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Research Article / Araştırma Makalesi

On The Cubic Bezier Curves In $\mathbf{E}^3$

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Abstract

In this study we have examined, the cubic Bezier curve based on the control points with matrix form in $\mathbf{E}^3$. Frenet vector fields and also curvatures of the cubic Bezier curve are examined in matrix form in $\mathbf{E}^3$. Also a simple way has been given to find the control points of any cubic Bezier curve.

Keywords: Bezier Curve, Cubic Bezier Curve, Frenet Apparatus

3 boyutlu Öklid Uzayında Kübik Bezier Eğrileri Üzerine

Öz

Bu çalışmamızda 3 boyutlu Öklid uzayında kontrol noktaları ile kübik Bezier eğrilerini matris formunda incelendi. Frenet vektör alanları ve eğrilikleri de matris formunda incelendi. Ayrıca herhangi bir kübik eğrinin kontrol noktalarının bulunması için basit bir yöntem de örnek ile verildi.

Anahtar Kelimeler: Bezier Curve, Cubic Bezier Curve, Frenet Apparatus

1. Introduction and Preliminaries

In 1962 Bézier curves was studied by the French engineer Pierre Bézier, who used them to design automobile bodies. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using de Casteljau's algorithm, a numerically stable method to evaluate Bézier curves. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion. To guarantee smoothness, the control point at which two curves meet must be on the line between the two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D curves

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for keyframe interpolation. We have been motivated by the following studies. First Bezier-curves with curvature and torsion continuity has been examined in Hagen (2001). Also in Michael (2003) and Zhang & Jieqing (2006) Bezier curves and surfaces has been given. In Kusak et al (2015) planar Bezier curves and Bishop Frame of Bezier Curves are examined, respectively. Recently equivalence conditions of control points and application to planar Bezier curves have been examined in Incesu & Gürsoy (2017). In this study we will define and work on Frenet apparatus of Bézier curves in $\mathbf{E}^3$. So we need the derivates of them.In this study we will define and work on Frenet apparatus of cubic Bézier curves in $\mathbf{E}^3$. A Bézier curve is defined by a set of control points P_0 through P_n, where n is called its order If $n=1$ for linear, If $n=2$ for quadratic Bézier curve, etc. The first and last control points are always the end points of the curve; however, the intermediate control points (if any) generally do not lie on the curve. Lets give the simple definitions of the kinds of Bézier curves. Generaly Béziers curve can be defined by $n+1$ control points $P_0, P_1, ..., P_n$ and has the following form:

Definition 1.1 The points P_I are called control points for the Bézier curve. The polygon formed by connecting the Bézier points with lines, starting with P_0 and finishing with P_n, is called the Bézier polygon (or control polygon). The convex hull of the Bézier polygon contains the Bézier curve. Bézier curve with $n+1$ control points $P_0, P_1, ..., P_n$ has the following equation Incesu & Gürsoy (2017):

$$\mathbf{B}(t) = \sum_{I=0}^{n} \binom{n}{I} t^I (1-t)^{n-I} (t)[P_I], \quad t \in [0,1]$$

$$\mathbf{B}(t) = \sum_{I=0}^{n} B_{n,I}(t)[P_I]$$

where $\mathbf{B}_{n,I}(t) = \binom{n}{I} t^I (1-t)^{n-I}$ and $\binom{n}{I}$ are the binomial coefficients, also expressed as

C_I^n is $\binom{n}{I} = \dfrac{n!}{I!(n-I)!}$. Given points P_0 and P_1, a linear Bézier curve is simply a straight line between those two points. Linear Bézier curve is given by

$$\mathbf{B}(t) = (1-t)P_0 + tP_1,$$

and also it has the matrix form with control points P_0 and P_1,

$$\mathbf{B}(t)=\begin{bmatrix} t & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \end{bmatrix}.$$

A quadratic Bézier curve is the path traced by the function $B(t)$, given points P_0 , P_1 and P_2, which can be interpreted as the linear interpolant of corresponding points on the linear Bézier curves from P_0 to P_1 and from P_1 to P_2 respectively.

$$B(t)=(1-t)^2 P_0+2t(1-t)P_1+t^2 P_2,$$

A quadratic Bézier has the matrix form with control points

$$\mathbf{B}(t)=\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \end{bmatrix}$$

