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Abstract Omega Algebra that Subsumes Tropical Min and Max Plus Algebras

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Abstract. In this paper abstract omega algebra is introduced and the definition is modeled in such a way that it subsumes almost all so called tropical min and max plus algebras. Concrete examples of distinct nature of these algebras are presented. As applications, symmetrized omega algebras are constructed and matrices with basic operations and some topological distances over them are defined.

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1. Introduction

Tropical geometry is a most recent but fast growing branch of mathematical sciences which is analytically based on idempotent analysis and algebraically on idempotent semirings or one may alternatively say on tropical semirings. These are basically extended sets of real numbers $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ and $\mathbb{R}_{-\infty} := \mathbb{R} \cup \{-\infty\}$ which are given monoidal structures by using min and max operations for addition, respectively, and in order to adhere the semiring structure, the additive operation of $\mathbb{R}$ is used as the multiplication operation. By these choices, both $\mathbb{R}_\infty$ and $\mathbb{R}_{-\infty}$ become idempotent semirings. In literatur, they are also termed as, min and max plus algebras, respectively. In both cases 0 of $\mathbb{R}$ becomes multiplicative identity and ∞ and $-\infty$ become additive identities of these semirings, respectively.

Interestingly, some authors associated $\mathbb{R}_{-\infty}$ to the tropical geometry while some other authors associated $\mathbb{R}_\infty$ to the tropical geometry (see for instance [2, 3, 5, 6]). In this paper, we unified the different terms and introuduce an original structure which in fact is an "abstract tropical algebra". We termed it as "omega algebra" or in short just, "$\omega-$ algebra". We will see that $\mathbb{R}_{-\infty}$ and $\mathbb{R}_\infty$ and their nearby structures, like min − max and max − times algebras, etc., are all subsumed under this newly defined structure. All these are idempotent semirings which sometimes also termed as dioids.

In the previous studies, for the construction of all such semirings, an ordered infinite abelian group is mandatory. In the newly introduced $\omega-$ algebra, the definition is extended to cyclically ordered abelian groups and also for finite

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sets under some suitable ordering. Note that cyclically ordered abelian groups are more general than that of ordered abelian groups [7].

In the following, first we give an abstract definition of omega algebras and supported them by presenting concrete examples, one from an ordered infinite set, another from a cyclically ordered infinite set, and a third one from a finite set. We also have constructed omega cartesian products and introduced omega homomorphisms. Finally, as applications, we have typically used some relations to construct symmetrized $\omega-$ algebras and defined over them matrices with their operations. Some topological distances over them are also constructed.

2. Omega Algebras

Let $(G, \circ, e)$ be an abelian group. Let A be a closed subset of G and $e \in A$. Then $(A, \circ, e)$ is a submonoid of G.

Assume that ω is an indeterminate (may belong to A or G, as we will see in Examples 2 & 3. Obviously, in this case ω is no longer an indeterminate). Because the terms are generated from tropical geometry, so such an indeterminate may be termed as a tropical indeterminate.

Definition 2.1. We say that $A_\omega = A \cup \{\omega\}$ is an omega algebra (in short $\omega-$ algebra) over the group G in case A_ω is closed under two binary operations,

$$\oplus, \otimes : A_\omega \times A_\omega \longrightarrow A_\omega,$$

such that $\forall a, b, c \in A$, the following axioms are satisfied:

(1) $a \oplus b = a \; or \; b$;

(2) $a \oplus \omega = a = \omega \oplus a$;

(3) $\omega \oplus \omega = \omega$;

(4) $a \otimes b = b \otimes a \in A$;

(5) $(a \otimes b) \otimes c = a \otimes (b \otimes c)$;

(6) $a \otimes e = a$;

(7) $a \otimes \omega = \omega \otimes a = \begin{cases} \omega & if \quad \omega \neq e \\ a & if \quad \omega = e \end{cases}$;

(8) $\omega \otimes \omega = \omega$;

(9) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

Remark 2.2. 1. $\oplus$ is a pairwise comparison operation, such as, max, min, inf, sup, up, down, lexicographic ordering, or any thing else that compairs two elements of A_ω. Obviously, it is associative and commutative and the tropical indeterminate ω playes the role of the identity. Hence $(A_\omega, \oplus, \omega)$ is a commutative monoid.

2. $\otimes$ is also associative and commutative on A_ω and e plays the role of the multiplicative identity of A_ω. Hence $(A_\omega, \otimes, e)$ is also a commutative monoid.

3. The left distributive law (8) also gives the right distributive law.

4. Every element of A_ω is an idempotent under $\oplus$.

5. Altogether, we write both structures as $A_\omega = (A_\omega, \oplus, \otimes, \omega, e)$. This is an idempotent semiring also called ”dioid” in literature.

Remark 2.3. An $\omega-$ algebra can similarly be defined over a commutative monoid or a ring or even on a semiring. More generally, one may construct analogously such algebras on other more weaker structures.

