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N-SPACES

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Abstract. In this paper, we introduce n-spaces constructed over an local ring with the maximal ideal (of non-unit elements). So, we give the example of an octonion n-space. Finally, we give two collineations of quaternion n-space.

1. Introduction and Preliminaries

In the early 1930s, P. Jordan, who is a physicist, has began to study with Jordan algebras. The algebra $\mathbf{H}(\mathbf{O}_3)$ is firstly used by Jordan, to define an octonion plane (over real octonion division algebra) [10]. Freudenthal, in [8], gave the same construction in [10]. Later, Springer, in [12], extended the construction given by Jordan and Freudenthal to the octonion (or Cayley) division algebras defined over a field whose characteristic is different from 2 and 3.

In [3], Bix deals with $\mathbf{J} = \mathbf{H}\ (\mathbf{O}_3, J\gamma)$, the set of 3 by 3 matrices with entries in an octonion algebra $\mathbf{O}$ defined over a local ring R with the maximal ideal I (of non-unit elements), that are symmetric with respect to the canonical involution $J\gamma : X \to \gamma^{-1}\overline{X}^{\mathbf{t}}\gamma$ where the γ_i are elements of $R \setminus I$ and $\gamma :=\mathrm{diag}\{\gamma_1, \gamma_2, \gamma_3\}$. Hence, any element X of $\mathbf{J}$ is of the form

$$X = \begin{pmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3\overline{a_2} \\ \gamma_1\overline{a_3} & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2\overline{a_1} & \alpha_3 \end{pmatrix} \text{ for } \alpha_i \in R \text{ and } a_i \in \mathbf{O}.$$

If it is defined a cubic form N such that $N(X) := \det X$, a quadratic mapping $X \to X^\sharp$:=adjoint of X, and a basepoint $C := I_3$ on $\mathbf{J}$ are defined, then the triple $(\mathbf{J}, N, C)$ is a quadratic (exceptional) Jordan algebra under the operator $U_X Y = T(X,Y)X - 2\left(X^\sharp \times Y\right)$ [11]. Then, for $X = \begin{pmatrix} \alpha_1 & \gamma_2 a_3 & \gamma_3\overline{a_2} \\ \gamma_1\overline{a_3} & \alpha_2 & \gamma_3 a_1 \\ \gamma_1 a_2 & \gamma_2\overline{a_1} & \alpha_3 \end{pmatrix}$ and $Y =$

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$\begin{pmatrix} \beta_1 & \gamma_2 b_3 & \gamma_3 \overline{b_2} \\ \gamma_1 \overline{b_3} & \beta_2 & \gamma_3 b_1 \\ \gamma_1 b_2 & \gamma_2 \overline{b_1} & \beta_3 \end{pmatrix} \in \mathbf{J}$, we can give the similar results to those given in [11, 3, 7]:

$$N(X) = \alpha_1\alpha_2\alpha_3 - \alpha_1\gamma_2\gamma_3 n(a_1) - \alpha_2\gamma_3\gamma_1 n(a_2) - \alpha_3\gamma_1\gamma_2 n(a_3) + \gamma_1\gamma_2\gamma_3 2t((a_1a_2)a_3),$$

$$X^\sharp = (X_{ij})_{3\times 3} \text{ for } X_{ii} = \alpha_j\alpha_k - \gamma_j\gamma_k n(a_i), x_{ij} = \gamma_i\gamma_k a_i a_j - \gamma_i \alpha_k \overline{a_k} \text{ and } X_{ji} = \overline{X_{ij}},$$

$$X \times Y = (z_{ij})_{3\times 3} \text{ for } \begin{cases} z_{ii} = \frac{1}{2}\left[\alpha_j\beta_k + \beta_j\alpha_k - 2\gamma_j\gamma_k n(a_i, b_i)\right], \\ z_{ij} = \frac{1}{2}\left(\gamma_j\left[\gamma_k\overline{(a_ib_j + b_ia_j)} - (\alpha_k b_k + \beta_k a_k)\right]\right),\ z_{ji} = \overline{z_{ij}} \end{cases},$$

$$T(X,Y) = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + 2\gamma_2\gamma_3 n(a_1,b_1) + 2\gamma_3\gamma_1 n(a_2,b_2) + 2\gamma_1\gamma_2 n(a_3,b_3),$$

where (i,j,k) is a cyclic permutation of $(1,2,3)$, n (defined by $n(x) := x\overline{x}$) is the norm (quadratic) form over $\mathbf{O}$, t (defined by $t(x) := \frac{1}{2}(x+\overline{x})$) is the trace (linear) form over $\mathbf{O}$ and finally $n(x,y)$ (defined by $n(x,y) := \frac{1}{2}[n(x+y) - n(x) - n(y)]$) is symmetric bilinear norm w.r.t. n.

Let Π denote the set of elements of rank 1 in $\mathbf{J}$. Then,

$$\Pi = \left\{ X \,\middle|\, X \in \mathbf{J} \setminus I\mathbf{J} \text{ and } X \times X = X^\sharp = 0 \right\}.$$

Note that, if $X \in \Pi$ and α is an element in $R \setminus I$, then $\alpha X \in \Pi$. For $X \in \Pi$, let X_* and X^* be two copies of the set $\{\alpha X \mid \alpha \in R \setminus I\}$.

Now, it is time to give the definition of an octonion plane $\mathbf{P}(\mathbf{J})$ from [3, 6].

