\infty} u_p(t) = a_0 + a_1 t + L^{-1}(g(t)) + L^{-1}\left(\int_a^t K(t,s)\left(\sum_{p=0}^{\infty} u_p(s)\right)ds\right) \quad (9)$$

Simply equation (9), we have:

$$\{u_0(t) + u_1(t) + u_2(t) + u_3(t) + \cdots = a_0 + a_1 t + L^{-1}(g(t)) + L^{-1}\left(\int_a^t K(t,s)u_0(s)ds\right) + L^{-1}\left(\int_a^t K(t,s)u_1(s)ds\right) L^{-1}\left(\int_a^t K(t,s)u_2(s)ds\right) + \cdots \quad (10)$$

To determine the unknown $u_0(t), u_1(t), u_2(t), u_3(t)$ .... of the solution $u(t)$, we set the recurrence relation

$$\{u_0(t) = a_0 + a_1 t + L^{-1}(g(t)) \quad u_1(t) = L^{-1}\left(\int_a^t K(t,s)u_0(s)ds\right) \quad u_2(t) = L^{-1}\left(\int_a^t K(t,s)u_1(s)ds\right) \quad (11) \quad u_3(t) = L^{-1}\left(\int_a^t K(t,s)u_2(s)ds\right) \quad \vdots \quad \vdots \quad u_{m+1}(t) = L^{-1}\left(\int_a^t K(t,s)u_m(s)ds\right)$$

where $m \geq 0$ and $L^{-1} = \int_a^t \int_a^t (.)dtdt$.

Having determined the components $u_i(t), i \geq 0$ the solution $u(t)$ of equation (1) is then obtained in a series form. Using equation (8), the obtained series converges to exact solution.

3.2. Exponentially fitted collocation approximation method (EFCAM)

Exponentially fitted collocation method was proposed by [13]. The whole idea is that we employ derivative of power series function $u(t)$, then substitute into second order integro-differential equation (1). Perturbation was slightly carried out which eventually collocate perturbed equation and form system of equations. Eventually, the unknown $a_0, a_1, a_2, a_3 \dots \dots a_N$ are determine using MAPLE 18 software.

Consider finite power series of the form:

$$u(t) = \sum_{k=0}^{N} a_k t^k \quad (12)$$

and exponentially fitted approximate solution

$$u(t) = \sum_{k=0}^{N} a_k t^k + \tau_2 e^t \quad (13)$$

Here $k \geq 0$, τ is a free parameter and 2 is the order of equation (1) and N is a finite computational length.

Taking derivative of (12) twice and substitute in equation (1), lead to:

$$\{\sum_{k=2}^{N} k(k-1)a_k t^{k-2} = g(t) + + \int_a^t K(t,s)\left(\sum_{k=0}^{N} a_k s^k\right)ds$$

$$\{\sum_{k=2}^{N} k(k-1)a_k t^{k-2} - + \int_a^t K(t,s)\left(\sum_{k=0}^{N} a_k s^k\right)ds = g(t) \quad (14)$$

Expand and collect the like terms, we have:

$$\{-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^0\right)ds\right)a_0$$
$$-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^1\right)ds\right)a_1 \left(2\right.$$
$$\left.-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^2\right)ds\right)\right)a_2$$
$$+\left(6t-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^3\right)ds\right)\right)a_3+$$
$$\vdots + \left(N(N-1)t^{N-2}\right.$$
$$\left.-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^N\right)ds\right)\right)a_N = g(t) \quad (15)$$

Slightly perturb and collocate equation (15), leads to:

$$\{-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^0\right)ds\right)a_0$$
$$-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^1\right)ds\right)a_1 \left(2\right.$$
$$\left.-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^2\right)ds\right)\right)a_2$$
$$+\left(6t-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^3\right)ds\right)\right)a_3+$$
$$\vdots + \left(N(N-1)t^{N-2}\right.$$
$$\left.-\left(\int_a^t K(t,s)\left(\sum_{k=0}^{N} s^N\right)ds\right)\right)a_N - H(t)$$
$$= g(t)$$