In this note, we confined ourselves to only $\omega-$ algebras over abelian groups and rings.

Proposition 2.4. *(i)* $\omega \in A_\omega$ *is unique.*

(ii) Let $\otimes|_A = \circ$. *Then* $\omega = e$ *if and only if* $\omega \in A$.

Proof. If $\omega \in A$, then ω^{-1} exists in G, and so

$$\omega \otimes \omega = \omega \Longrightarrow \omega \circ \omega = \omega \Longrightarrow \omega = e.$$

The rest is trivial. □

Definition 2.5. Let $A_\omega = (A_\omega, \oplus, \otimes, \omega, e)$ be an $\omega-$ algebra over an abelian group $G = (G, \circ, e)$. Let B be a non-empty closed subset of A such that $e \in B$. Then B_ω is said to be an $\omega-$ subalgebra of A_ω in case $B_\omega = (B_\omega, \oplus, \otimes, \omega, e)$ itself is an $\omega-$ algebra over G.

The trivial $\omega-$ subalgebra of A_ω is $O = (\{\omega, e\}, \oplus, \otimes, \omega, e)$.
An $\omega-$ algebra is called simple if its only proper $\omega-$ subalgebra is O.

The following is obvious.

Proposition 2.6. *Let $A_\omega = (A_\omega, \oplus, \otimes, \omega, e)$ be a tropical algebra over an abelian group $G = (G, \circ, e)$. Then a subset B_ω of A_ω is an $\omega-$subalgebra of A_ω if and only if*
(i) B is a non-empty closed subset of A such that $e \in B$.
(ii) B_ω is closed under the binary operations $\oplus$ and $\otimes$.

3. Examples

Example 3.1. Max-plus algebra, min-plus algebra and all such "so called" algebras are particular cases of the $\omega-$ algebra over the ring $\mathbb{R}$ or its associated subrings.

A simpler example is following.
In the ring $(\mathbb{Z}, +, \cdot)$, for any integer n, we set $W(n) = \{0, n, 2n, \cdots\}$. This is an additive submonoid of $(\mathbb{Z}, +)$. Let $\omega = -\infty$, $a \oplus b = \max(a, b)$ and $a \otimes b = a + b, \forall a, b \in W(n)$. Then

$$W(n)_{-\infty} = \{W(n)_{-\infty}, \oplus, \otimes, -\infty, 0\}$$

is $-\infty -$ algebra over the ring of integers $\mathbb{Z}$.
Hence we have a sequence of $\omega-$ subalgebras

$$W(n) \geq W(2n) \geq \cdots .$$

Example 3.2. (A cyclically ordered abelian group)
This example is constructed exclusively over an abelian group.
A cyclically ordered abelian group is more general than that of a linearly ordered abelian group. Every linearly ordered abelian group is cyclically ordered but the converse in general is not true. Following example is that of a cyclically ordered abelian group which is not an ordered abelian group [7]. For more details about a cyclically ordered abelian groups see [1].
Consider the cyclically ordered abelian group in the form of the unit circle

$$C = \{z \in \mathbb{C} \mid |z| = 1\}.$$

Let

$$W = \{0, 1, 2, \cdots\}.$$

For some $\theta \in [0, 1)$, define $\rho_x = e^{2\pi i\theta x}$, where $x \in W$, in particular, $\rho_0 = 1$.
Set

$$A := \{\rho_x \,| x \in W\} \subset C.$$

Because $\rho_x\rho_y = \rho_{x+y}, \forall x, y \in W,\ A$ is multiplicatively closed.

Theorem 3.3. *$A_{\rho_0} = A \cup \{\rho_0\}$ is an omega algebra with the identical additive and multiplicative identities. This omega algebra contains infinite omega subalgebras.*

Proof. Define $\oplus$ on A by

$$\rho_x \oplus \rho_y = \rho_z \text{ where } x, y, z \in W, \text{ with } z = \max(x, y)$$

and define $\otimes$ on A by

$$\rho_x \otimes \rho_y = \rho_{x+y}, \text{ where } x, y \in W.$$

Clearly, both operations are associative and as $\rho_0 \oplus \rho_x = \rho_x$ and $\rho_0 \otimes \rho_x = \rho_x$ so $(A, \oplus, \rho_0)$ and $(A, \otimes, \rho_0)$ are monoids.
Finally, $\forall x, y, z \in W$,

$$\begin{aligned} \rho_x \otimes (\rho_y \oplus \rho_z) &= \rho_x \otimes \rho_{\max(y,z)} \\ &= \rho_{x+\max(y,z)} \\ &= \rho_{\max(x+y,x+z)} \\ &= (\rho_{x+y} \oplus \rho_{x+z}) \\ &= (\rho_x \otimes \rho_y) \oplus (\rho_x \otimes \rho_z). \end{aligned}$$

As, $\forall x \in W$,

$$\rho_x \oplus \rho_x = \rho_x$$

and $\rho_0 = 1$ we conclude that, $A = (A, \oplus, \otimes, 1, 1)$ is an omega algebra.

