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Improved bounds for the number of spanning trees of graphs

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Abstract. For a given a simple connected graph, we present some new bounds via a new approach for the number of spanning trees. Usage this approach presents an advantage not only to derive old and new bounds on this topic but also gives an idea how some previous results in similar area can be developed.

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1. Introduction

Let G be a simple connected graph with n vertices and m edges. Let $V(G) = \{v_1, v_2, \cdots, v_n\}$ be the vertex set and $E(G) = \{e_1, e_2, \cdots, e_m\}$ be the edge set of G. If any two vertices v_i and v_j of G are adjacent, that is, $v_iv_j \in E(G)$, then we use the notation $v_i \sim v_j$. For $v_i \in V(G)$, the degree of the vertex v_i, denoted by d_i, is the number of the vertices adjacent to v_i.

Let $R_\alpha = R_\alpha(G) = \sum\limits_{v_i \sim v_j} (d_id_j)^\alpha$ be the *general Randić index* ([3]) of the graph G, where $\alpha \neq 0$ is a fixed real number. Note that the *Randić index* $R_{-1} = R_{-1}(G) = \sum\limits_{v_i \sim v_j} \frac{1}{d_id_j}$ is also well studied in the literature. Fore more details on R_{-1}, see ([6, 22]).

The *Laplacian matrix* of the graph G is the matrix $L(G) = D(G) - A(G)$, where $A(G)$ and $D(G)$ are the $(0, 1)$-*adjacency matrix* and the *diagonal matrix* of vertex degrees of G, respectively. The *normalized Laplacian matrix* of G is defined as $l(G) = D(G)^{-\frac{1}{2}} L(G) D(G)^{\frac{1}{2}}$,where $D(G)^{-\frac{1}{2}}$ is the matrix which is obtained by taking $\left(-\frac{1}{2}\right)$ power of each entry of $D(G)$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the *Laplacian eigenvalues* and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ be the *normalized Laplacian eigenvalues*. For more details on Laplacian and normalized Laplacian eigenvalues, see [8, 10, 18, 19].

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Among various indices in mathematical chemistry, the *Kirchhoff index* $K_f(G)$ and a relative of it, the *degree Kirchhoff index* $K'_f(G)$, have received a great deal of attention, recently. For a connected undirected graph G, the *Kirchhoff index* was defined by Klein and Randić ([16]) as

$$K_f = K_f(G) = \sum_{i<j} r_{ij}\,,$$

where r_{ij} is the effective resistance of the edge v_iv_j. We refer the reader to the references ([1, 16]), and their bibliographies, to get a taste of the variety of approaches used to study this descriptor. In [26], Zhou et al. studied the extremal graphs with given matching number, connectivity and the minimal Kirchhoff index. Also in [21] and [23], the authors determined independently the extremality on the unicyclic graphs with respect to the Kirchhoff index. Moreover, in [25], Zhou et al. presented some lower bounds for the Kirchhoff index of a connected (molecular) graph via the number of vertices (atoms), the number of edges (bands), valency (maximum vertex degree), connectivity and chromatic number.

The *degree Kirchhoff index* was proposed by Chen and Zhang in [7], is defined as

$$K'_f(G) = \sum_{i\sim j} d_i d_j r_{ij}.$$

The *degree Kirchhoff index* has been taken attention as much as the *Kirchhoff index*. It can been seen the reference [7] for some bounds on the degree Kirchhoff index and for some relations on the degree Kirchhoff and Kirchhoff indices. We finally give the reference [20] for further studies over degree Kirchhoff index.

We just want to remind the expression of Kirchhoff index in terms of the Laplacian eigenvalues (see [15]) as in the equality

$$K_f = K_f(G) = n\sum_{i=1}^{n-1}\frac{1}{\mu_i}\,. \tag{1.1}$$

Moreover, in [7], by considering normalized Laplacian eigenvalues, the degree Kirchhoff index is defined as

$$K'_f = K'_f(G) = 2m\sum_{i=1}^{n-1}\frac{1}{\lambda_i}\,. \tag{1.2}$$

Hence, by taking into account (1.1) and (1.2), we can easily conclude that the degree Kirchhoff index is the normalized Laplacian analogue of the ordinary Kirchhoff index.