$$\{-\left(\int_a^t K(t_i,s)\left(\sum_{k=0}^{N} s^0\right)ds\right)a_0$$
$$-\left(\int_a^t K(t_i,s)\left(\sum_{k=0}^{N} s^1\right)ds\right)a_1 \left(2\right.$$
$$\left.-\left(\int_a^t K(t_i,s)\left(\sum_{k=0}^{N} s^2\right)ds\right)\right)a_2$$
$$+\left(6t-\left(\int_a^t K(t_i,s)\left(\sum_{k=0}^{N} s^3\right)ds\right)\right)a_3+$$
$$\vdots + \left(N(N-1){t_i}^{N-2}\right.$$
$$\left.-\left(\int_a^t K(t_i,s)\left(\sum_{k=0}^{N} s^N\right)ds\right)\right)a_N - T_N(t_i)\tau_1$$
$$- T_{N-1}(t_i)\tau_2 = g(t_i) \quad (16)$$

Here τ_1 and τ_2 are free tau parameters to be determined, $T_N(t_i)$ and $T_{N-1}(t_i)$ are the Chebyshev polynomials of degree N define in [16] and $t_i = a + \frac{(b-a)i}{N+2}$, $i = 1,2, \ldots . N+1$.

Evaluating the integrals at the right side (16) and using few terms from both sides and collecting the coefficients, equation (16) gives rise to ($N+2$) algebraic linear system of equations in ($N+2$) unknown constants. Two extra equations are obtained from the given initial conditions.

Thus, MAPLE 18 software is used to obtain the unknown constants: $a_0, a_1, a_2, a_3, \ldots, a_N$ and τ_1, τ_2.

This is then substitute into the exponential fitted approximate solution (13).

4. NUMERICAL EXPERIMENT

In this section, four examples are presented to show the applicability of the proposed methods for the numerical assessment of symmetric and non-symmetric kernel functions on second order non homogenous Volterra integro-differential equations.

Example 1: Consider symmetric kernel function of second order non-homogenous Volterra integro-differential equation [14].

$$\{\frac{d^2u(t)}{dt^2} = -\frac{1}{2}t^2 - \frac{2}{3}t^3 + \int_0^t K(t,s)u(s)ds \qquad (17)$$

subject to initial conditions:

$$u(0) = 1, \quad u'(0) = 4 \qquad (18)$$

when $K(t,s) = K(s,t) = \{(t-s)^2\,(s-t)^2$.

ADM

Consider algorithm (10), we obtain the following:

$$\{u_0(t) = 1 + 4t - \frac{1}{24}t^4 - \frac{1}{30}t^5 \quad u_1(t) = \frac{1}{60}t^5 + \frac{1}{90}t^6 - \frac{1}{181440}t^9 - \frac{1}{453600}t^{10} \quad u_2(t) = \{\frac{1}{907200}t^{10} + \frac{1}{2494800}t^{11} - \frac{1}{21794572800}t^{14} + \frac{1}{81729648000}t^{15}$$

The solution in closed form is given as

$$u(t) \approx u_0(t) + u_1(t) + u_2(t) \ldots.$$

$$u(t) \approx \{1 + 4t - \frac{1}{24}t^4 - \frac{1}{30}t^5 + \frac{1}{60}t^5 + \frac{1}{90}t^6 - \frac{1}{181440}t^9 - \frac{1}{907200}t^{10} + \frac{1}{907200}t^{10} + \frac{1}{2494800}t^{11} - \frac{1}{21794572800}t^{14} + \frac{1}{81729648000}t^{15} \qquad (19)$$

EFCAM

Consider equation (16), we obtain the following:

$$\{a_0 = 0.99999999. \quad a_1 = 4.000000000 \; a_2 = -1.43341437\;10^{-8}, \; a_3 = 2.527456460\;10^{-7} \; a_4 = -0.041669441\;, \; a_5 = -0.01664654052 \; a_6 = 0.01101120870\;, \; a_7 = 0.0003472824498 \; a_8 = -0.0008566083\,, a_9 = 0.001498756008 \; a_{10} = -0.00186646807\,, a_{11} = 0.001595032809 \; a_{12} = -0.0008935941, \; a_{13} = 0.0002953130446 \; a_{14} = -0.000043614257\,, \tau_1 = 1.90579645\;10^{-10} \; \tau_2 = 1.520953686\;10^{-10}$$

