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Investigation of Lorenz Equation System with Variable Step Size Strategy

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Abstract. In this study, variable step size strategy has been considered to analyze the numerical solution of the Lorenz system with chaotic structure. Phase portraits have been obtained for this chaotic system. The effectiveness of the variable step size strategy for the solution of this chaotic system has been discussed.

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Keywords: Lorenz systems, numerical solution, variable step size.

1. Introduction

As a meteorologist, Edward Norton Lorenz studied to analyze the weather forecasts in 1963 and he developed a mathematical model consisting of three non-linear differential equations. In his paper "Deterministic Nonperiodic Flow", Lorenz explained that these equations are sensitive to the initial conditions. Thus, the idea of "deterministic chaos" was discovered. In later years, chaos was thought to be very useful for many technological disciplines and was studied extensively in science and engineering ([1, 6, 13, 14]). Therefore, numerical and approximate solutions of the Lorenz system are analyzed in many studies ([2, 7, 8, 11, 15]).

The mathematical model of the Lorenz system has given as following:

$$\frac{dx}{dt} = \sigma(y - x), \frac{dy}{dt} = rx - y - z, \frac{dz}{dt} = xy - bz \tag{1.1}$$

where x, y and z are state variables; σ is the prandtl number, b and r are positive constant parameters. In the Lorenz system, x is proportional to the rate of convection, y to the horizontal temperature variation and z to the vertical temperature variation ([11, 12]).

If typical parameter values for the Lorenz System are chosen as $\sigma = 10$, $r = 28$ and $b = 8/3$, then the Lorenz system exhibits chaotic orbits. In chaotic case, the behavior of the solution changes rapidly, so the step size must be so small. This means numerical calculations have difficulties and long arithmetical operations.

In [4] and [5], a variable step size strategy has been proposed for numerical solutions of nonlinear differential equation systems such as the Lorenz System. This strategy, allows deciding by performing error checking at each step, where to use the small step size and where the large step size should be used. So, it is possible to calculation at the

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desired error level. In this study, as a different perspective, we have wanted to investigate the behavior of the Lorenz system with the variable step size strategy in [4] and [5]. We have obtained phase portraits for the Lorenz system with the chaotic structure.

2. Variable Step Size Strategy for Nonlinear Differential Equation System

Let us consider the Cauchy problem

$$X'(t) = AX(t) + \varphi(t, X), X(t_0) = X_0, \tag{2.1}$$

and assume that the solution of (2.1) is unique and exists on the region $D = (t, X) : t \in [t_0, T], |x_j - x_{j0}| \leq b_j$, where $A = (a_{ij}) \in \mathbf{R}^{N\times N}$, $X \in \mathbf{R}^N$ and $\varphi \in \mathbf{C}^1([t_0 - T, t_0 + T] \times N)$.

For the i-th step of the numerical integration of the Cauchy problem (2.1), step size is obtained as

$$h_i = N^{-1/4}\left(\frac{2\delta_L}{N^2\alpha^2\beta_{i-1} + N\alpha\gamma_{i-1} + \zeta_{i-1}}\right)^{1/2} \tag{2.2}$$

such as the local error is smaller than δ_L- error level ([4, 5]).

The parameters herein are defined as follows.

- δ_L is the error level determined by user,
- N is the dimension of the matrix $A = (a_{ij}) \in \mathbf{R}^{N\times N}$,
- Y_i is the numerical solution in i-th step,
- $Z(t)$ is the solution of the Cauchy problem $Z'(t) = AZ(t) + \varphi(t, Z(t)), Z(t_{i-1}) = Y_{i-1}$,
- $\alpha = \max_{1\leq i,j\leq N} |a_{ij}|$,
- $\max_{1\leq j\leq N}(\sup_{t_{i-1}\leq\tau_i<t_i} |z_j(\tau_i)|) \leq \beta_{i-1}$, $\tau_i \in [t_{i-1}, t_i)$,
- $\max_{1\leq j\leq N}(\sup_{t_{i-1}\leq\tau_i<t_i} |\varphi_j(\tau_i, z(\tau_i))|) \leq \gamma_{i-1}$, $\tau_i \in [t_{i-1}, t_i)$,
- $\max_{1\leq j\leq N}(\sup_{t_{i-1}\leq\tau_i<t_i} |\frac{\varphi_j}{dt}(\tau_i, z(\tau_i))|) \leq \zeta_{i-1}$, $\tau_i \in [t_{i-1}, t_i)$.