Definition 1. *The octonion plane* $\mathbf{P}(\mathbf{J}) = (\mathbf{P}, \mathbf{L}, |, \backsimeq)$ *consists of the incidence structure* $(\mathbf{P}, \mathbf{L}, |)$ *(points, lines, and incidence), and the connection relation is defined as follows:*

$\mathbf{P} = \{X_* \mid X \in \Pi\}$, $\mathbf{L} = \{X^* \mid X \in \Pi\}$,

$X_* | Y^*$, *X_* is on Y^*, if* $V_{Y,X} = 0$, *that is,* $V_{Y,X} =: \{1XY\} = \{X1Y\} = \{XY1\} = X \cdot Y = 0$ *where* $X \cdot Y = \frac{1}{2}(XY + YX)$ *(Jordan multiplication).*

$X_* \backsimeq Y_*$, *X_* is connected to Y_* if* $X \times Y \in I\mathbf{J}$,

$X^* \backsimeq Y^*$, *X^* is connected to Y^* if* $X \times Y \in I\mathbf{J}$,

$X_* \backsimeq Y^*$, *X_* is connected (or near) to Y^* if* $T(X,Y) \in I$.

Now, we recall some informations on projective Klingenberg and Moufang-Klingenberg planes from [2].

Definition 2. *Let* $\mathbb{M} = (\mathbf{P}, \mathbf{L}, \in', \sim')$ *consist of an incidence structure* $(\mathbf{P}, \mathbf{L}, \in')$ *(points, lines, incidence) and an equivalence relation '$\sim'$' (neighbour relation) on* $\mathbf{P}$ *and on* $\mathbf{L}$*. Then* $\mathbb{M}$ *is called a projective Klingenberg plane (PK-plane), if it satisfies the following axioms:*

(PK1) If P, Q are non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are non-neighbour lines, then there is a unique point $g \wedge h$ on both g and h.

(PK3) There is a projective plane $\mathbb{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in')$ *and incidence structure epimorphism* $\Psi : \mathbb{M} \to \mathbb{M}^*$*, such that the conditions*

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim' Q,\ \Psi(g) = \Psi(h) \iff g \sim' h$$

hold for all $P, Q \in \mathbf{P},\ g, h \in \mathbf{L}$.

A point $P \in' \mathbf{P}$ is called near a line $g \in' \mathbf{L}$ iff there exists a line h such that $P \in' h$ for some line $h \sim' g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a collineation of $\mathbb{M}$.

A Moufang-Klingenberg plane (MK-plane) is a PK-plane $\mathbb{M}$ that generalizes a Moufang plane, and for which $\mathbb{M}^*$ is a Moufang plane (for the details see [2]).

In [9, Chapter III.2, Theorem 1], Jacobson showed that the fact that $(\mathbf{D}_n, J\gamma)$ is a Jordan algebra implies that $\mathbf{D}$ is associative if $n \geq 4$ but alternative with its symmetric elements in the nucleus if $n = 3$. Therefore, in [1], in the case of $n \geq 4$ we were able to study the elements of the quaternion division algebra $\mathbb{Q}$ over a field F, which is associative. For this reason, we could not continue studying by elements of an octonion algebra. But, without the need for Jordan matrix algebras, the obtained results in [1] show the existence of the following two possibilities: either the definition of the octonion plane (octonion 2-space) may be extended to an (octonion) $n-$space or a new geometric structure may be obtained. We need to recall some results in the case $n = 4$ from [1] for better understanding of the construction of the new structure which we call n-space.

Consider $\mathcal{A} := \mathbb{Q} + \mathbb{Q}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1 b_1 + (a_1 b_2 + a_2 b_1)\varepsilon, \quad (a_i, b_i \in \mathbb{Q},\ i = 1, 2)$$

Then $\mathcal{A}$ is a (not commutative) local ring with the maximal ideal $\mathbf{I} = \mathbb{Q}\varepsilon$ of non-units.

$\mathbf{J'} = \mathbf{H}\ (\mathcal{A}_4, J\gamma)$, the set of 4 by 4 matrices, with entries from $\mathcal{A}$, that are symmetric with respect to the canonical involution $J\gamma : X \to \gamma^{-1}\overline{X}^{\mathbf{t}}\gamma$ where the γ_i are non-zero elements of F and $\gamma :=$diag$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$. Hence, any element X of $\mathbf{J'}$ is of the form

$$X = [x_{ij}] = \begin{pmatrix} \alpha_1 & \gamma_2 a_{12} & \gamma_3 \overline{a_{13}} & \gamma_4 a_{14} \\ \gamma_1 \overline{a_{12}} & \alpha_2 & \gamma_3 a_{23} & \gamma_4 \overline{a_{24}} \\ \gamma_1 a_{13} & \gamma_2 \overline{a_{23}} & \alpha_3 & \gamma_4 a_{34} \\ \gamma_1 \overline{a_{14}} & \gamma_2 a_{24} & \gamma_3 \overline{a_{34}} & \alpha_4 \end{pmatrix} \text{ for } \alpha_i \in F \text{ and } a_i \in \mathcal{A}.$$

If we take a quartic form N such that $N(X) := \det X$, a cubic mapping $X \to X^{\sharp} :=$adjoint of X, and a basepoint $C := I_4$ on $\mathbf{J}$, then: it is clear that

$$\begin{aligned} T(X, Y) \quad = \quad & \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + \alpha_4\beta_4 \\ & + 2\gamma_1\gamma_2 n(a_{12}, b_{12}) + 2\gamma_1\gamma_3 n(a_{13}, b_{13}) + 2\gamma_1\gamma_4 n(a_{14}, b_{14}) \\ & + 2\gamma_2\gamma_3 n(a_{23}, b_{23}) + 2\gamma_2\gamma_4 n(a_{24}, b_{24}) + 2\gamma_3\gamma_4 n(a_{34}, b_{34}), \end{aligned}$$

as $T(X,Y) := T(X{\cdot}Y) =$trace$(X{\cdot}Y)$. Moreover, $X{\times}Y := \frac{1}{6}\left[(X+Y)^{\#} - X^{\#} - Y^{\#}\right]$ because of $X \times X = X^{\#}$.