Substitute into equation (13) while computational length N=14, the solution is given in closed form:

$$u(t) \approx \{0.99999999 + 4.000000000t - 1.43341437\;10^{-8}t^2 + 2.527456460\;10^{-7}t^3 - 0.041669441\,t^4 - 0.0166465405t^5 + 0.01101120870t^6 + 0.0003472824t^7 - 0.0008566083t^8 \qquad (20)$$
$$+ 0.001498756008t^9 - 0.00186646807t^{10} + 0.001595032809t^{11} - 0.0008935941t^{12} + 0.0002953130446t^{13} - 0.000043614257t^{14}\;1.520953686\;10^{-10}e^t$$

Table 1. Symmetric kernel on Volterra integro-differential Equation

t	Analytical	ADM	EFCAM
0	1.000000000	1.000000000	1.000000000
0.1	1.399995322	1.399995677	1.399995678
0.2	1.799916622	1.799928711	1.799928711
0.3	2.199532900	2.199630100	2.199630102
0.4	2.598375823	2.598808176	2.598808176
0.5	2.995659736	2.997048599	2.997048599
0.6	3.390193677	3.393822338	3.393822340
0.7	3.780285445	3.788501632	3.788501632
0.8	4.163637751	4.180383887	4.180383888
0.9	4.537236514	4.568723507	4.568723505
1.0	4.897231441	4.952771565	4.952771565

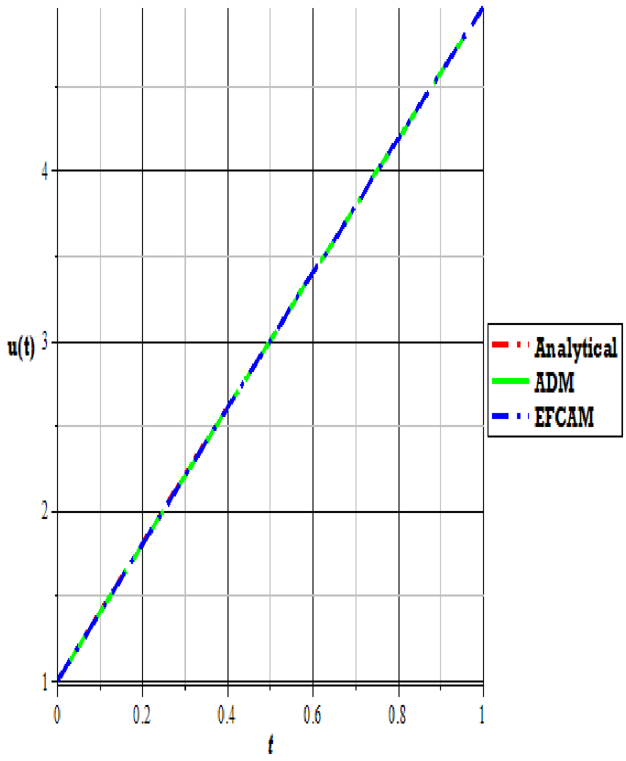

Figure 1. Symmetric kernel function for Example 1

Example 2: Consider symmetric kernel function of second order non-homogenous Volterra integro-differential equation [14]

$$\{\frac{d^2u(t)}{dt^2} = 6t^2 + t + 1 + \int_0^t K(t,s)u(s)ds \qquad (21)$$

subject to initial conditions:

$$u(0) = 0, \quad u'(0) = 2 \qquad (22)$$

when $K(t,s) = K(s,t) = \{s^2t^2 + st + 1\ t^2s^2 + ts + 1$

ADM

Consider algorithm (10), we obtain the following:

$$\{u_0(t) = 2t + \frac{1}{2}t^4 + \frac{1}{6}t^3 + \frac{1}{2}t^2\ u_1(t) = \{\frac{1}{12}t^4 + \frac{1}{120}t^5 + \frac{17}{720}t^6 + \frac{3}{560}t^7 + \frac{1}{105}t^8 + \frac{11}{4320}t^9 + \frac{1}{3240}t^{10} + \frac{1}{1540}t^{11}\ u_2(t) = \{\frac{1}{40}t^5 + \frac{1}{120}t^7 + \frac{1}{40320}t^8 + \frac{17}{362880}t^9 + \frac{1}{134400}t^{10} + \frac{1}{103950}t^{11} + \frac{1}{518400}t^{12} + \frac{1}{5559840}t^{13} + \frac{1}{3363360}t^{14}$$