Finally, consider $W(n) = \{0, n, 2n, \cdots\}$, where $n = 1, 2, \cdots$. For each n, one can construct an omega subalgebra. □

Example 3.4. (A Lexicographic Ordering.)

Consider the binary linear code of length 2;

$$\mathbb{Z}_2^{(2)} = \{00,\ 01,\ 10,\ 11\}.$$

Under componentwise addition + and componentwise multiplication $\circ$, $(\mathbb{Z}_2^{(2)}, +, \circ)$ is a ring with code-words $0 = 00$ and $1 = 11$ as additive and multiplicative identities.

We define the lexicographic ordering on the elements of $\mathbb{Z}_2^{(2)}$ and arrange them as:

$$00 < 01 < 10 < 11$$

Let $A = \{00, 01\}$. Consider $\omega = 11$.

Note that, in this example, $\omega \notin A$ but $\omega \in G$.

We define addition on $A_\omega = \{00, 01, 11\}$ by:

$$a \oplus b = \min(a, b).$$

Hence we get the table:

$\oplus$	00	01	11
00	00	00	00
01	00	01	01
11	00	01	11

.

Define multiplication as the boolean sum, namely,

$$0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 1 = 1.$$

Hence we get the table:

$\otimes$	00	01	11
00	00	01	11
01	01	01	11
11	11	11	11

.

We conclude that: $(A_\omega, \oplus, 11)$ and $(A_\omega, \otimes, 00)$ are the additive and multiplicative monoides.

Clearly, this is a simple $\omega-$ algebra.

Example 3.5. (Cartesian products of omega algebras)

In this example we explain a construction of an omega algebra from other given omega algebras.

Let $\{(G_i, \circ_i, e_i) : i = 1, \cdots, n\}$ be abelian groups and $\{(A_{\omega_i}, \oplus_i, \otimes_i, \omega_i, e_i) : i = 1, \cdots, n\}$ be a respective family of omega algebras, where ω_i are tropical indeterminates. As usual, we define the cartesian product as

$$\mathcal{X}_\omega = A_{\omega_1} \times \cdots \times A_{\omega_n} = \{(a_1, \cdots a_n) : a_i \in A_{\omega_i}; i = 1, \cdots, n\}.$$

In order to provide a convenient technique to give an additive structure to $\mathcal{X}_\omega$, we assume that the $n - tuples$ $\mathbf{a} = (a_1, \cdots a_n)$, $\mathbf{b} = (b_1, \cdots b_n) \in \mathcal{X}_\omega$ are in lexicographic ordering. Then define the sum

$$\mathbf{a} \oplus \mathbf{b} = \mathbf{a} \text{ or } \mathbf{b} \tag{3.1}$$

by using the following rules:

$$\text{If } a_1 \oplus_1 b_1 = a_1 \text{ then } \mathbf{a} \oplus \mathbf{b} = \mathbf{a}. \tag{3.2}$$

$$\text{If } a_i = b_i \text{ for } 1 \le i \le k \le n, \text{ and } a_{k+1} \oplus_{k+1} b_{k+1} = a_{k+1}, \text{ then } \mathbf{a} \oplus \mathbf{b} = \mathbf{a}. \tag{3.3}$$

Similarly, rules for $\mathbf{a} \oplus \mathbf{b} = \mathbf{b}$ can be determined.

Multiplication can be define componenetwise. Thus

$$\mathbf{a} \otimes \mathbf{b} = (a_1 \otimes_1 b_1, \cdots, a_n \otimes_n b_n).$$

The other rules can straightforwardely be verified. Hence $(\mathcal{X}_\omega, \oplus, \otimes, \omega, e)$, where $\omega = (\omega_1, \cdots, \omega_n)$ is the additive identity and $e = (e_1, \cdots, e_n)$ is the multiplicative identity of $\mathcal{X}_\omega$, is an omega algebra over the cartesian product of abelian groups $G_1 \times \cdots \times G_n$.

4. Omega Homomorphisms

Definition 4.1. Let $h : (G_1, \circ, e_1) \longrightarrow (G_2, *, e_2)$ be an abelian groups homomorphism. Let ω_1 and ω_2 be tropical indeterminates such that $A_{1\omega_1} = (A_{1\omega_1}, \oplus_1, \otimes_1, \omega_1, e_1)$ and $A_{2\omega_2} = (A_{2\omega_2}, \oplus_2, \otimes_2, \omega_2, e_2)$ be omega algebras over G_1 and G_2, respectively. The homomorphism h is called an *omega homomorphism* if in case

$$Im(h|_{A_{1\omega_1}}) \subseteq A_{2\omega_2},$$

the following conditions are satisfied:
(*i*) $h|_{A_{1\omega_1}}(\omega_1) = \omega_2$;
(*ii*) $\forall a, b \in A_{1\omega_1},\ h|_{A_{1\omega_1}}(a \oplus_1 b) = h(a) \oplus_2 h(b)$;
(*iii*) $\forall a, b \in A_{1\omega_1},\ h|_{A_{1\omega_1}}(a \otimes_1 b) = h(a) \otimes_2 h(b)$.