So, it is obtained the following results for the quaternion 3-space $\mathbf{P}(\mathbf{J}') = (\mathbf{P}, \mathbf{L}, |\ , \backsimeq)$ where $\mathbf{J}'$ is the 56−dimensional special Jordan matrix algebra:

The set of points $\mathbf{P}$ consists of the following four classes (which we call as points of types 1,2,3 and 4, respectively):

$$\left\{ P_1 = \begin{pmatrix} 1 & \gamma_1^{-1}\gamma_2\overline{x_2} & \gamma_1^{-1}\gamma_3\overline{x_3} & \gamma_1^{-1}\gamma_4\overline{x_4} \\ x_2 & \gamma_1^{-1}\gamma_2 n(x_2) & \gamma_1^{-1}\gamma_3 x_2\overline{x_3} & \gamma_1^{-1}\gamma_4 x_2\overline{x_4} \\ x_3 & \gamma_1^{-1}\gamma_2 x_3\overline{x_2} & \gamma_1^{-1}\gamma_3 n(x_3) & \gamma_1^{-1}\gamma_4 x_3\overline{x_4} \\ x_4 & \gamma_1^{-1}\gamma_2 x_4\overline{x_2} & \gamma_1^{-1}\gamma_3 x_4\overline{x_3} & \gamma_1^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} 1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}^t \mid x_i \in \mathcal{A} \right\} \cup$$

$$\left\{ P_2 = \begin{pmatrix} \gamma_2^{-1}\gamma_1 n(x_1) & x_1 & \gamma_2^{-1}\gamma_3 x_1\overline{x_3} & \gamma_2^{-1}\gamma_4 x_1\overline{x_4} \\ \gamma_2^{-1}\gamma_1\overline{x_1} & 1 & \gamma_2^{-1}\gamma_3\overline{x_3} & \gamma_2^{-1}\gamma_4\overline{x_4} \\ \gamma_2^{-1}\gamma_1 x_3\overline{x_1} & x_3 & \gamma_2^{-1}\gamma_3 n(x_3) & \gamma_2^{-1}\gamma_4 x_3\overline{x_4} \\ \gamma_2^{-1}\gamma_1 x_4\overline{x_1} & x_4 & \gamma_2^{-1}\gamma_3 x_4\overline{x_3} & \gamma_2^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} x_1 \\ 1 \\ x_3 \\ x_4 \end{pmatrix}^t \mid x_1 \in \mathbf{I},\ x_3, x_4 \in \mathcal{A} \right\} \cup$$

$$\left\{ P_3 = \begin{pmatrix} \gamma_3^{-1}\gamma_1 n(x_1) & \gamma_3^{-1}\gamma_2 x_1\overline{x_2} & x_1 & \gamma_3^{-1}\gamma_4 x_1\overline{x_4} \\ \gamma_3^{-1}\gamma_1 x_2\overline{x_1} & \gamma_3^{-1}\gamma_2 n(x_2) & x_2 & \gamma_3^{-1}\gamma_4 x_2\overline{x_4} \\ \gamma_3^{-1}\gamma_1\overline{x_1} & \gamma_3^{-1}\gamma_2\overline{x_2} & 1 & \gamma_3^{-1}\gamma_4\overline{x_4} \\ \gamma_3^{-1}\gamma_1 x_4\overline{x_1} & \gamma_3^{-1}\gamma_2 x_4\overline{x_2} & x_4 & \gamma_3^{-1}\gamma_4 n(x_4) \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \\ 1 \\ x_4 \end{pmatrix}^t \mid x_1, x_2 \in \mathbf{I},\ x_4 \in \mathcal{A} \right\} \cup$$

$$\left\{ P_4 = \begin{pmatrix} \gamma_4^{-1}\gamma_1 n(x_1) & \gamma_4^{-1}\gamma_2 x_1\overline{x_2} & \gamma_4^{-1}\gamma_3 x_1\overline{x_3} & x_1 \\ \gamma_4^{-1}\gamma_1 x_2\overline{x_1} & \gamma_4^{-1}\gamma_2 n(x_2) & \gamma_4^{-1}\gamma_3 x_2\overline{x_3} & x_2 \\ \gamma_4^{-1}\gamma_1 x_3\overline{x_1} & \gamma_4^{-1}\gamma_2 x_3\overline{x_2} & \gamma_4^{-1}\gamma_3 n(x_3) & x_3 \\ \gamma_4^{-1}\gamma_1\overline{x_1} & \gamma_4^{-1}\gamma_2\overline{x_2} & \gamma_4^{-1}\gamma_3\overline{x_3} & 1 \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{pmatrix}^t \mid x_i \in \mathbf{I} \right\},$$

the set of lines $\mathbf{L}$ consists of the following four classes (which we call as lines of types 1,2,3 and 4, respectively):

$$\left\{ l_1 = \begin{bmatrix} 1 & -m_2 & -m_3 & -m_4 \\ -\gamma_2^{-1}\gamma_1\overline{m_2} & \gamma_2^{-1}\gamma_1 n(m_2) & \gamma_2^{-1}\gamma_1\overline{m_2}m_3 & \gamma_2^{-1}\gamma_1\overline{m_2}m_4 \\ -\gamma_3^{-1}\gamma_1\overline{m_3} & \gamma_3^{-1}\gamma_1\overline{m_3}m_2 & \gamma_3^{-1}\gamma_1 n(m_3) & \gamma_3^{-1}\gamma_1\overline{m_3}m_4 \\ -\gamma_4^{-1}\gamma_1\overline{m_4} & \gamma_4^{-1}\gamma_1\overline{m_4}m_2 & \gamma_4^{-1}\gamma_1\overline{m_4}m_3 & \gamma_4^{-1}\gamma_1 n(m_4) \end{bmatrix} =: \begin{bmatrix} 1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}^t \mid m_i \in \mathbf{I} \right\} \cup$$