The number of spanning trees, $t(G)$, of a graph G is equal to the total number of distinct spanning subgraphs of G that are trees. This quantity is also known as the complexity of G, and is given by the following formula in terms of the Laplacian eigenvalues ([9]):

$$t(G) = \frac{1}{n}\prod_{i=1}^{n-1}\mu_i.$$

It is well known that the number of spanning trees of G is can be expressed by the normalized Laplacian eigenvalues as ([8, 9])

$$t(G) = \left(\frac{\Delta'}{2m}\right)\prod_{i=1}^{n-1}\lambda_i$$

where $\Delta' = \prod_{i=1}^{n} d_i$.

In the literature ([11–13, 17, 24]), it was obtained so many bounds on $t(G)$.

We organize this paper in the following way. In Section 2, we give some previously known results which will be needed later. In Section 3, we obtain some bounds on the number of spanning trees of connected graphs in terms of the number of vertices, the number of edges, degree Kirchhoff index and Randić index (R_{-1}). We also improve some bounds which was obtained previously for the number of spanning trees of graphs.

2. Preliminaries

In this section, we give some useful lemmas which will be used later.

Let $a_{1,}a_{2,}...,a_r$ be positive real numbers. For a positive number k among the values $1 \le k \le r$, let us suppose that each P_k is defined as in the following:

$$\begin{aligned} P_1 &= \frac{a_1 + a_2 + \cdots + a_r}{r}, \\ P_2 &= \frac{a_1a_2 + a_1a_3 + \cdots + a_1a_r + a_2a_3 + \cdots + a_{r-1}a_r}{\frac{1}{2}r(r-1)}, \\ &\vdots \\ P_{r-1} &= \frac{a_1a_2\cdots a_{r-1} + a_1a_2\cdots a_{r-2}a_r + \cdots + a_2a_3\cdots a_{r-1}a_r}{r}, \\ P_r &= a_1a_2\cdots a_r. \end{aligned}$$

Hence the *arithmetic mean* is simply P_1 while the *geometric mean* is $P_r^{1/r}$. In fact the following famous lemma (see [2]) gives a relationship among them.

Lemma 2.1 (Maclaurin's Symmetric Mean Inequality [2])**.** *For $a_1, a_2, \cdots, a_r \in \mathbb{R}^+$, it is true that*

$$P_1 \ge P_2^{1/2} \ge P_3^{1/3} \ge \cdots \ge P_r^{1/r}.$$

Equality among them holds if and only if $a_1 = a_2 = \cdots = a_r$.

Let K_n and $K_{p,q}$ $(p + q = n)$ denote the complete graph and the complete bipartite graph, respectively.

Lemma 2.2 ([10])**.** *Let G be a graph with n vertices and without isolated vertices. Then $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ if and only if G is a complete graph K_n.*

Lemma 2.3 ([10])**.** *Let G be a connected graph with $n \ge 2$ vertices. Then $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1}$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$.*

3. Main Results

Now we present our main results on $t = t(G)$.

Theorem 3.1. *Let G be a connected graph with n vertices, m edges and degree Kirchhoff index K'_f. Then*

$$(2m)^{n-2}\Delta'\left(\frac{n-1}{K'_f}\right)^{n-1} \le t(G) \le \frac{\Delta'(n-1)}{K'_f}\left(\frac{n}{n-1}\right)^{n-2} \tag{3.1}$$

with equality holding if and only if $G \cong K_n$.

Proof. Upper bound: Setting $r = n-1$ and $a_i = \lambda_i,\ i = 1, 2, ..., n-1$, by Lemma 2.1, we get

$$P_1 \ge P_{n-2}^{1/n-2},$$

where

$$P_1 = \frac{\sum\limits_{i=1}^{n-1}\lambda_i}{n-1}$$

and

$$\begin{aligned} P_{n-2} &= \frac{\sum\limits_{i=1}^{n-1}\prod\limits_{j=1, j\neq n-i+1}^{n-1}\lambda_j}{n-1} \\ &= \frac{\prod\limits_{j=1}^{n-1}\lambda_j}{n-1} \times \sum_{i=1}^{n-1}\frac{1}{\lambda_i} = \left[\frac{tK'_f}{\Delta'(n-1)}\right]. \end{aligned}$$

From the above,

$$\frac{n}{n-1} \geq \left[\frac{tK'_f.}{\Delta'(n-1)}\right]^{\frac{1}{n-2}},$$

that is,

$$t \leq \frac{\Delta'(n-1).}{K'_f}\left(\frac{n}{n-1}\right)^{n-2}$$

which gives the upper bound in (3.1). First part of the proof is over.