Definition 4.2. An omega homomorphism $h : G_1 \longrightarrow G_2$, satisfying

$$h|_{A_{1\omega_1}} : A_{1\omega_1} \longrightarrow A_{2\omega_2}$$

is said to be an omega ismorphism if $h|_{A_{1\omega_1}}$ is an ismorphism.

As usual we display an omega isomorphism by

$$A_{1\omega_1} \cong A_{2\omega_2}.$$

Note that in above definition we have not assumed that $h : G_1 \longrightarrow G_2$ is an isomorphism. If h is an isomorphism of abelian groups and if h is also an omega isomorphism we may then say that h is a strong omega isomorphism.

Proposition 4.3. *Let $h : G_1 \longrightarrow G_2$ be an isomorphism. Then $A_{1\omega_1} \cong A_{2\omega_2}$ if and only if h is an omega epimorphism.*

Proof. By definition $h|_{A_{1\omega_1}} : A_{1\omega_1} \longrightarrow A_{2\omega_2}$ is an epimorphism and it is monic as h is monic. Hence $A_{1\omega_1} \cong A_{2\omega_2}$. The converse is obvious. □

Omega monomorphisms and omega epimorphisms can analogously be defined.

Example 4.4. In Example 3.1, $\mathbb{Z} \ncong \mathbb{R}$ as additive groups. But under inclusion maps

$$n \longrightarrow n;\ n \longrightarrow 2n;\ \cdots$$

we get the omega isomorphisms,

$$W(n)_{-\infty} \cong W(n)_{-\infty} \cong W(2n)_{-\infty} \cdots$$

respectively, which are not strong, of course.

Example 4.5. Consider Example 3.2, $(\mathbb{Z}_2^{(2)}, +)$ is an additive abelian group and the map that exchanges the generators, namely,

$$01 \longrightarrow 10;\quad 10 \longrightarrow 01$$

is an automorphism. In this group we notice that we have two simple omega algebras,

$$A_\omega = \{00, 01, 11\} \text{ and } B_\omega = \{00, 10, 11).$$

Both are simple and omega isomorphic. This is a strong omega isomorphism.

Example 4.6. In the cartesian product of omega algebras, we can define an omega injection by using the arrow

$$A_{\omega_i} \longrightarrow \mathcal{X}_\omega$$

via

$$a_i \longmapsto (\omega_1, \cdots, \omega_{i-1}, a_i, \omega_{i+1}, \cdots, \omega_n)$$

and an omega surjection map

$$\mathcal{X}_\omega \longrightarrow A_{\omega_i}$$

via

$$(a_1, \cdots, a_n) \longmapsto a_i.$$

Similarly, one observes that

$$\{\omega_1\} \times \cdots \times \{\omega_{i-1}\} \times A_{\omega_i} \times \{\omega_{i+1}\} \times \cdots \times \{\omega_n\} \leq \mathcal{X}_\omega$$

is an omega subalgebra.

5. The Symmetrized Omega Algebra

Let $(G, \circ, e)$ be an abelian group and $(A_\omega, \oplus, \otimes, \omega, e)$ an $\omega-$algebra over the group G . Following the method used in constructing integers from the naturl numbers, we consider the set of ordered pairs $\mathcal{P}_\omega = A_\omega^2$ with componentwise addition $\oplus$, for all (a, b), $(c, d) \in \mathcal{P}_\omega$,

$$(a, b) \oplus (c, d) = (a \oplus c, b \oplus d) \tag{5.1}$$

Because of the four possibilities (a, b), (a, d) , (c, d) or (c, b) for the result, the addition in (4) is ambiguous. As our goal from constructing the algebra of pairs is the construction of the symmetrized omega algebra of A_ω, we are in front of two possibilities: One is to use Example 3.5 for $n = 2$ and define an equivalence relation $\sim$ on the $\omega-$algebra of pairs which is compatible with relevant operations and the other is to define an equivalence relation on the set $\mathcal{P}_\omega$ that allows the componentwise addition to be defined in the quotient set.

First construction

(1) Let $\leq$ be the ordering defined on A_ω by the relation

$$a \leq b \Longleftrightarrow a \oplus b = b \tag{5.2}$$

which gives a total order on A_ω and for all $a \in A_\omega$, we have $\omega \leq a$. For $a \neq b$, such that $a \oplus b = b$, we denote by $a < b$.

Under the ordering $\leq$, rules (2) and (3) defined in Example 3.5 are satisfied on $\mathcal{P}_\omega = A_\omega^2$ and so $\mathcal{P}_\omega$ is an $\omega-$algebra under the addition defined in (1) and the componentwise multiplication.