$$\left\{ l_2 = \begin{bmatrix} \gamma_1^{-1}\gamma_2 n(m_1) & -\gamma_1^{-1}\gamma_2\overline{m_1} & \gamma_1^{-1}\gamma_2\overline{m_1}m_3 & \gamma_1^{-1}\gamma_2\overline{m_1}m_4 \\ -m_1 & 1 & -m_3 & -m_4 \\ \gamma_3^{-1}\gamma_2\overline{m_3}m_1 & -\gamma_3^{-1}\gamma_2\overline{m_3} & \gamma_3^{-1}\gamma_2 n(m_3) & \gamma_3^{-1}\gamma_2\overline{m_3}m_4 \\ \gamma_4^{-1}\gamma_2\overline{m_4}m_1 & -\gamma_4^{-1}\gamma_2\overline{m_4} & \gamma_4^{-1}\gamma_2\overline{m_4}m_3 & \gamma_4^{-1}\gamma_2 n(m_4) \end{bmatrix} =: \begin{bmatrix} m_1 \\ 1 \\ m_3 \\ m_4 \end{bmatrix}^t \mid m_1 \in \mathcal{A},\ m_3, m_4 \in \mathbf{I} \right\} \cup$$

$$\left\{ l_3 = \begin{bmatrix} \gamma_1^{-1}\gamma_3 n(m_1) & \gamma_1^{-1}\gamma_3\overline{m_1}m_2 & -\gamma_1^{-1}\gamma_3\overline{m_1} & \gamma_1^{-1}\gamma_3\overline{m_1}m_4 \\ \gamma_2^{-1}\gamma_3\overline{m_2}m_1 & \gamma_2^{-1}\gamma_3 n(m_2) & -\gamma_2^{-1}\gamma_3\overline{m_2} & \gamma_2^{-1}\gamma_3\overline{m_2}m_4 \\ -m_1 & -m_2 & 1 & -m_4 \\ \gamma_4^{-1}\gamma_3\overline{m_4}m_1 & \gamma_4^{-1}\gamma_3\overline{m_4}m_2 & -\gamma_4^{-1}\gamma_3\overline{m_4} & \gamma_4^{-1}\gamma_3 n(m_4) \end{bmatrix} =: \begin{bmatrix} m_1 \\ m_2 \\ 1 \\ m_4 \end{bmatrix}^t \mid m_1, m_2 \in \mathcal{A},\ m_4 \in \mathbf{I} \right\} \cup$$

$$\left\{ l_4 = \begin{bmatrix} \gamma_1^{-1}\gamma_4 n(m_1) & \gamma_1^{-1}\gamma_4\overline{m_1}m_2 & \gamma_1^{-1}\gamma_4\overline{m_1}m_3 & -\gamma_1^{-1}\gamma_4\overline{m_1} \\ \gamma_2^{-1}\gamma_4\overline{m_2}m_1 & \gamma_2^{-1}\gamma_4 n(m_2) & \gamma_2^{-1}\gamma_4\overline{m_2}m_3 & -\gamma_2^{-1}\gamma_4\overline{m_2} \\ \gamma_3^{-1}\gamma_4\overline{m_3}m_1 & \gamma_3^{-1}\gamma_4\overline{m_3}m_2 & \gamma_3^{-1}\gamma_4 n(m_3) & -\gamma_3^{-1}\gamma_4\overline{m_3} \\ -m_1 & -m_2 & -m_3 & 1 \end{bmatrix} =: \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ 1 \end{bmatrix}^t \mid m_i \in \mathcal{A} \right\}.$$

The incidence relation "|", equivalent to $X \cdot Y = 0$, is obtained as follows:

$$\begin{aligned} [1, k_2, k_3, k_4] \quad = \quad & \{(k_2 + k_3y_3 + k_4y_4, 1, y_3, y_4) |\, y_3, y_4 \in \mathcal{A}\} \cup \\ & \{(k_2z_2 + k_3 + k_4z_4, z_2, 1, z_4) |\, z_2 \in \mathbf{I}, z_4 \in \mathcal{A}\} \cup \\ & \{(k_2t_2 + k_3t_3 + k_4, t_2, t_3, 1) |\, t_2, t_3 \in \mathbf{I}\}, \end{aligned}$$

$$\begin{aligned} [l_1, 1, l_3, l_4] \quad = \quad & \{(1, l_1 + l_3x_3 + l_4x_4, x_3, x_4) |\ x_3, x_4 \in \mathcal{A}\} \cup \\ & \{(z_1, l_1z_1 + l_3 + l_4z_4, 1, z_4) |\, z_1 \in \mathbf{I}, z_4 \in \mathcal{A}\} \cup \\ & \{(t_1, l_1t_1 + l_3t_3 + l_4, t_3, 1) |\, t_1, t_3 \in \mathbf{I}\}, \end{aligned}$$

$$\begin{aligned}[m_1,m_2,1,m_4] &= \{(1,x_2,m_1+m_2x_2+m_4x_4,x_4)|\,x_2,x_4\in\mathcal{A}\}\cup\\ &\quad\{(y_1,1,m_1y_1+m_2+m_4y_4,y_4)|\,y_1\in\mathbf{I},y_4\in\mathcal{A}\}\cup\\ &\quad\{(t_1,t_2,m_1t_1+m_2t_2+m_4,1)|\,t_1,t_2\in\mathbf{I}\},\\ [n_1,n_2,n_3,1] &= \{(1,x_2,x_3,n_1+n_2x_2+n_3x_3,)|\,x_2,x_3\in\mathcal{A}\}\cup\\ &\quad\{(y_1,1,y_3,n_1y_1+n_2+n_3y_3,)|\,y_1\in\mathbf{I},y_3\in\mathcal{A}\}\cup\\ &\quad\{(z_1,z_2,1,n_1z_1+n_2z_2+n_3)|\,z_1,z_2\in\mathbf{I}\}.\end{aligned}$$