Lower Bound: Now setting $r = n-1$ and $a_i = \lambda_i$, $i = 1, 2, ..., n-1$, in Lemma 2.1, we have

$$P_{n-2}^{1/n-2} \geq P_{n-1}^{1/n-1}$$

where

$$P_{n-2} = \frac{\prod_{j=1}^{n-1}\lambda_j}{n-1} \times \sum_{i=1}^{n-1}\frac{1}{\lambda_i} = \frac{tK'_f.}{\Delta'(n-1)}$$

and

$$P_{n-1} = \prod_{i=1}^{n-1}\lambda_i = \frac{2mt}{\Delta'}.$$

From this, we write

$$\left(\frac{2mt}{\Delta'}\right)^{\frac{1}{n-1}} \leq \left(\frac{tK'_f}{\Delta'(n-1)}\right)^{\frac{1}{n-2}}$$

and hence we obtain the required result.

Now suppose that both the equality holds in (3.1). Then all the inequalities in above must be equalities.

The inequality for both lower and upper bounds, $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$, by Lemma 2.1. Consequently, from Lemma 2.2, we have $G \cong K_n$.

Conversely, one can see easily that the equality holds in (3.1) for complete graph K_n. □

In [4], for a non-zero real number α, it was obtained the sum of the α-th power of the non-zero normalized Laplacian eigenvalues, $s_\alpha(G)$, of a graph G without isolated vertices as

$$s_\alpha(G) = \sum_{i=1}^{h}\lambda_i^\alpha$$

where h is the number of non-zero normalized Laplacian eigenvalues of G.

Now we present the following result which contains s_{-2} particularly for $\alpha = -2$.

Theorem 3.2. *Let G be a connected graph with n vertices, m edges and degree Kirchhoff index K'_f. Then*

$$\frac{\Delta'}{2m}\left[\frac{(n-1)(n-2)}{\left(\frac{K'_f}{2m}\right)^2 - s_{-2}}\right]^{\frac{n-1}{2}} \leq t(G) \leq \frac{(\frac{n}{n-1})^{n-3}\Delta'(n-1)(n-2)}{2m\left[\left(\frac{K'_f}{2m}\right)^2 - s_{-2}\right]} \tag{3.2}$$

with equality holding if and only if $G \cong K_n$.

Proof. Upper Bound : Setting $r = n-1$ and $a_i = \lambda_i$, $i = 1, 2, ..., n-1$, by Lemma 2.1, we get

$$P_1 \geq P_{n-3}^{1/n-3},$$

where

$$P_1 = \frac{\sum_{i=1}^{n-1}\lambda_i}{n-1}$$

and

$$P_{n-3}=\frac{\prod_{i=1}^{n-1}\lambda_i}{(n-1)(n-2)}\left[\left(\sum_{i=1}^{n-1}\frac{1}{\lambda_i}\right)^2-\sum_{i=1}^{n-1}\frac{1}{\lambda_i^2}\right].$$

From the above

$$\frac{n}{n-1}\geq\left\{\frac{2mt}{\Delta'(n-1)(n-2)}\left[\left(\frac{K'_f}{2m}\right)^2-s_{-2}\right]\right\}^{\frac{1}{n-3}},$$

that is,

$$t(G)\leq\frac{(\frac{n}{n-1})^{n-3}(n-1)(n-2)}{\left(\frac{K'_f}{2m}\right)^2-s_{-2}}\frac{\Delta'}{2m}$$

which gives the upper bound in (3.2). First part of the proof is over.