Let $\triangledown$ be the relation defined on $\mathcal{P}_\omega$ as follows: for all (a, b), $(c, d) \in \mathcal{P}_\omega$

$$(a, b) \triangledown (c, d) \Longleftrightarrow a \oplus d = b \oplus c.$$

Then $\triangledown$ is reflexive and symmetric but not transitive for A_ω contains more than 4 elements.

In fact, let a, b, c, $d \in A_\omega$ such that $a < b < c < d$, then we have

$$a \oplus d = d = b \oplus d = c \oplus d \text{ and } a \oplus c = c \neq b = b \oplus b$$

which give $(a, b) \triangledown (d, d)$ and $(d, d) \triangledown (b, c)$, but there is no relation between (a, b) and (b, c).

As $\triangledown$ is not an equivalence relation, we cannot use it to obtain the quotient $\omega-$algebra $\frac{\mathcal{P}_\omega}{\triangledown}$ (like the one to obtain integers from the natural numbers).

Definition 5.1. Let $\sim$ be the equivalence relation close to $\triangledown$ defined as follows: for all (a, b), $(c, d) \in \mathcal{P}_\omega$,

$$(a, b) \sim (c, d) \iff \begin{cases} (a, b) \triangledown (c, d) & \text{if } a \neq b \text{ and } c \neq d \\ (a, b) = (c, d) & \text{otherwise} \end{cases}.$$

In addition to the class element $\overline{\omega} = \overline{(\omega, \omega)}$; for all $a \in A_\omega$, with $a \neq \omega$, we have three kinds of equivalence classes:

(a) $\overline{(a, \omega)} = \{(a, b) \in \mathcal{P}_\omega, b < a\}$, called positive $\omega-$element.

(b) $\overline{(\omega, a)} = \{(b, a) \in \mathcal{P}_\omega, b < a\}$, called negative $\omega-$element.

(c) $\overline{(a, a)}$ called balenced $\omega-$element.

Unfortunately, the addition defined by (1) and rules (2) and (3) in Example 3.5 is not compatible with the equivalence relation in $\mathcal{P}_\omega$, because for (a, ω), (a, b), (ω, c), $(d, c) \in \mathcal{P}_\omega$, such that

$$\begin{cases} (a, \omega) \sim (a, b) \\ (\omega, c) \sim (d, c) \end{cases},$$

we have

$$(a, \omega) \oplus (\omega, c) \sim (a, b) \oplus (d, c) \text{ iff } (a, b) \oplus (d, c) = (a, b)$$

$$\text{and if } (a, b) \oplus (d, c) = (d, c),$$

then there is no compatibility. So the omega algebra of pairs cannot produce the symmetrized omega algebra.

Second construction

Proposition 5.2. *The addition operation* $\overline{\oplus}$ *defined by*

$$\overline{(a,b)}\overline{\oplus}\overline{(c,d)} = \overline{(a\oplus c, b\oplus d)}$$

on the quotient set $\frac{\mathcal{P}_\omega}{\sim}$ *is well defined and satisfies the axioms* (1), (2) *and* (3) *of definition of omega algebra.*

Proof. By using the previous equivalence classes, for all a, $b \in A_\omega$, we have

(1)

$$\overline{(a,\omega)}\overline{\oplus}\overline{(b,\omega)} = \overline{(a\oplus b, \omega\oplus\omega)} = \overline{(a,\omega)} \text{ or } \overline{(b,\omega)};$$

(2)

$$\overline{(a,\omega)}\overline{\oplus}\overline{(\omega,b)} = \overline{(a\oplus\omega, \omega\oplus b)} = \overline{(a,b)} = \begin{cases} \overline{(a,\omega)} & \text{if } b<a \\ \overline{(\omega,b)} & \text{if } a<b \end{cases} = \overline{(a,\omega)} \text{ or } (\omega,b);$$

(3)

$$\overline{(a,\omega)}\overline{\oplus}\overline{(b,b)} = \overline{(a\oplus b, \omega\oplus b)} = \overline{(a\oplus b, b)} = \begin{cases} \overline{(a,\omega)} & \text{if } b<a \\ \overline{(b,b)} & \text{if } a<b \end{cases} = \overline{(a,\omega)} \text{ or } \overline{(b,b)};$$

(4)

$$\overline{(\omega,a)}\overline{\oplus}\overline{(b,b)} = \overline{(\omega\oplus b, a\oplus b)} = \overline{(b, a\oplus b)} = \begin{cases} \overline{(\omega,a)} & \text{if } b<a \\ \overline{(b,b)} & \text{if } a<b \end{cases} = \overline{(\omega,a)} \text{ or } \overline{(b,b)}.$$

A direct check shows that the axioms (1), (2) and (3) of definition of omega algebra are satisfied with the zero class element $\overline{\omega}$. □