Finally; the connection relation "$\backsimeq$", equivalent to $X\times Y\in I\mathbf{J}$, is obtained as follows:

$$\begin{aligned}(x_1,x_2,x_3,x_4) &\backsimeq (y_1,y_2,y_3,y_4)\Leftrightarrow x_i-y_i\in\mathbf{I}\text{ for }i=1,2,3,4,\\ [k_1,k_2,k_3,k_4] &\backsimeq [n_1,n_2,n_3,n_4]\Leftrightarrow k_i-n_i\in\mathbf{I}\text{ for }i=1,2,3,4.\end{aligned}$$

Besides, from types of points on lines, it is clear that a point and a line of same type is not connected (near). Moreover, the result is equivalent to $T(X,Y)\notin I=\{0\}$ for a point (or line) X and a line (or point) Y, respectively. In the other cases, we say that they are connected (near).

Now, we are ready to construct the n-space.

2. n-Spaces

Let $\mathbf{R}$ be a local ring with the maximal ideal $\mathbf{I}$ (of non-unit elements). Then $\mathbb{S}_n(\mathbf{R})=(\mathbf{P},\mathbf{L},\in,\sim)$ is the incidence structure with neighbour relation defined as follows.

The set of points $\mathbf{P}$ consists of the following $n+1$ points (which we call as points of types 1,2,3,..., $n+1$; respectively):

$$\mathbf{P}=\{P_i=(x_1,...,x_{i-1},1,x_{i+1},...,x_{n+1})\mid x_1,...,x_{i-1}\in\mathbf{I}\text{ and }x_{i+1},...,x_{n+1}\in\mathbf{R}\}.$$

The set of lines $\mathbf{L}$ consists of the following $n+1$ lines (which we call as lines of types 1,2,3,..., $n+1$; respectively):

$$\mathbf{L}=\{M_i=[m_1,...,m_{i-1},1,m_{i+1},...,m_{n+1}]\mid m_1,...,m_{i-1}\in\mathbf{R}\text{ and }m_{i+1},...,m_n\in\mathbf{I}\}.$$

The incidence relation "$\in$" is defined as follows:

$$\begin{aligned}M_1 &= [1,m_2,m_3,m_4,m_5,...,m_{n-1},m_n,m_{n+1}]\\ &= \{(m_2+m_3y_3+\cdots+m_{n+1}y_{n+1},1,y_3,...,y_{n+1})|\ y_3,...,y_{n+1}\in\mathbf{R}\}\cup\\ &\quad\left\{\begin{array}{c}(m_2z_2+m_3+m_4z_4+\cdots+m_{n+1}z_{n+1},z_2,1,z_4,...,z_{n+1})|\\ z_2\in\mathbf{I},z_4,...,z_{n+1}\in\mathbf{R}\end{array}\right\}\cup\\ &\quad\left\{\begin{array}{c}(m_2t_2+m_3t_3+m_4+m_5t_5+\cdots+m_{n+1}t_{n+1},t_2,t_3,1,t_5,...,t_{n+1})|\\ t_2,t_3\in\mathbf{I},t_5,...,t_{n+1}\in\mathbf{R}\end{array}\right\}\cup\\ &\quad\vdots\\ &\quad\left\{\begin{array}{c}(m_2k_2+\cdots+m_{n-1}k_{n-1}+m_n+m_{n+1}k_{n+1},k_2,k_3,...,k_{n-1},1,k_{n+1})|\\ k_2,...,k_{n-1}\in\mathbf{I},\ k_{n+1}\in\mathbf{R}\end{array}\right\}\cup\\ &\quad\{(m_2l_2+m_3l_3+\cdots+m_nl_n+m_{n+1},l_2,l_3,l_4,...,l_n,1)|\ l_2,...,l_n\in\mathbf{I}\},\end{aligned}$$

$$\begin{aligned}
M_2 &= [m_1, 1, m_3, m_4, m_5, ...m_{n-1}, m_n, m_{n+1}] \\
&= \{(1, m_1 + m_3 y_3 + \cdots + m_{n+1} y_{n+1}, y_3, ..., y_{n+1}) |\ y_3, ..., y_{n+1} \in \mathbf{R}\} \cup \\
& \left\{ \begin{array}{c} (z_1, m_1 z_1 + m_3 + m_4 z_4 + \cdots + m_{n+1} z_{n+1}, 1, z_4, ..., z_{n+1}) | \\ z_1 \in \mathbf{I}, z_4, ..., z_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
& \left\{ \begin{array}{c} (t_1, m_1 t_1 + m_3 t_3 + m_4 + m_5 t_5 + \cdots + m_{n+1} t_{n+1}, t_3, 1, t_5, ..., t_{n+1}) | \\ t_1, t_3 \in \mathbf{I}, t_5, ..., t_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
& \vdots \\
& \left\{ \begin{array}{c} (k_1, m_1 k_1 + m_3 k_3 + \cdots + m_{n-1} k_{n-1} + m_n + m_{n+1} k_{n+1}, k_3, ..., k_{n-1}, 1, k_{n+1}) | \\ k_1, k_3, ..., k_{n-1} \in \mathbf{I},\ k_{n+1} \in \mathbf{R} \end{array} \right\} \cup \\
& \{(l_1, m_1 l_1 + m_3 l_3 + \cdots + m_n l_n + m_{n+1}, l_3, l_4, ..., l_n, 1) |\ l_1, l_3, ..., l_n \in \mathbf{I}\},
\end{aligned}$$