Lower Bound: Now we take $r=n-1$ and $a_i=\lambda_i,\ i=1,2,...,n-1$, in Lemma 2.1, we have

$$P_{n-3}^{1/n-3}\geq P_{n-1}^{1/n-1},$$

where

$$P_{n-1}=\prod_{i=1}^{n-1}\lambda_i$$

and

$$P_{n-3}=\frac{\prod_{i=1}^{n-1}\lambda_i}{(n-1)(n-2)}\left[\left(\sum_{i=1}^{n-1}\frac{1}{\lambda_i}\right)^2-\sum_{i=1}^{n-1}\frac{1}{\lambda_i^2}\right].$$

From this, we write

$$\left(\frac{2mt}{\Delta'}\right)^{\frac{1}{n-1}}\leq\left\{\frac{2mt}{\Delta'(n-1)(n-2)}\left[\left(\frac{K'_f}{2m}\right)^2-s_{-2}\right]\right\}^{\frac{1}{n-3}}$$

where $s_{-2}=\sum_{i=1}^{n-1}\frac{1}{\lambda_i^2}$. Hence we obtain the required result.

Suppose that both the equality holds in (3.2). Then all the inequalities in above must be equalities.

The inequalitiy for both lower and upper bounds, $\lambda_1=\lambda_2=\cdots=\lambda_{n-1}$, by Lemma 2.1. Consequently, from Lemma 2.2, we have $G\cong K_n$.

Conversely, one can see easily that the equality holds in (3.2) for complete graph $K_{n.}$ □

Now we give an another upper bound for $t(G)$ that we think as good result.

Theorem 3.3. *Let G be a connected graph with n vertices, m edges, degree Kirchhoff index K'_f and Randić index R_{-1}. Then*

$$t(G)\leq\frac{\Delta'(n-1)}{K'_f}\left[\frac{n^2-2-2R_{-1}}{(n-1)(n-2)}\right]^{\frac{n-2}{2}}\tag{3.3}$$

with equality holding if and only if $G\cong K_n$.

Proof. Taking $r=n-1$ and $a_i=\lambda_i,\ i=1,2,...,n-1$ and using $P_2^{1/2}\geq P_{n-2}^{1/n-2}$, by Lemma 2.1, where

$$\begin{aligned}P_2&=\frac{1}{(n-1)(n-2)}\sum_{i=1}^{n-1}\sum_{j=1,j\neq i}^{n-1}\lambda_i\lambda_j\\&=\frac{1}{(n-1)(n-2)}\left[\left(\sum_{i=1}^{n-1}\lambda_i\right)^2-\sum_{i=1}^{n-1}\lambda_i^2\right]\\&=\frac{1}{(n-1)(n-2)}\left[n^2-n-2R_{-1}\right]\end{aligned}$$

as $\sum_{i=1}^{n-1}\lambda_i = n$ and $\sum_{i=1}^{n-1}\lambda_i^2 = n + 2R_{-1}$

and

$$\begin{aligned} P_{n-2} &= \frac{\sum_{i=1}^{n-1}\prod_{j=1, j\neq n-i+1}^{n-1}\lambda_j}{n-1} \\ &= \frac{\prod_{j=1}^{n-1}\lambda_j}{n-1}\times\sum_{i=1}^{n-1}\lambda_i = \frac{t\,K_f'}{\Delta'\,(n-1)}, \end{aligned}$$

we obtain the inequality (3.3).

The equality holds in (3.3) if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1}$ from Lemma 2.1. Also, by Lemma 2.2, $G \cong K_n$. Conversely the equality follows easily. □

Remark 3.4. We take $r = n-1$ and $a_i = \lambda_i,\ i = 1, 2, ..., n-1$, in Lemma 2.1. If we consider the inequality $P_1 \geq P_{n-1}^{1/n-1}$, we have the following result for $t(G)$ which was obtained in [14]

$$t(G) \leq \frac{\Delta'}{2m}\left(\frac{n}{n-1}\right)^{n-1}. \tag{3.4}$$

Similarly, if we consider the inequality $P_2^{1/2} \geq P_{n-2}^{1/n-2}$, we get the following upper bound

$$\frac{\Delta'}{2m}\left[\frac{n^2-(n-2)(n+2R_{-1})}{(n-1)}\right]^{\frac{n-1}{2}} \leq t(G) \leq \frac{\Delta'}{2m}\left[\frac{n^2-n-2R_{-1}}{(n-1)(n-2)}\right]^{\frac{n-1}{2}} \tag{3.5}$$

which was obtained in [5].

We now consider connected bipartite graphs.