The solution in closed form is given as:

$$u(t) \approx u_0(t) + u_1(t) + u_2(t) \ldots.$$

$$u(t) \approx \{2t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{7}{12}t^4 + \ + \frac{1}{30}t^5 + \frac{17}{720}t^6 + \frac{23}{1680}t^7 + \frac{11}{1152}t^8 + \frac{941}{362880}t^9 \quad (23) \quad \frac{1147}{3628800}t^{10} + \frac{137}{207900}t^{11} + \frac{1}{518400}t^{12} + \frac{1}{5559840}t^{13} + \frac{1}{3363360}t^{14}$$

EFCAM

Consider equation (16), we obtain the following:

$$\{a_0 = 3.41029653510^{-9}, \ a_1 = 2.000000002\ a_2 = 0.499999766, \ a_3 = 0.1666704294\ a_4 = 0.5832961941, \ a_5 = 0.008572137303\ a_6 = 0.02257332574, \ a_7 = 0.008873873203\ a_8 = 0.0029836925, a_9 = 0.01245032147\ a_{10} = -0.009466627967, a_{11} = 0.007364861819\ a_{12} = -0.002654643617, a_{13} = 0.0005423347641\ \tau_1 = -3.410296535\ 10^{-9}, \tau_2 = -2.306787544\ 10^{-9}$$

Substitute into equation (13) while computational length N=13, the solution is given in closed form:

$$u(t) \approx$$

$$\{3.410296535\ 10^{-9} + 2.000000002t\ 0.4999997661\ t^2 + 0.1666704294t^3 + 0.5832961941\ t^4 + 0.008572137303\ t^5 + 0.02257332574\ t^6 + 0.008873873203t^7\ \ 0.002983692582\ t^8 + 0.01245032147\ t^9 \quad (24)\ - 0.009466627967\ t^{10} + 0.007364861819\ t^{11} - 0.002654643617t^{12} + 0.0005423347641\ t^{13} - 2.306787544\ 10^{-9} e^t$$

Table 2. Symmetric kernel on Volterra integro-differential Equation

t	Analytical	ADM	EFCAM
0	0.0000000000	0.0000000000	0.0000000000
0.1	0.2052251076	0.2052253584	0.2052251080
0.2	0.4222709439	0.4222790453	0.4222709435
0.3	0.6542644055	0.6543268872	0.6542644044
0.4	0.9057985281	0.9060674732	0.9057985261
0.5	1.1830094170	1.183852209	1.183009413
0.6	1.4937040730	1.495867665	1.493704072
0.7	1.8475656370	1.852409718	1.847565655
0.8	2.2564756030	2.266290682	2.256475787
0.9	2.7350108960	2.753436687	2.735012043
1.0	3.3011998730	3.333754134	3.301205663

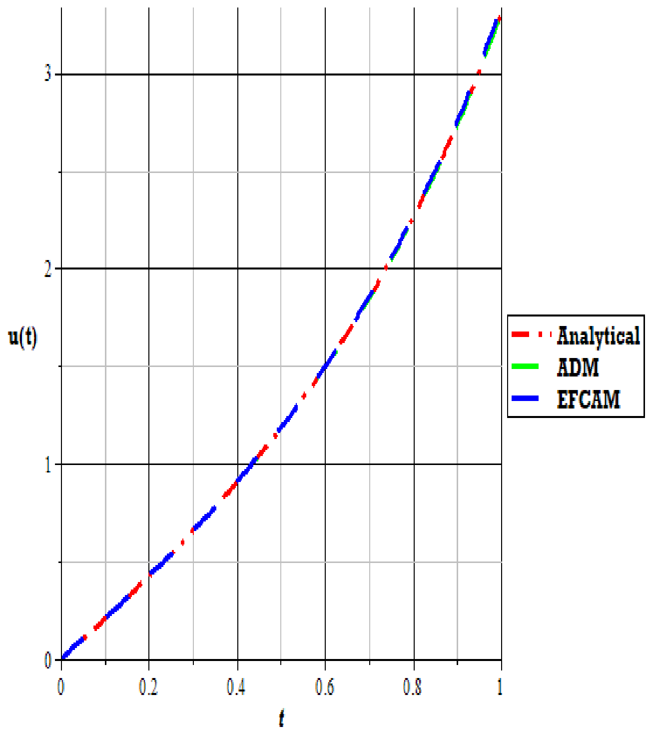

Figure 2. Symmetric kernel function for Example 2

Example 3: Consider non-symmetric kernel function of second order non-homogenous Volterra integro-differential equation [14].