Step size given by (2.2) is obtained for Euler method in [4]and [5].

Algorithm 2.1. Step Size Algorithm (SSA). Algorithm SSA calculates the step sizes given by equation (2.2) and the numerical solution of the Cauchy problem (2.1) using these step sizes ([4, 5]).

Step 0 (Input). $t_0, T, b, h^*, \delta_L, X_0, \varphi(t, X), A$.

Step 1. Calculate α and $\frac{d\varphi(t,X)}{dt}$.

Step 2. Calculate β_{i-1}, γ_{i-1} and ζ_{i-1}.

Step 3. Calculate step size $\hat{h}_i$.

Step 4. Control step size with K-algorithm (see, [4] for K-algorithm).

Step 5. Calculate $t_i = t_{i-1} + h_i$ and $Y_i = (I + h_iA)Y_{i-1} + h_i\varphi(t_{i-1}, Y_{i-1})$.

3. Numerical Solution of Lorenz System with Variable Step Size Strategy

To examine the solution of the Lorenz system with a variable step size strategy, let us write the system in the form (2.1). Then we get the system

$$X' = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} X + \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix},$$

where $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$ and $\varphi(t, X) = \begin{pmatrix} 0 \\ -xz \\ xy \end{pmatrix}$. To obtain the step size at each step of the numerical iteration, the parameters $\alpha, \beta_{i-1}, \gamma_{i-1}$ and ζ_{i-1} are calculated as follows:

- $\alpha = \max_{1\leq i,j\leq N} |a_{ij}| = \max \sigma, r, 1, b$,
- $\beta_{i-1} = \max |x_{i-1}|, |y_{i-1}|, |z_{i-1}|$,
- $\gamma_{i-1} = \max |x_{i-1}z_{i-1}|, |x_{i-1}y_{i-1}|$,
- $\zeta_{i-1} = \max |\sigma x_{i-1}z_{i-1} - \sigma x_{i-1}y_{i-1} - x_{i-1}^2 y_{i-1} + bz_{i-1}|, |\sigma y_{i-1}^2 - \sigma y_{i-1}x_{i-1} + rx_{i-1}^2 - y_{i-1}x_{i-1} - x_{i-1}^2 y_{i-1}|$

([10]).

4. Numerical Experiment

Lorenz system exhibits chaotic trajectories in case of $\sigma = 10$, $r = 28$ and $b = 8/3$. Therefore, let us calculate the solution according to these values to see the effectiveness of the variable step size strategy. Let the initial condition be $X(0) = (-15.8, -17.48, 35.64)^T$ for $t \in [0, 10]$ and δ_L- error level be 10^{-1}.

The numerical solutions obtained using SSA and the 3D view of the Lorenz system are shown in Fig. 1 and Fig. 2, respectively. The phase portraits of the Lorenz system are given in Fig. 3. The pictures in Fig. 2 and Fig. 3 are the classical image of the Lorenz system encountered in the literature ([3, 6, 9, 16]). This shows that the variable step size strategy and algorithm are suitable for numerical solutions of chaotic systems such as the Lorenz system.

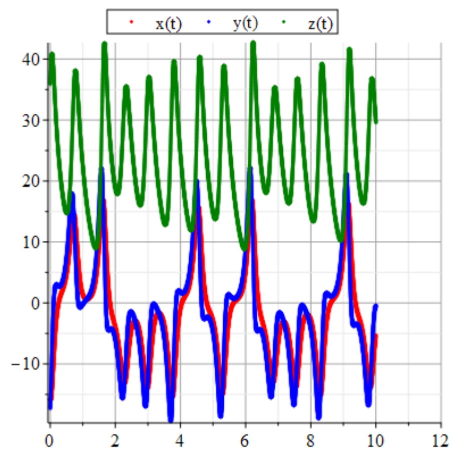

Figure 1. The numerical solutions of the Lorenz system for $\sigma = 10$, $r = 28$ and $b = 8/3$.