Proposition 5.3. *(i) The set* $\frac{\mathcal{P}_\omega}{\sim}$ *is closed under the binary multiplication operation* $\overline{\otimes}$ *defined as follows: for all* $\overline{(a,b)}, \overline{(c,d)} \in \frac{\mathcal{P}_\omega}{\sim}$*;*

$$\overline{(a,b)}\overline{\otimes}\overline{(c,d)} = \overline{((a\otimes c)\oplus(b\otimes d), (a\otimes d)\oplus(b\otimes c))}$$

and satisfies axioms from (4) *to* (9) *of definition of omega algebra with the unit class element* $\overline{e} = \overline{(e,\omega)}$*.*

(ii) In addition, we have for all a, $b \in A_\omega$

(1)

$$\overline{(a,\omega)}\overline{\otimes}\overline{(b,\omega)} = \overline{(a\otimes b,\omega)};$$

(2)

$$\overline{(a,\omega)}\overline{\otimes}\overline{(\omega,b)} = \overline{(\omega,a\otimes b)};$$

(3)

$$\overline{(a,\omega)}\overline{\otimes}\overline{(b,b)} = \overline{(a\otimes b,a\otimes b)};$$

(4)

$$\overline{(\omega,a)}\overline{\otimes}\overline{(b,b)} = \overline{(a\otimes b,a\otimes b)}.$$

Proof. A routine but direct calculations give the desired results. □

Definition 5.4. The omega algebra $\left(\frac{\mathcal{P}_\omega}{\sim}, \overline{\oplus}, \overline{\otimes}, \overline{\omega}, \overline{e}\right)$ is called the symmetrized ω−algebra over the abelian group $G\times G$ and we denote it by $\mathbb{S}_\omega$.

In the coming sections just for simplicity we will only use $\oplus$ and $\otimes$ instead the operations $\overline{\oplus}$ and $\overline{\otimes}$, respectively.

Remark 5.5. 1. Despite the nature of the positive and the negative ω−elements, they are not the inverses of each other for the additive operation $\overline{\oplus}$,

2. Proposition 2.6 shows that we have three symmetrized ω−subalgebras of $\mathbb{S}_\omega$,

$$\begin{aligned} \mathbb{S}_\omega^{(+)} &= \left\{\overline{(a,\omega)}, a\in A_\omega\right\}, \\ \mathbb{S}_\omega^{(-)} &= \left\{\overline{(\omega,a)}, a\in A_\omega\right\}, \\ \mathbb{S}_\omega^{(0)} &= \left\{\overline{(a,a)}, a\in A_\omega\right\}. \end{aligned}$$

3. The three symmetrized ω−subalgebras of $\mathbb{S}_\omega$ are connected by the zero class element $\overline{\omega}$.

6. Rules of Calculation in Omega and The Omega -Absolute Value

Let $a \in \mathbb{A}_\omega$. Then we admit the following notations:

$$+a. = \overline{(a,\omega)}, -a. = \overline{(\omega,a)}, \ \cdot a = \overline{(a,a)}.$$

By results in Proposition 5.2 and Proposition 5.3 and the above notation, it is easy to verify the rules of calculation in the following proposition.

Proposition 6.1. *For all a,b $\in A_\omega$, we have*
(i) $(+a) \oplus (+b) = +(a \oplus b)$;
(ii) $(+a) \oplus (-b) = \begin{cases} +a & \text{if } b < a \\ -b & \text{if } b > a \\ \cdot a & \text{if } b = a \end{cases}$;
(iii) $(\pm a) \oplus (\cdot b) = \begin{cases} \pm a & \text{if } b < a \\ \cdot b & \text{if } b > a \end{cases}$:
(iv) $(-a) \oplus (-b) = -(a \oplus b)$;
(v) $(+a) \otimes (+b) = +(a \otimes b)$;
(vi) $(+a) \otimes (-b) = -(a \otimes b)$;
(vii) $(\pm a) \otimes (\cdot b) = \cdot (a \otimes b)$;
(viii) $(-a) \otimes (-b) = +(a \otimes b)$.

From the previous rules, we can notice that the sign of the result in the addition operation follows the greater element in A_ω. While in the multiplication operation, the balance sign is the strong one (has priority).

From Proposition 6.1, we can deduce the following.

Proposition 6.2. *The map* $|.|_\omega : \mathbb{S}_\omega \longrightarrow A_\omega$*, such that for all* $a \in A_\omega$*,*

$$|+a|_\omega = |-a|_\omega = |\cdot a|_\omega = a$$

is an absolute value on $\mathbb{S}_\omega$*. We call it the* $\omega-$ *absolute value.*

Proposition 6.3. *Let* $(A_\omega, \oplus, \otimes, \omega, e)$ *be an* $\omega-$*algebra over an abelian group* $(G, \circ, e)$ *and A a subgroup of G, such that the* $\otimes|A = \circ$*. Then* A_ω *is a* $\mathbb{Z}-$*semimodule.*

Proof. As G is an abelian group, then it is a $\mathbb{Z}-$module (considered as an additive group), which yields to A to be a subsemimodule. Then A_ω is a $\mathbb{Z}-$semimodule. □

Let $sign(.)$ denote one of the three signs of an element in $\mathbb{S}_\omega$. Under conditions of Proposition 6.4, we can define the $\omega-$power and the $\omega-$multiple of an element in $\mathbb{S}_\omega$.