$$\{\frac{d^2u(t)}{dt^2} = t + 1 + \int_0^t K(t,s)u(s)ds \qquad (25)$$

subject to initial conditions:

$$u(0) = 1, \quad u'(0) = 1 \qquad (26)$$

when

$$\{\ K(t,s) = (t-s)\ K(s,t) = (s-t)$$

Consider equation (10), we obtain the following:

$$\{k(t,s) = \{u_0(t) = 1 + t + \tfrac{1}{2!}t^2 + \tfrac{1}{3!}t^3\ u_1(t) = \tfrac{1}{4!}t^4 + \tfrac{1}{5!}t^5 + \tfrac{1}{6!}t^6 + \tfrac{1}{7!}t^7\ u_2(t) = \tfrac{1}{8!}t^8 + \tfrac{1}{9!}t^9 + \tfrac{1}{10!}t^{10} + \tfrac{1}{11!}t^{11}\ \ k(s,t) = \{\ \ u_0(t) = 1 + t + \tfrac{1}{2!}t^2 + \tfrac{1}{3!}t^3\ \ \ u_1(t) = -\tfrac{1}{4!}t^4 - \tfrac{1}{5!}t^5 - \tfrac{1}{6!}t^6 - \tfrac{1}{7!}t^7\ u_2(t) = \tfrac{1}{8!}t^8 + \tfrac{1}{9!}t^9 + \tfrac{1}{10!}t^{10} + \tfrac{1}{11!}t^{11}$$

The solution in closed form is given as:

$$u(t) \approx u_0(t) + u_1(t) + u_2(t)\ ....$$

$$\{k(t,s) = u(t) \approx \{1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \frac{1}{7!}t^7 + \frac{1}{8!}t^8 + \frac{1}{9!}t^9 + \frac{1}{10!}t^{10} + \frac{1}{11!}t^{11}\ \ k(s,t) = u(t) \approx \{1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 - \frac{1}{4!}t^4 - \frac{1}{5!}t^5 - \frac{1}{6!}t^6 - \frac{1}{7!}t^7 - \frac{1}{8!}t^8 - \frac{1}{9!}t^9 - \frac{1}{10!}t^{10} - \frac{1}{11!}t^{11} \quad (27)$$

EFCAM

Consider equation (16), we obtain the following:

$$
\begin{aligned}
k(t,s) = \{a_0 &= 1.000000004\,,\ a_1 = 1.000000000\ a_2 \\
&= 0.4999998473, a_3 \\
&= 0.1666685047\ a_4 \\
&= 0.04165385733,\ a_5 \\
&= 0.008388697652\ a_6 \\
&= 0.00123467016, a_7 \\
&= 0.0004796759526\ a_8 \\
&= -0.0003086880157,\ a_9 \\
&= 0.0002502211379\ a_{10} \\
&= -0.0001040673428,\ a_{11} \\
&= 0.00001910346056\ \tau_1 \\
&= -3.757219228\ 10^{-9}.\tau_2 \\
&= -4.290656975\ 10^{-10}
\end{aligned}
$$

$$
\begin{aligned}
k(s,t) = \{a_0 &= 1.000000006\,,\ a_1 = 1.000000000\ a_2 \\
&= 0.499999751, a_3 = 0.1666696625\ a_4 \\
&= -0.0416875595,\ a_5 \\
&= -0.008243120393\ a_6 \\
&= -0.001639615056, a_7 \\
&= 0.0002574463432\ a_8 \\
&= -0.0005136976419, a_9 \\
&= 0.0004006094653\ a_{10} \\
&= -0.0001665747911, a_{11} \\
&= 0.00003029601045\ \tau_1 \\
&= -6.254569677\ 10^{-9}.\tau_2 \\
&= -3.393171724\ 10^{-10}
\end{aligned}
$$