Theorem 3.5. *Let G be a connected bipartite graphs with $n > 2$ vertices, m edges and degree Kirchhoff index K_f'. Then*

$$\frac{\Delta'}{m}\left[\frac{n-2}{\frac{K_f'}{2m}-\frac{1}{2}}\right]^{n-2} \leq t(G) \leq \frac{\Delta'}{m}\left[\frac{n-2}{\frac{K_f'}{2m}-\frac{1}{2}}\right] \tag{3.6}$$

with equality holding if and only if $G \cong K_{p,q}$.

Proof. Lower Bound : Setting $r = n-2$ and $a_i = \lambda_i,\ i = 2, 3, ..., n-1$, by Lemma 2.1, we have

$$P_{n-3}^{1/n-3} \geq P_{n-2}^{1/n-2} \tag{3.7}$$

where

$$\begin{aligned} P_{n-3} &= \frac{\sum_{i=2}^{n-1}\prod_{j=2, j\neq n-i+1}^{n-1}\lambda_j}{n-2} \\ &= \frac{\prod_{j=1}^{n-1}\lambda_j}{n-2}\times\sum_{i=2}^{n-1}\frac{1}{\lambda_i} = \frac{2mt}{\Delta'\lambda_1(n-2)}\left[\frac{K_f'}{2m}-\frac{1}{\lambda_1}\right] \end{aligned}$$

and

$$P_{n-2} = \prod_{i=2}^{n-1}\lambda_i = \frac{2mt}{\Delta'\lambda_1}.$$

Since G is bipartite, we also have $\lambda_1 = 2$ [8]. Then, we combine this fact and the inequality (3.7), we arrive the result.

Upper Bound: Now using $P_1 \geq P_{n-3}^{1/n-3}$ with $r = n-2$ and $a_i = \lambda_i,\ i = 2, 3..., n-1$ from Lemma 2.1, for bipartite graph G, where

$$P_1 = \frac{\prod_{i=2}^{n-1} \lambda_i}{n-2} = \frac{n-\lambda_1}{n-2}$$

and P_{n-3} is as in the proof of lower bound, we have the result.

All equalities hold if and only if $\lambda_2 = \lambda_3 = \cdots = \lambda_{n-1}$.

Now we suppose that all equalities hold. Then, by Lemma 2.3, we conclude that $G \cong K_{p,q}$.

Conversely, we can easily see that the equalities hold for the complete bipartite graph $K_{p,q}$. □

Example 3.6. Let G be a graph with $V_G = \{1, 2, 3, 4\}$ and $E_G = \left\{ \begin{array}{c} \{1,2\}, \{2,3\}, \\ \{2,4\}, \{3,4\} \end{array} \right\}$ and H be a graph with $V_H = \{1, 2, 3, 4\}$ and $E_H = \{\{1, 2\}, \{2, 3\}, \{2, 4\}\}$. For these graphs $t(G) = 3$ and $t(H) = 1$. The lower and upper bounds for $t(G)$ are as follows:

Upper bounds	(3.1)	(3.2)	(3.3)	(3.4)	(3.5)
G	3.14	3	3	3.55	3.308
H	1.07	1	1	1.185	1.075

Lower bounds	(3.1)	(3.2)	(3.5)
G	2.76	2.76	2.6
H	0.864	0.918	0.769

The above tablo shows that in some cases, the upper bounds (3.2) and (3.3) are the best among the above mentioned bounds for $t(G)$. Moreover we see that the upper bounds (3.2) and (3.3), in many graph examples, are equal to $t(G)$. We actually see that our results is better than the results which was obtained previously.

Now we give some results for K_f' which are obtained from the bounds (3.1), (3.2) and (3.3), respectively.

Corollary 3.7. *Let G be a connected graph with n vertices, m edges and t spanning trees. Then*

$$(n-1)\left[\frac{(2m)^{n-2}\Delta'}{t}\right]^{1/n-1} \leq K_f' \leq \frac{\Delta'(n-1)}{t}\left(\frac{n}{n-1}\right)^{n-2}, \tag{3.8}$$

$$2m\sqrt{(n-1)(n-2)\left(\frac{\Delta'}{2mt}\right)^{\frac{2}{n-1}} + s_{-2}} \leq K_f' \tag{3.9}$$

$$K_f' \leq 2m\sqrt{\frac{\left(\frac{n}{n-1}\right)^{n-3}\Delta'(n-1)(n-2) + s_{-2}}{2mt}} \tag{3.10}$$

and

$$K_f' \leq \frac{\Delta'(n-1)}{t}\left(\frac{n^2-n-2R_{-1}}{(n-1)(n-2)}\right)^{\frac{n-2}{2}}. \tag{3.11}$$

with equality holding if and only if $G \cong K_n$.