$$\vdots$$

$$\begin{aligned}
M_{n+1} &= [m_1, m_2, m_3, m_4, ...m_{n-1}, m_n, 1] \\
&= \{(1, y_2, y_3, ..., y_n, m_1 + m_2 y_2 + \cdots + m_n y_n) |\ y_2, ..., y_n \in \mathbf{R}\} \cup \\
& \left\{ \begin{array}{c} (z_1, 1, z_3, z_4, ..., z_n, m_1 z_1 + m_2 + m_3 z_3 + \cdots + m_n z_n) | \\ z_1 \in \mathbf{I}, z_3, ..., z_n \in \mathbf{R} \end{array} \right\} \cup \\
& \left\{ \begin{array}{c} (t_1, t_2, 1, t_4, ..., t_n, m_1 t_1 + m_2 t_2 + m_3 + m_4 t_4 + \cdots + m_n t_n) | \\ t_1, t_2 \in \mathbf{I}, t_4, ..., t_n \in \mathbf{R} \end{array} \right\} \cup \\
& \vdots \\
& \left\{ \begin{array}{c} (k_1, k_2, ..., k_{n-2}, 1, k_n, m_1 k_1 + \cdots + m_{n-2} k_{n-2} + m_{n-1} + m_n k_n) | \\ k_1, k_2, ..., k_{n-2} \in \mathbf{I},\ k_n \in \mathbf{R} \end{array} \right\} \cup \\
& \{(l_1, l_2, ..., l_{n-1}, 1, m_1 l_1 + m_2 l_2 + \cdots + m_{n-1} l_{n-1} + m_n) |\ l_1, l_2, ..., l_{n-1} \in \mathbf{I}\}.
\end{aligned}$$

The connection relation "$\sim$" is defined as follows:

$$\begin{aligned}
P &= (x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_{n+1}) \sim (y_1, ..., y_{i-1}, y_i, y_{i+1}, ..., y_{n+1}) = Q \\
&\iff x_i - y_i \in \mathbf{I}\ \ (1 \leq i \leq n+1), \forall P, Q \in \mathbf{P}; \\
g &= [m_1, ..., m_{i-1}, m_i, m_{i+1}, ..., m_{n+1}] \sim [p_1, ..., p_{i-1}, p_i, p_{i+1}, ..., p_{n+1}] = h \\
&\iff m_i - p_i \in \mathbf{I}\ \ (1 \leq i \leq n+1), \forall g, h \in \mathbf{L}.
\end{aligned}$$

If we more closely examine the case $n = 2$, then $\mathbb{S}_2(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ is obtained as follows:

The set of points $\mathbf{P}$ consists of the following three points (which we call as points of types 1,2,3; respectively):

$$\begin{aligned}
\mathbf{P} &= \{P_1 = (1, x_2, x_3)\ |\ x_2, x_3 \in \mathbf{R}\} \cup \\
& \quad \{P_2 = (x_1, 1, x_3)\ |\ x_1 \in \mathbf{I}, x_3 \in \mathbf{R}\} \cup \\
& \quad \{P_3 = (x_1, x_2, 1)\ |\ x_1, x_2 \in \mathbf{I}\}.
\end{aligned}$$

The set of lines $\mathbf{L}$ consists of the following three lines (which we call as lines of types 1,2,3; respectively):

$$\begin{aligned} \mathbf{L} \;=\; & \{ M_1 = [1, m_2, m_3] \mid m_2, m_3 \in \mathbf{I}\} \cup \\ & \{ M_2 = [m_1, 1, m_3,] \mid m_1 \in \mathbf{R}, m_3 \in \mathbf{I}\} \\ & \{ M_3 = [m_1, m_2, 1] \mid m_1, m_2 \in \mathbf{R}\}. \end{aligned}$$

The incidence relation "$\in$" is as follows:

$$M_1 = [1, m_2, m_3] = \{ (m_2 + m_3 y_3, 1, y_3) | \; y_3 \in \mathbf{R}\} \cup \{ (m_2 z_2 + m_3, z_2, 1) | \; z_2 \in \mathbf{I}\},$$

$$M_2 = [m_1, 1, m_3] = \{ (1, m_1 + m_3 y_3, y_3) | \; y_3 \in \mathbf{R}\} \cup \{ (z_1, m_1 z_1 + m_3, 1) | \; z_1 \in \mathbf{I}\},$$

$$M_3 = [m_1, m_2, 1] = \{ (1, y_2, m_1 + m_2 y_2) | \; y_2 \in \mathbf{R}\} \cup \{ (z_1, 1, m_1 z_1 + m_2) | \; z_1 \in \mathbf{I}\}.$$

The connection relation "$\sim$" is as follows:

$$\begin{aligned} P &= (x_1, x_2, x_3) \sim (y_1, y_2, y_3) = Q \iff x_i - y_i \in \mathbf{I} \;\; (i = 1, 2, 3), \forall P, Q \in \mathbf{P}; \\ g &= [m_1, m_2, m_3] \sim [p_1, p_2, p_3] = h \Leftrightarrow m_i - p_i \in \mathbf{I} \;\; (i = 1, 2, 3)), \forall g, h \in \mathbf{L}. \end{aligned}$$

So, we have obtained a PK-plane (2-space), isomorphic to the PK-plane given in [2], in the case $n = 2$.