Substitute into equation (13) while computational length N=11, the solution is given in close form

$$
\begin{aligned}
&\{k(t,s) = u(t) \\
&\approx \{1.000000004 + t + 0.4999998473\ t^2\ 0.166668506t^3 \\
&+ 0.04165385733\ t^4\ 0.00838869765t^5 \\
&+ 0.0012346701\ t^6\ 0.00047967595t^7 \\
&- 0.00030868803\ t^8\ 0.000250221138t^9 \\
&- 0.00010406734t^{10}\ \ 0.00001910346t^{11} \\
&- \ 4.29065698\ 10^{-10} e^t\ \ k(s,t) = u(t) \\
&\approx \{1.000000006 + t \\
&+ 0.4999997515t^2 \quad (29)\ 0.1666696625t^3 \\
&- 0.04168755958t^4\ - 0.008243120393t^5 \\
&- 0.001639615056t^6\ 0.000257446343t^7 \\
&- 0.000513697642t^8\ 0.000400609465t^9 \\
&- 0.000166574791\ t^{10}\ 0.000030296012t^{11} \\
&- 3.393171724\ 10^{-10}\ e^t
\end{aligned}
$$

Table 3. Non- symmetric kernel on Volterra integro-differential Equation

T	Analytical	ADM	EFCAM
0	1.000000000	1.000000000	1.000000000
0.1	1.105170918	1.105170918	1.105170921
0.2	1.221402758	1.221402759	1.221402759
0.3	1.349858808	1.349858808	1.349858809
0.4	1.491824698	1.491824698	1.491824699
0.5	1.648721271	1.648721270	1.648721270
0.6	1.822118800	1.822118801	1.822118800
0.7	2.013752707	2.013752707	2.013752705
0.8	2.225540928	2.225540929	2.225540925
0.9	2.459603111	2.459603111	2.459603109
1.0	2.718281828	2.718281828	2.718281825

Table 4. Non-symmetric kernel n Volterra integro-differential Equation

t	Analytical	ADM	EFCAM
0	1.000000000	1.000000000	1.000000006
0.1	1.105162415	1.105162416	1.105162421
0.2	1.221263909	1.221263907	1.221263912
0.3	1.349141196	1.349141196	1.349141199
0.4	1.489508669	1.489508670	1.489508669
0.5	1.642945601	1.642945600	1.642945600
0.6	1.809882092	1.809882093	1.809882090
0.7	1.990583724	1.990583725	1.990583722
0.8	2.185134863	2.185134863	2.185134859
0.9	2.393420585	2.393420585	2.393420577
1.0	2.615107221	2.615107224	2.615107202

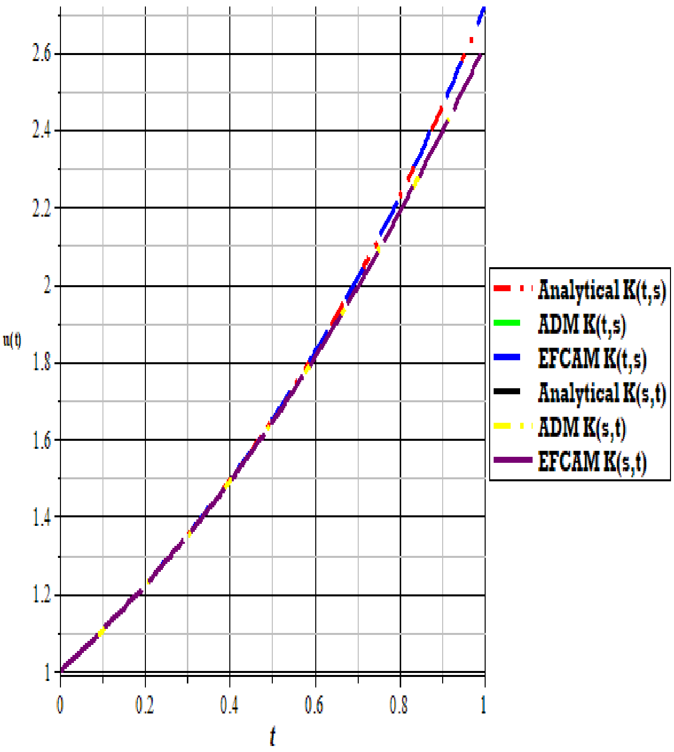

Figure 3. Non-symmetric kernel function for Example 3

Example 4: Consider non-symmetric kernel function of second order non-homogenous Volterra integro-differential equation [14].