For trees and unicyclic graphs, we obtain the following results over K_f'

Corollary 3.8. *Let T be a tree of order n with m edges. Then*

$$(n-1)\left[(2m)^{n-2}\Delta'\right]^{1/n-1} \leq K_f'(T) \leq \Delta'(n-1)\left(\frac{n}{n-1}\right)^{n-2},$$

$$2m\sqrt{(n-1)(n-2)\left(\frac{\Delta'}{2m}\right)^{\frac{2}{n-1}} + s_{-2}} \leq K_f'(T)$$

$$K_f'(T) \leq 2m\sqrt{\frac{\left(\frac{n}{n-1}\right)^{n-3}\Delta'(n-1)(n-2)}{2m} + s_{-2}}$$

and

$$K_f'(T) \leq \Delta'(n-1)\left(\frac{n^2-n-2R_{-1}}{(n-1)(n-2)}\right)^{\frac{n-2}{2}}.$$

Proof. Since T is a tree, $t=1$. From (3.8), (3.9), (3.10) and (3.11) we get the required result. □

Corollary 3.9. *Let U be a unicyclic graph of order n with m edges. Then*

$$(n-1)\left[\frac{(2m)^{n-2}\Delta'}{n}\right]^{1/n-1} \leq K_f'(U) \leq \frac{\Delta'(n-1)}{3}\left(\frac{n}{n-1}\right)^{n-2},$$

$$2m\sqrt{(n-1)(n-2)\left(\frac{\Delta'}{2mn}\right)^{\frac{2}{n-1}}} + s_{-2} \leq K_f'(U)$$

$$K_f'(U) \leq 2m\sqrt{\frac{\left(\frac{n}{n-1}\right)^{n-3}\Delta'(n-1)(n-2)}{6m}} + s_{-2}$$

and

$$K_f'(U) \leq \frac{\Delta'(n-1)}{3}\left(\frac{n^2-n-2R_{-1}}{(n-1)(n-2)}\right)^{\frac{n-2}{2}}.$$

Proof. For unicyclic graph U, $3 \leq t \leq n$. From (3.8), (3.9), (3.10) and (3.11), we get the required result. □

In [4] , it was obtained the following results for K_f' :

$$K_f' \geq \frac{2m}{P} + 2(n-2)m\left(\frac{\Delta' P}{2mt}\right)^{\frac{1}{n-2}} \tag{3.12}$$

and

$$K_f' \geq \frac{2m}{P} + \frac{2(n-2)^2 m}{P} \tag{3.13}$$

where $P = 1 + \sqrt{\frac{2}{n(n-1)}R_{-1}}$.

For graphs in Example 3.6, we consider the following table where $K_f'(G) = 20.3$ and $K_f'(H) = 15$.

Lower Bounds	(3.8)	(3.9)	(3.12)	(3.13)
$K_f'(G)$	19.05	20.26	19.09	18.01
$K_f'(H)$	14.28	14.73	14.39	13.52

Upper bounds	(3.8)	(3.10)	(3.11)
$K_f'(G)$	21.3	20.33	20.33
$K_f'(H)$	16	15	15

Finally, from (3.6), considering connected bipartite graphs, we give the following result for K_f'.

Corollary 3.10. *Let G be a connected bipartite graph with n vertices, m edges and t spanning trees. Then*

$$m + 2(n-2)m\left(\frac{\Delta'}{mt}\right)^{\frac{1}{n-2}} \leq K_f' \leq m + 2(n-2)\frac{\Delta'}{t}$$

with equality holding if and only if $G \cong K_{p,q}$.

Remark 3.11. We note that the lower bound which is obtained in Corollary 3.10 is same with the lower bound in [4, Corollary 3.6].

Remark 3.12. Note that if G is a $k-$regular graph, then $\lambda_i = \frac{\mu_i}{k}$for $i = 1, 2, .., n$ (see [8]). Hence we have $K_f = \frac{n}{2m}\frac{1}{k^{n-1}}K_f'$ for any k-regular graph. Therefore, in the case of G is regular, results obtained for K_f' can be immeadiately re-stated for K_f.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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