Definition 6.4. The *$\omega-$power and the $\omega-$multiple of an element in* S_ω

Let $a \in \mathbb{S}_\omega$ and $n \in \mathbb{N}^*$.

(1) The $\omega-$power of a is defined by the rule:

$$\begin{aligned} \text{If } a &\neq \omega, \text{ then } a^{\otimes n} = \underbrace{a \otimes ... \otimes a}_{n-\text{times}} = (sign)^n \underbrace{(\alpha \circ ... \circ \alpha)}_{n-\text{times}}, \text{ where } a = sign(\alpha) \\ \omega^{\otimes n} &= \omega. \end{aligned}$$

(2) The $\omega-$multiple of a by n is defined by the rule:

$$\begin{aligned} \text{If } a &\neq \omega, \text{ then } n \otimes a = na = sign(n\alpha) \text{ for } \alpha \in A \\ n \otimes \omega &= \omega. \end{aligned}$$

7. Matrix Operations

Definition 7.1. Let $(A_\omega, \oplus, \otimes, \omega, e)$ be an $\omega-$algebra over an abelian group $(G, \circ, e)$ and A a subgroup of G, such that the $\otimes|A = \circ$. Let $X = \left(x_{ij}\right)_{m\times n}$, $Y = \left(y_{ij}\right)_{m\times n}$, $Z = \left(z_{jk}\right)_{n\times p}$ be three matrices over $\mathbb{S}_\omega$ and $\alpha \in \mathbb{S}_\omega$. Then, we define the sum $X \oplus Y$, the product $X \otimes Z$ and the product by a scalar $\alpha \otimes X$ over the group G as follow:

$$X \oplus Y = \left(x_{ij} \oplus y_{ij}\right)_{m\times n}$$

$$X \otimes Z = \left(\underset{j=1}{\overset{n}{\oplus}} \left(x_{ij} \otimes z_{jk}\right)\right)_{m\times p}$$

$$\alpha X = \left(\alpha \otimes x_{ij}\right)_{m\times n}$$

Remark 7.2. 1. From the previous definition, (it is easy to show that) the zero matrix and the identity matrix are respectively: $\Omega = (\omega)_{m\times n}$ and E is a square matrix consists of $+e$ on the diagonal and ω on the off diagonal.

2. We can define the $\omega-$algebraic norm $\|.\|_\omega$ of a matrix $X = \left(x_{ij}\right)_{m\times n}$ over $\mathbb{S}_\omega$ as follows:

$$\|X\|_\omega = \underset{j=1}{\overset{n}{\oplus}} \underset{i=1}{\overset{m}{\oplus}} \left(\left|x_{ij}\right|_\omega\right)$$

3. If $\otimes|A \neq \circ$, then we must take the operation defined on A_ω and all possible induced operations into consideration.

Example 7.3. If the $\omega-$algebra is the max-plus-algebra, then, the previous operations become the max, + and scalar multiplication over $\mathbb{R}_\epsilon$, where $\varepsilon = -\infty$.

8. Some Topological Distances Over Omega-Algebras

8.1. **The inner distance on $\mathbb{S}_\omega$.** Some metrics were first time introduced and some algebraic and topological properties were studied in the symmetrized max-plus-algebra [4]. Let A_ω be an $\omega-$algebra over an abelian group $(G, \circ)$, such that the restriction $\otimes|A = \circ$. By the fundamental theorem of abelian groups, the group G is a direct sum (direct product) of its cyclic groups, which make it isomorphic to a direct sum of copies of the cyclic groups $\mathbb{Z}$ of integers and/or isomorphic to a direct sum of cyclic groups of the quotients of $\mathbb{Z}$ (according to the group is infinite or finite). In all cases we can represent an element of G by n-tuple of elements of $\mathbb{Z}$ for some natural number n via that isomorphism. From this point of view, we will define metrics (or semimetrics) on our $\omega-$algebra via the distance in $\mathbb{Z}$. Let Φ be a such isomorphism. For any $a \in G$, there exists $n \in \mathbb{N}$ and there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}$ (also they can be the representatives of classes in $\mathbb{Z}$), such that $\Phi(a) = (\alpha_1, ..., \alpha_n)$. Let us extend Φ on A_ω as follows:

$$\Phi(a) = \begin{cases} (\alpha_1, ..., \alpha_n) & \text{for } a \in A, n \in \mathbb{N} \\ 0 & \text{for } a = \omega \end{cases}$$