If we take $\mathbf{R} := \mathbb{O} + \mathbb{O}\varepsilon$ where $\mathbb{O}$ is the Cayley division algebra over a field F and $\varepsilon \notin \mathbb{O}$, then $\mathbb{S}_2(\mathbf{R})$ is an octonion plane and also the MK-plane, introduced by Blunck in [4]. Moreover, for $n > 2$, $\mathbb{S}_n(\mathbf{R})$ is the example of n-space (or octonion n-space). Note that the (quaternion) n-space $\mathbb{S}_n(\mathbb{Q} + \mathbb{Q}\varepsilon)$ is a subspace of the (octonion) n-space $\mathbb{S}_n(\mathbb{O}+\mathbb{O}\varepsilon)$. Besides, it is well-known that there is no a projective space constructed over non-associative division rings, and therefore a epimorphism onto an ordinary projective n-space can not exist. This means that the space $\mathbb{S}_n(\mathbb{O} + \mathbb{O}\varepsilon)$ for $n > 2$ is not a PK structure. For this reason, we tend to construct some collineations of the space $\mathbb{S}_n(\mathbb{Q} + \mathbb{Q}\varepsilon)$.

Finally, we would like to complete this paper by giving two collineations of the quaternion n-space $\mathbb{S}_n(\mathbb{Q} + \mathbb{Q}\varepsilon)$.

$$\begin{aligned} \mathrm{T}_{a_2,0,\ldots,0,0} \quad &: \quad \text{for } a_2 \in \mathbb{Q}, \\ (1, x_2, x_3, ..., x_n, x_{n+1}) \quad &\to \quad (1, x_2 + a_2, x_3 + 0, ..., x_n + 0, x_{n+1} + 0), \\ (x_1, 1, x_3, ..., x_{n+1}) \quad &\to \quad (x_1, 1, x_3 - (x_3 a_2) x_1, ..., x_{n+1} - (x_{n+1} a_2) x_1), \\ (x_1, x_2, 1, x_4, ..., x_n, x_{n+1}) \quad &\to \quad (x_1, x_2 + a_2 x_1, 1, x_4, ..., x_n, x_{n+1}), \\ &\;\;\vdots \\ (x_1, x_2, x_3, ...x_{n-1}, 1, x_{n+1}) \quad &\to \quad (x_1, x_2 + a_2 x_1, x_3, ..., x_{n-1}, 1, x_{n+1}), \end{aligned}$$

$$(x_1, x_2, , ...x_{n-1}, x_n, 1) \quad \to \quad (x_1, x_2 + a_2 x_1, x_3, ..., x_{n-1}, x_n, 1)\,,$$

$$\begin{array}{rcl} [m_1, m_2, ..., m_n, 1] & \to & [m_1 - m_2 a_2, m_2, ..., m_n, 1] \\ [m_1, m_2, ..., 1, m_{n+1}] & \to & [m_1 - m_2 a_2, m_2, ..., 1, m_{n+1}] \\ & \vdots & \\ [m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] & \to & [m_1 - m_2 a_2, m_2, 1, m_4, ..., m_n, m_{n+1}] \\ [m_1, 1, m_3, ..., m_n, m_{n+1}] & \to & [m_1 + a_2, 1, m_3, ..., m_n, m_{n+1}] \\ [1, m_2, m_3, ..., m_n, m_{n+1}] & \to & [1, m_2, m_3, ..., m_n, m_{n+1}] \end{array}$$

Similarly, the transformation $\mathrm{T}_{0,a_3,0,\ldots,0}$ can be defined in the following way: for any $a_3 \in \mathbb{Q}$,

$$\begin{array}{rcl} (1, x_2, x_3, x_4, ..., x_{n+1}) & \to & (1, x_2 + 0, x_3 + a_3, x_4 + 0, ..., x_{n+1} + 0) \\ (x_1, 1, x_3, x_4, ..., x_{n+1}) & \to & (x_1, 1, x_3 + a_3 x_1, x_4, ..., x_{n+1})\,, \\ (x_1, x_2, 1, x_4, ..., x_{n+1}) & \to & (x_1, x_2 - (x_2 a_3)\, x_1, 1, x_4 - (x_4 a_3)\, x_1, ..., x_{n+1} - (x_{n+1} a_3)\, x_1)\,, \\ & \vdots & \\ (x_1, x_2, x_3, ...x_{n-1}, 1, x_{n+1}) & \to & (x_1, x_2, x_3 + a_3 x_1, x_4, ..., x_{n-1}, 1, x_{n+1})\,, \\ (x_1, x_2, , ..., x_{n-1}, x_n, 1) & \to & (x_1, x_2, x_3 + a_3 x_1, x_4, ..., x_{n-1}, x_n, 1)\,, \end{array}$$

$$\begin{array}{rcl} [m_1, m_2, ..., m_n, 1] & \to & [m_1 - m_3 a_3, m_2, ..., m_n, 1]\,, \\ [m_1, m_2, ..., m_{n-1}, 1, m_{n+1}] & \to & [m_1 - m_3 a_3, m_2, ..., m_{n-1}, 1, m_{n+1}]\,, \\ & \vdots & \\ [m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] & \to & [m_1 + a_3, m_2, 1, m_4, ..., m_n, m_{n+1}]\,, \\ [m_1, 1, m_3, ..., m_n, m_{n+1}] & \to & [m_1 - m_3 a_3, 1, m_3, ..., m_n, m_{n+1}]\,, \\ [1, m_2, m_3, ..., m_n, m_{n+1}] & \to & [1, m_2, m_3, ..., m_n, m_{n+1}]\,. \end{array}$$