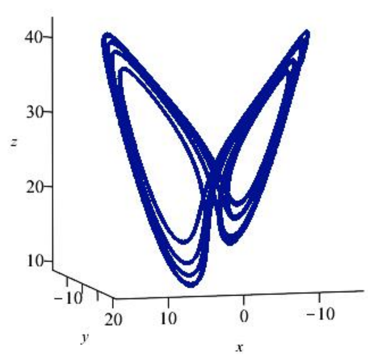

Figure 2. 3D view of the Lorenz system in the xyz plane for $\sigma = 10$, $r = 28$ and $b = 8/3$.

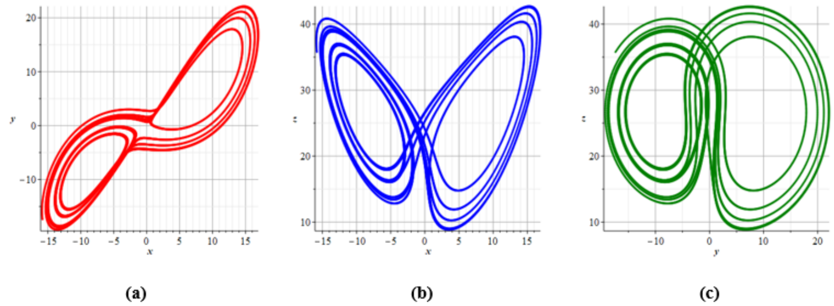

FIGURE 3. Phase portrait of system for $\sigma = 10$, $r = 28$ and $b = 8/3$ (**a**) Projection in x-y plane, (**b**) Projection in x-z plane, (**c**) Projection in y-z plane.

The step sizes obtained from SSA have been summarized in Table 1. The obtained step sizes at the each step of the numerical integration are demonstrated in Fig. 4. The step sizes range between $h_{min} = 0.5654184091e-3$ and $h_{max} = 0.1329369677e-2$.

i- Iteration number	h_i- Step size
1	0.6181019260e-3
2	0.6171672763e-3
3	0.6162485392e-3
⋮	⋮
140	0.5959058535e-3
⋮	⋮
690	0.1003313023e-2
⋮	⋮
900	0.6454555753e-3
⋮	⋮
1600	0.1220910232e-2
⋮	⋮
4100	0.8335061633e-3
⋮	⋮
12367	0.7166318553e-3
12368	0.554534e-3

TABLE 1. Some values of h_i obtained from SSA.

If a fixed step size was used in numerical integration, h_{min} should be used to obtain a numerical solution at the desired error level. This means an increase in the number of iterations. It is not desirable to increase the number of iterations in the solution of a system having a chaotic structure such as the Lorenz system. Because small changes in initial values can result in very large errors. Since the variable step size strategy provides a numerical solution with less iteration, it is seen that it provides advantage in solutions of chaotic systems.

Another advantage of the variable step width strategy is that it is based on the Euler method. Because the calculation with Euler method is more practical than other one-step numerical methods. Normally, the Euler method is not preferred for these chaotic systems because it has a larger error. However, considering the variable step size strategy, theoretically, the calculations with the Euler method are occured at the desired error level.

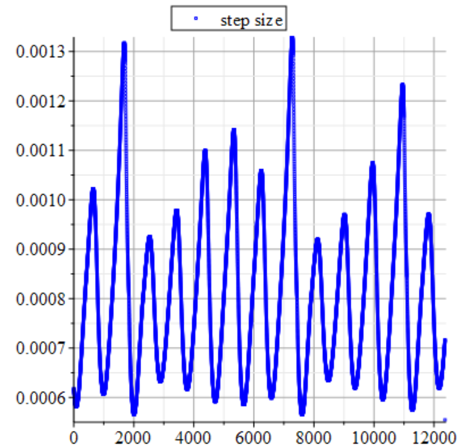

FIGURE 4. The step sizes obtained from SSA

5. CONCLUSIONS

In this paper, we have used the variable step size strategy for the numerical solution of the Lorenz system to observe and verify the existing chaotic dynamic behaviors of the Lorenz system. By using the variable step size strategy, we have obtained the phase portraits of the chaotic Lorenz system in accordance with the literature .The used strategy allows to computation at the desired error level. It is seen that the variable step size strategy increases the effectiveness of the Euler method and enable it to be used successfully for the chaotic Lorenz system.

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