$$\{\frac{d^2u(t)}{dt^2} = 1 + t - \frac{t^3}{31} + \int_0^t K(t,s)u(s)ds \qquad (30)$$

$$subject\ to\ initial\ conditions:$$

$$u(0) = 0 \qquad u^{/}(0) = 2 \qquad (31)$$

$$\{\ K(t,s) = t^3s^2 + 1\ K(s,t) = s^3t^2 + 1$$

ADM

Consider equation (10), we obtain the following:

$\{k(t,s) = \{u_0(t) = \{2t + \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{620}t^5\ \ u_1(t) = \{\frac{1}{12}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 - \frac{1}{208320}t^8 + \frac{1}{180}t^9 + \frac{t^{10}}{1080} + \frac{1}{4620}t^{11} - \frac{1}{870480}t^{13}\ \ k(s,t) = \{u_0(t) = \{2t + \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{620}t^5\ \ u_1(t) = \{\frac{1}{12}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 - \frac{1}{208320}t^8 + \frac{1}{144}t^9 + \frac{1}{900}t^{10} + \frac{1}{3960}t^{11} - \frac{1}{773760}t^{13}$

The solution in close formed is given as

$$u(t) \approx u_0(t) + u_1(t) + \cdots.$$

$$\{k(t,s) = u(t) \approx \{2t + \frac{1}{2}t^2 + + \frac{1}{6}t^3 - \frac{1}{620}t^5 + \frac{1}{12}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 - \frac{1}{208320}t^8 + \frac{1}{180}t^9 + \frac{t^{10}}{1080} + \frac{1}{4620}t^{11} - \frac{1}{870480}t^{13}\ \ k(s,t) = u(t) \approx \{\ 2t + \frac{1}{2}t^2 + \frac{1}{6}t^3 - \frac{1}{620}t^5 + \frac{1}{12}t^4 + \frac{1}{120}t^5 + \frac{1}{720}t^6 - \frac{1}{208320}t^8 + \frac{1}{144}t^9 + \frac{1}{900}t^{10} + \frac{1}{3960}t^{11} - \frac{1}{773760}t^{13} \qquad (32)$$

EFCAM

Consider equation (16), we obtain the following:

$$k(s,t) = \{a_0 = -3.24787640710^{-9}, a_1 = 2.000000007\ a_2 = 0.5000003604, a_3 = 0.1666613521\ a_4 = 0.0833789638, a_5 = 0.006476916042\ a_6 = 0.00222628486, a_7 = -0.001476579128\ a_8 = 0.002670665105,\ \ a_9 = 0.003464478311\ a_{10} = 0.00129650136, a_{11} = 0.001096040923\ a_{12} = -0.000707666673, a_{13} = 0.000231470098\ \tau_1 = 3.24787640710^{-9}, \tau_2 = -6.59335272310^{-9}$$

Substitute into equation (13) while computational length N=13, the solution is given in closed form

$$\{k(t,s) = u(t) \approx \{1.17645503810^{-10} + 2.000000010t + 0.5000001012\,t^2 + 0.1666654340t^3 + 0.08333942803t^4 + 0.006727115765t^5 + 0.00115497992t^6 + 0.00169299751t^7 + -0.003871054346t^8 + 0.01424531284t^9 - 0.00772113217t^{10} + 0.007014853163t^{11} - 0.00291650618t^{12} + 0.000611187901t^{13} - 1.17645503810^{-10}e^t \;. \qquad .$$

$$= u(t) \approx \{-3.24787640710^{-9} + 2.000000007t + 0.5000003604t^2 + 0.1666613521t^3 + 0.08337896382t^4 + 0.006476916042t^5 + 0.00222628487t^6 - 0.00147657913t^7 + 0.002670665105t^8 + 0.0034644783t^9 + 0.001296501364\,t^{10} + 0.00109604092t^{11} + -0.0007076667t^{12} + 0.000231470092t^{13} + 3.24787640710^{-9}e^t$$