And, continuing on like this, finally, the transformation $\mathrm{T}_{0,0,\ldots,0,a_{n+1}}$ can be defined in the following manner: for any $a_{n+1} \in \mathbb{Q}$,

$$\begin{array}{rcl} (1, x_2, x_3, ..., x_n, x_{n+1}) & \to & (1, x_2 + 0, x_3 + 0, ..., x_n + 0, x_{n+1} + a_{n+1}) \\ (x_1, 1, x_3, ..., x_n, x_{n+1}) & \to & (x_1, 1, x_3, ..., x_n, x_{n+1} + a_{n+1} x_1)\,, \\ (x_1, x_2, 1, x_4, ..., x_n, x_{n+1}) & \to & (x_1, x_2, 1, x_4, ..., x_n, x_{n+1} + a_{n+1} x_1)\,, \\ & \vdots & \\ (x_1, x_2, x_3, ..., x_{n-1}, 1, x_{n+1}) & \to & (x_1, x_2, x_3, ..., x_{n-1}, 1, x_{n+1} + a_{n+1} x_1)\,, \\ (x_1, x_2, , ..., x_n, 1) & \to & (x_1, x_2 - (x_2 a_{n+1})\, x_1, ..., x_n - (x_n a_{n+1})\, x_1, 1)\,, \end{array}$$

$$[m_1, m_2, ..., m_n, 1] \quad \to \quad [m_1 + a_{n+1}, m_2, ..., m_n, 1]\,,$$

$$\begin{array}{rcl}
[m_1, m_2, ..., 1, m_{n+1}] & \rightarrow & [m_1 - m_{n+1}a_{n+1}, m_2, ..., 1, m_{n+1}], \\
& \vdots & \\
[m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] & \rightarrow & [m_1 - m_{n+1}a_{n+1}, m_2, 1, m_4, ..., m_n, m_{n+1}], \\
[m_1, 1, m_3, ..., m_n, m_{n+1}] & \rightarrow & [m_1 - m_{n+1}a_{n+1}, 1, m_3, ..., m_n, m_{n+1}], \\
[1, m_2, m_3, ..., m_n, m_{n+1}] & \rightarrow & [1, m_2, m_3, ..., m_n, m_{n+1}].
\end{array}$$

So, in this case, we have the translation transformation $\mathrm{T}_{a_2,a_3,...,a_{n-1},a_n,a_{n+1}}$ of $\mathbb{S}_n(\mathbb{Q}+\mathbb{Q}\varepsilon)$. The other transformation F_a is defined as follows:

$$\begin{array}{rcl}
\mathrm{F}_a & : & \text{for } a \notin \mathbb{Q}\varepsilon, \\
(1, x_2, x_3, ..., x_n, x_{n+1}) & \rightarrow & (1, ax_2a, x_3a, ..., x_na, x_{n+1}a) \\
(x_1, 1, x_3, ..., x_n, x_{n+1}) & \rightarrow & \left(a^{-1}x_1a^{-1}, 1, x_3a^{-1}, ..., x_na^{-1}, x_{n+1}a^{-1}\right) \\
(x_1, x_2, 1, x_4, ..., x_n, x_{n+1}) & \rightarrow & \left(a^{-1}x_1, ax_2, 1, x_4, ..., x_n, x_{n+1}\right) \\
& \vdots & \\
(x_1, x_2, x_3, ...x_{n-1}, 1, x_{n+1}) & \rightarrow & \left(a^{-1}x_1, ax_2, x_3, ...x_{n-1}, 1, x_{n+1}\right) \\
(x_1, x_2, x_3, ..., x_n, 1) & \rightarrow & \left(a^{-1}x_1, ax_2, x_3, ..., x_n, 1\right)
\end{array}$$

$$\begin{array}{rcl}
[m_1, m_2, m_3, ..., m_n, 1] & \rightarrow & \left[m_1a, m_2a^{-1}, m_3, ..., m_n, 1\right] \\
[m_1, m_2, m_3, ..., m_{n-1}, 1, m_{n+1}] & \rightarrow & \left[m_1a, m_2a^{-1}, m_3, ..., m_{n-1}, 1, m_{n+1}\right] \\
& \vdots & \\
[m_1, m_2, 1, m_4, ..., m_n, m_{n+1}] & \rightarrow & \left[m_1a, m_2a^{-1}, 1, m_4, ..., m_n, m_{n+1}\right] \\
[m_1, 1, m_3, ..., m_n, m_{n+1}] & \rightarrow & [am_1a, 1, am_3, ..., am_n, am_{n+1}] \\
[1, m_2, m_3, ..., m_n, m_{n+1}] & \rightarrow & \left[1, a^{-1}m_2a^{-1}, a^{-1}m_3, ..., a^{-1}m_n, a^{-1}m_{n+1}\right]
\end{array}$$

To show that the transformations $\mathrm{T}_{a_2,0,...0,0,0}$, $\mathrm{T}_{0,a_3,...0,0,0}$, $\mathrm{T}_{0,0,...,0,a_{n+1}}$ (and as a result, $\mathrm{T}_{a_2,a_3,...,a_{n-1},a_n,a_{n+1}}$ which is the combination of the all above transformations) and F_a are collineations of $\mathbb{S}_n(\mathbb{Q}+\mathbb{Q}\varepsilon)$, it is basically enough to prove Lemma 3 given in [5]. And also, we will often need the two results that $\mathbb{Q}+\mathbb{Q}\varepsilon$ is associative and that multiplication of any elements in the ideal $\mathbf{I} = \mathbb{Q}\varepsilon$ is equal to zero. Hence, we obtain that it is possible to study in the spaces by means of the collineations, analogous of the collineations given for showing 4-transitivity on the class of MK-plane in [5].

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