Theorem 1.1 *The derivatives of the any Bézier curve* $\mathbf{B}(t)$ *is*

$$\mathbf{B}'(t)=\sum_{I=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-i-1}Q_i$$

where

$$Q_0=n(P_1-P_0),\ Q_1=n(P_2-P_1),\ Q_2=n(P_3-P_2),...,Q_n=n(P_{i+1}-P_i).$$

Proof. Computing higher order derivatives of a Bézier curve is a simple matter. Once the control points are known, the control points of its derivative curve can be obtained immediately. Since the control points are constants and independent of the variable t, computing the derivative curve $\mathbf{B}'(t)$ reduces to the computation of the derivatives of $\mathbf{B}_{n,i}(t)$'s, we have the following result

$$\frac{d\mathbf{B}(t)}{dt}=\left[\sum_{I=0}^{n}\binom{n}{i}t^i(1-t)^{n-i}P_i\right]',$$

$$\mathbf{B}'(t)=\sum_{I=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-i-1}\left[n(P_{i+1}-P_i)\right].$$

Where $Q_0=n(P_1-P_0),Q_1=n(P_2-P_1),Q_2=n(P_3-P_2),...,Q_n=n(P_{i+1}-P_i).$
Therefore, the derivative of $\alpha(t)$ is a Bézier curve of degree $n-1$ defined by n control

points $n(P_1-P_0), n(P_2-P_1), n(P_3-P_2), ..., n(P_n-P_{n-1})$. This derivative curve is usually referred to as the hodograph of the original Bézier curve. Note that $P_{i+1}-P_i$ is the direction vector from P_i to P_{i+1} and $n(P_{i+1}-P_i)$ is n times longer than the direction vector. Recall that the derivative of $\mathbf{B}(t)$ is the following:

$$\mathbf{B}'(t)=\sum_{I=0}^{n-1} \mathbf{B}_{n-1,i}(t)[Q_i]$$

where

$$\mathbf{B}_{n-1,i}(t)=\binom{n-1}{i}t^i(1-t)^{n-1-i}.$$

Also, after some simple algebraic operations we can show that the derivative of a Bézier curve is the difference of two Bézier curves of degree $n-1$, thus,

$$\mathbf{B}'(t)=n\left(\sum_{I=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-1-i}[P_{i+1}]-\sum_{I=0}^{n-1}\binom{n-1}{i}t^i(1-t)^{n-1-i}[P_i]\right).$$

Applying the derivative formula to the above Bézier curve we find the second derivative of the original Bézier curve:

$$\mathbf{B}''(t)=\sum_{I=0}^{n-2}\binom{n-2}{i}t^i(1-t)^{n-2-i}\left[n-1(Q_{i+1}-Q_i)\right]$$

$$=\sum_{I=0}^{n-2}\binom{n-2}{i}t^i(1-t)^{n-2-i}\left[n(n-1)((P_{i+2}-2P_{i+1}+P_i))\right].$$

with control points $n(n-1)((P_{i+2}-2P_{i+1}+P_i))$, $0<i<n-2$.

Theorem 1.2 The set, whose elements are Frenet vector fields and the curvatures of a curve $\alpha(t)\subset IE^3$, is called Frenet apparatus of the curves. Let $\alpha(t)$ be the curve, with $\eta=\|\alpha'(t)\|\neq 1$ and Frenet apparatus are $\{T(t),N(t),B(t),\kappa(t),\tau(t)\}$. Frenet vector fields are given for a non arc-lengthed curve

$$T(t)=\frac{\alpha'(t)}{\|\alpha'(t)\|},\quad N(t)=B(t)\Lambda T(t),\quad B(t)=\frac{\alpha'(t)\Lambda\alpha''(t)}{\|\alpha'(t)\Lambda\alpha''(t)\|},$$

where curvature functions are defined by

$$\kappa(t)=\frac{\left\|\alpha^{'}(t)\Lambda\alpha^{''}(t)\right\|}{\left\|\alpha^{'}(t)\right\|^{3}},\ \tau(t)=\frac{\left\langle \alpha^{'}(t)\Lambda\alpha^{''}(t),\alpha^{'''}(t)\right\rangle}{\left\|\alpha^{'}(t)\Lambda\alpha^{''}(t)\right\|^{2}}.$$

Also Frenet formulae are well known as

$$\begin{bmatrix} T^{'} \\ N^{'} \\ B^{'} \end{bmatrix}=\begin{bmatrix} 0 & \eta\kappa & 0 \\ -\eta\kappa & 0 & \eta\tau \\ 0 & -\eta\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

2. Cubic Bezier Curve and Frenet Apparatus in $\mathbf{E}^3$

Definition 2.1 Four points P_0, P_1, P_2 and P_3 in the plane or in higher-dimensional space define a cubic Bézier curve with the following equation

$$\mathbf{B}(t)=(1-t)^3P_0+3t(1-t)^2P_1+3t^2(1-t)P_2+t^3P_3.$$

Those curve starts at P_0 going toward P_1 and arrives at P_3 coming from the direction of P_2. Usually, it will not pass through P_1 or P_2; these points are only there to provide directional information. The distance between P_0 and P_1 determines "how long" the curve moves into direction P_2 before turning towards P_3.