Table 5. Non- symmetric kernel on Volterra Integro-differential Equation

t	Analytical	ADM	EFCAM
0	0.000000000	0000000000	0.000000000
0.1	0.205175086	0.205175069	0.205175070
0.2	0.421470011	0.421468909	0.421468918
0.3	0.650205199	0.650192458	0.650192579
0.4	0.892948544	0.892876066	0.892877125
0.5	1.151564198	1.151285222	1.151291369
0.6	1.428286475	1.427449669	1.427476700
0.7	1.725830112	1.723722238	1.723819881
0.8	2.047555014	2.042895706	2.043199983
0.9	2.397713108	2.388422212	2.389267401
1.0	2.781817795	2.764801302	2.766942728

Table 6. Non- symmetric kernel on Volterra Integro-differential Equation

t	Analytical	ADM	EFCAM
0	0.000000000	0.000000000	0.000000000
0.1	0.205175069	0.205175069	0.205175071
0.2	0.421468914	0.421468909	0.421468919
0.3	0.650192547	0.650192487	0.650192554
0.4	0.892876734	0.892876451	0.892876743
0.5	1.151288446	1.151288132	1.151288459
0.6	1.427461417	1.427464916	1.427461431
0.7	1.723757695	1.723784226	1.723757717
0.8	2.042989624	2.043105095	2.042989691
0.9	2.388649276	2.389036150	2.388649643
1.0	2.765316717	2.766411307	2.765318799

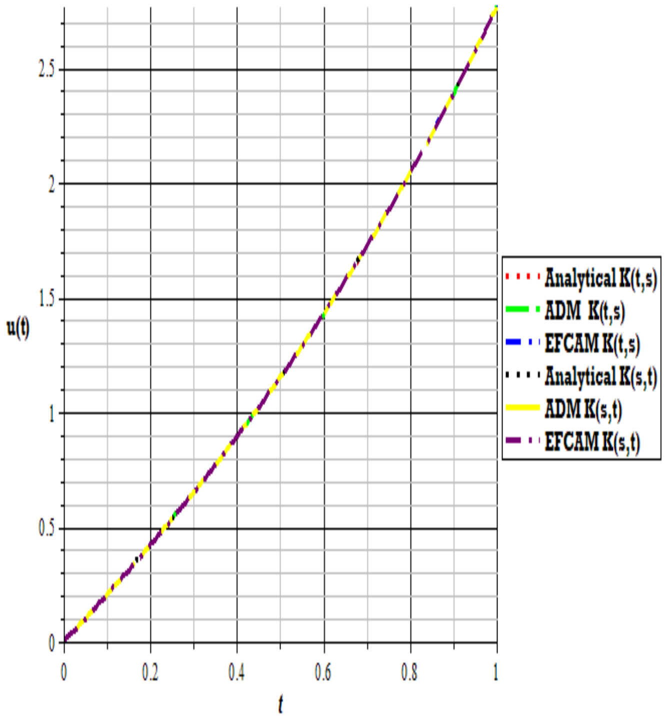

Figure 4. Non-symmetric kernel function for Example 4

5. DISCUSSION AND CONCLUSION

In this paper, we present two numerical techniques for numerical solutions of symmetric and non-symmetric kernel functions on Volterra non homogenous second order integro-differential equations. Table 1, Table 2, Figure 1 and Figure 2 depict no significant difference on symmetric kernels $K(t,s) = K(s,t)$ while Table 3, Table 4, Table 5, Table 6, Figure 3, Figure 4, Figure 5 and Figure 6 depict non- symmetric kernels $K(t,s) \neq K(s,t)$ which established in definition (1). From numerical solutions obtained it was demonstrated that proposed methods are powerful and efficient in given approximation solutions in closed form. However, comparison certified that EFCM gives closed numerical solutions to that of analytical solutions as evident in all Tables of numerical solutions obtained. Therefore recommend for solving similar problems in applied sciences.

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Authors' Contribution

K.I.F: Conceptualization and carry out all computation works of all examples considered.

I.G.B: Vetting the literature searches and typing the article.

A.T.T: Validate the methods and coordinate the final write up.

The Declaration of Ethics Committee Approval

This article does not require ethics committee

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Ethics.

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