When there is a valued structure $(\mathbb{S}, v)$ which originally is an additive group, Then the valuation v (the absolute value) can produce a distance d such that for all a, $b \in \mathbb{S}$, $d(a, b) = v(a + (-b)) = v(a - b)$. It said that v produces a distance in the usual way. The $\omega-$absolute value cannot produce a distance on $\mathbb{S}_\omega$ in the usual way, because the negative elements of $\mathbb{S}_\omega$ are not their inverses, which gives: for $a \neq \omega$,

$$d(+a, -a) = |+a \oplus (-a)|_\omega = |\cdot a|_\omega = a \neq \omega.$$

For this reason, we define a distance $d_{\mathbb{S}_\omega}$ in $\mathbb{S}_\omega$ via a distance $d_\mathbb{Z}$ in copies of $\mathbb{Z}$ as follows:

$$d_{\mathbb{S}_\omega}(a, b) = d_\mathbb{Z}(\Phi(a), \Phi(b)) = \|\Phi(a) - \Phi(b)\|$$

where $\|.\|$ is a norm on the $\mathbb{Z}-$semimodule $\Phi(A_\omega)$. We call thedistance $d_{\mathbb{S}_\omega}$ inner distance.

8.2. **The Frobenius and the Euclidean distances on $\mathbb{S}_\omega$.** We suppose that the group G is a direct sum of its n cyclic subgroups (then G is a finitely generated group). Then for every $a \in A$, $\Phi(a) = (\varphi_1(a), ..., \varphi_n(a))$, where $\varphi_1, ..., \varphi_n$ are the projections of $\Phi(G)$ on the corresponding sub-cyclic groups of $\mathbb{Z}$ or its quotients. Let c_a be the circulant matrix defined by $(\varphi_1(a), ..., \varphi_n(a))$. As circulant matrices are diagonalizable in a same basis, we can benefit from this property to define a metric on $\mathbb{S}_\omega$ by using Frobenius norm $\|\|_F$ (it is a matrix norm). Let θ be a cube root of unity, say, $\theta = \frac{-1+\sqrt{3}i}{2}$. As elements of $\mathbb{S}_\omega$ are defined by three signs, then we can emerge from $\mathbb{S}_\omega$ into the algebra of circulant matrices by the map ϕ defined by: for all $a \in A_\omega$,

$$\phi(+a) = \theta \exp((c_a)), \phi(-a) = \theta^2 \exp((c_a)), \phi(\cdot a) = \exp((c_a))$$

Definition 8.1. Let $d_F : \mathbb{S}_\omega \times \mathbb{S}_\omega \longrightarrow \mathbb{R}$ be defined by

$$d_F(a, b) = \|\phi(a) - \phi(b)\|_F$$

Then d_F is a distance on $\mathbb{S}_\omega$, we call it the omega Frobenius distance.

Definition 8.2. $d_E : \mathbb{S}_\omega \times \mathbb{S}_\omega \longrightarrow \mathbb{R}$ be defined by

$$d_E(a, b) = d_e(\Phi(|a|_\omega), \Phi(|b|_\omega))$$

where d_E is the Euclidean distance of $\mathbb{R}^n$. We call d_E the Euclidean distance.

Remark 8.3. It is clear that both "Euclidean" and "inner"distances induce the same topology on $\mathbb{S}_\omega$, which will be called the usual topology on $\mathbb{S}_\omega$.

Problem What is the important topological properties of the space of circulant matrices over real numbers? How to benefit from omega Frobenius distance to translate those properties from the space of circulant matrices to $\mathbb{S}_\omega$?

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] Giraudet, M., Leloup, G., Lucas, F., *First order theory of cyclically ordered groups*, hal-00879429. 3.2

[2] Litvinov, G.L., *The Maslov dequantization, idempotent and tropical mathematics: a brief introduction*, Journal of Mathematical Sciences, **140**(2007), 426–444. 1

[3] Maclagan, D., Sturmfels, B., Introduction to Tropical Geometry, Graduate Studies in Mathematics, Vol 161, American Mathematical Society, 2015. 1

[4] Ozel, C., Piekosz, A., Wajch, E., Zekraoui, H., *The minimizing vector theorem in symmetrized max-plus algebra*, Journal of convex analysis, **26(2)**(2019), 661–686. 8.1

[5] Pin, J. E., Tropical Semirings, Idempotency (Bristol, 1994), 50–69, Publ. Newton Inst., Vol. 11, Cambridge Univ. Press, Cambridge, 1998. 1

[6] Simon, I., *Recognizable sets with multiplicities in the tropical semiring*, pages 107–120 in Mathematical Foundations of Computer Science (Carlsbad, 1988), Lecture Notes in Computer Science, Vol. 324, Springer, Berlin, 1988. 1

[7] Swierczkowski, S., *On cyclically ordered groups*, Fundamenta Mathematicae, **47**(1959) 161–166. 1, 3.2