Theorem 2.1 Let α be a cubic Bézier curve with control points $P_0, P_1, P_2,$ and P_3. The matrix form of the cubic Bezier curve with is

$$\alpha(t)=\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

the corresponding matrix of the cubic Bezier curve is

$$\left[cm\alpha^3\right]=\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

called as coefficients's Matrix of cubic Bezier curve with inverse:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Proof. Let $\alpha(t)$ be a cubic Bézier curve with four control points P_0, P_1, P_2 and P_3 in $\mathbf{E}^3$, couse of the definition, a cubic Bézier curve is

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3 P_3$$

$$= -t^3 P_0 + 3t^3 P_1 - 3t^3 P_2 + t^3 P_3 + 3t^2 P_0 - 6t^2 P_1 + 3t^2 P_2 P - 3tP_0 + 3tP_{10}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -P_0 + 3P_1 - 3P_2 + P_3 \\ 3P_0 - 6P_1 + 3P_2 \\ -3P_0 + 3P_1 \\ P_0 \end{bmatrix}$$

it is easy to write the matrix product form as in the proof. In differantial geometry to calculate the Frenet apparatus we need the derivatives. Also we will give their matrix form as in the following theorems.

Theorem 2.2 The first derivative of a cubic Bézier curve $\alpha,$ has the following equation and matrix form is

$$\alpha'(t) = \begin{bmatrix} t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}.$$

Where

$$Q_0 = 3(P_1 - P_0) = (x_0, y_0, z_0), Q_1 = 3(P_2 - P_1) = (x_1, y_1, z_1), Q_2 = 3(P_3 - P_2) = (x_2, y_2, z_2)$$

are the control points of the first derivative of cubic Bezier curve .The corresponding matrix of the first derivative of cubic Bezier curve is called as the coefficients's Matrix of the first derivative of a cubic Bézier curve;

$$\left[d_1^3\alpha\right]=\begin{bmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix} \text{ with inverse matrix}$$

$$\begin{bmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}^{-1}=\begin{bmatrix}0 & 0 & 1\\ 0 & \frac{1}{2} & 1\\ 1 & 1 & 1\end{bmatrix}$$

Also it can be written based on the control points

$$\alpha^{'}(t)=\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}x_0-2x_1-x_2 & y_0-2y_1-y_2 & z_0-2z_1-z_2\\ 2x_1-2x_0 & 2y_1-2y_0 & 2z_1-2z_0\\ x_0 & y_0 & z_0\end{bmatrix}$$

$$\alpha^{'}(t)=\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}d_{11} & d_{12} & d_{13}\\ d_{21} & d_{22} & d_{23}\\ d_{31} & d_{32} & d_{33}\end{bmatrix}.$$

Proof. According to the definition of the derivation of a cubic Bézier curve

$$\alpha(t)=P_0-3tP_0+3tP_1+3t^2P_0-6t^2P_1-t^3P_0+3t^2P_2+3t^3P_1-3t^3P_2+t^3P_3$$

we get

$$\alpha^{'}(t)=t^2Q_2+Q_0\left(-2t+t^2+1\right)+Q_1\left(2t-2t^2\right)$$

$$=Q_0-2tQ_0+2tQ_1+t^2Q_0-2t^2Q_1+t^2Q_2$$

It is easy to write its matrix form as

$$\alpha^{'}(t)=\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{bmatrix}\begin{bmatrix}Q_0\\ Q_1\\ Q_2\end{bmatrix}.$$

Theorem 2.3 The second derivative of a cubic Bézier curve and matrix form is

$$\alpha''(t) = \begin{bmatrix} t & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix}$$

where the control points of the second derivative of cubic Bezier curve are $R_0 = 6(P_2 - 2P_1 + P_0) = 6(x_1 - x_0, y_1 - y_0, z_1 - z_0)$ and $R_1 = 6(P_3 - 2P_2 + P_1) = 6(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. And also the matrix;

$$\left[d_2^3\alpha\right] = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

is the matrix of coefficients of the second derivative with inverse matrix

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Proof. According to the definition of the second derivation of a cubic Bézier curve α, we have

$$\alpha''(t) = (1-t)2(Q_1 - Q_0) + t2(Q_2 - Q_1) = (1-t)2[Q_0Q_1] + t2[Q_1Q_2]$$
$$\alpha''(t) = (1-t)R_0 + tR_1.$$

Hence it is easy to write its matrix form with control points R_0 and R_1 as in the theorem. Also it can be written

$$\alpha''(t) = [t,1] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6(P_2 - 2P_1 + P_0) \\ 6(P_3 - 2P_2 + P_1) \end{bmatrix}$$

$$\alpha''(t) = 6[t\ 1] \begin{bmatrix} (x_2 - 2x_1 + x_0) & (y_2 - 2y_1 + y_0) & (z_2 - 2z_1 + z_0) \\ (x_1 - x_0) & (y_1 - y_0) & (z_1 - z_0) \end{bmatrix}$$

Theorem 2.4 The third derivative of a cubic Bézier curve and its matrix form is;

$$\alpha'''(t) = [R_0R_1],$$
$$\alpha'''(t) = 6[P_3 - 3P_2 + 3P_1 - P_0],$$
$$\alpha'''(t) = 6((x_2 - 2x_1 + x_0), (y_2 - 2y_1 + y_0), (z_2 - 2z_1 + z_0))$$

where the control point of the third derivative of cubic Bezier curve is

$$[R_0R_1] = R_1 - R_0 = 2[Q_1Q_2] - 2[Q_0Q_1] = 6[P_3 - 3P_2 + 3P_1 - P_0].$$

Proof. According to the definition of the third derivation of a cubic Bézier curve α, we have

$$\alpha'''(t) = ((1-t)R_0 + tR_1)',$$
$$\alpha'''(t) = [R_0R_1],$$
$$= R_1 - R_0$$

where the control point is $[R_0R_1] = 2[Q_1Q_2] - 2[Q_0Q_1]$ or it can be written

$$[R_0R_1] = 6(P_3 - 3P_2 + 3P_1 - P_0).$$

Since

$$R_0 = 2[Q_0Q_1] = 6(x_1 - x_0, y_1 - y_0, z_1 - z_0),$$

$$R_1 = 2[Q_1Q_2] = 6(x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

we have the following result too

$$[R_0R_1] = 6((x_2 - 2x_1 + x_0), (y_2 - 2y_1 + y_0), (z_2 - 2z_1 + z_0)).$$

2.1 Frenet Apparatus of A Cubic Bezier Curve

Theorem 2.6 Tangent vecror field of a cubic Bezier curve has the following matrix form

$$T(t) = \frac{1}{\eta}\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

where

$$Q_0 = 3(P_1 - P_0) = 3[P_0P_1] = (x_0, y_0, z_0),\ Q_1 = 3(P_2 - P_1) = 3[P_1P_2] = (x_1, y_1, z_1),$$

$$Q_2 = 3(P_3 - P_2) = 3[P_2P_3] = (x_2, y_2, z_2) \text{ and } \eta = \|\alpha'\|.$$

Proof. Since $T(t) = \dfrac{\alpha'(t)}{\|\alpha'(t)\|}$, we get

$$T(t)=\frac{\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}d_1 & B^3\end{bmatrix}\begin{bmatrix}3P_0P_1\\3P_1P_2\\3P_2P_3\end{bmatrix}}{\eta}$$

$$T(t)=\frac{1}{\eta}\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\-2 & 2 & 0\\1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_0 & y_0 & z_0\\x_1 & y_1 & z_1\\x_2 & y_2 & z_2\end{bmatrix}$$

Where let η has the follwing matrix form

$$\eta=\left\|\alpha'\right\|=\left(\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}6 & -6 & 1\\-6 & 8 & -2\\1 & -2 & 1\end{bmatrix}\begin{bmatrix}x_0 & y_0 & z_0\\x_1 & y_1 & z_1\\x_2 & y_2 & z_2\end{bmatrix}\begin{bmatrix}x_0 & y_0 & z_0\\x_1 & y_1 & z_1\\x_2 & y_2 & z_2\end{bmatrix}^T\begin{bmatrix}t^2 & t & 1\end{bmatrix}^T\right)^{\frac{1}{2}}$$

Theorem 2.7 Binormal vector field of a cubic Bezier curve is,

$$B(t)=\frac{6}{m}\begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & b_{13}\\b_{21} & b_{22} & b_{23}\\b_{31} & b_{32} & b_{33}\\b_{41} & b_{42} & b_{43}\end{bmatrix}$$

where $m=\left\|\alpha'(t)\Lambda\alpha''(t)\right\|$ and

$b_{11}=2y_0z_2-2z_0y_2-4y_1z_2+4y_2z_1$, $b_{21}=y_0z_1-y_1z_0-3y_0z_2+3z_0y_2+3y_1z_2-3y_2z_1$,
$b_{31}=-2y_0z_1+2y_1z_0+y_0z_2-z_0y_2$, $b_{41}=y_0z_1-y_1z_0$,
$b_{12}=-2x_0z_2+2x_2z_0+4x_1z_2-4x_2z_1$, $b_{22}=-x_0z_1+x_1z_0+3x_0z_2-3x_2z_0-3x_1z_2+3x_2z_1$,
$b_{32}=2x_0z_1-2x_1z_0-x_0z_2+x_2z_0$, $b_{42}=-x_0z_1+x_1z_0$
$b_{13}=2x_0y_2-2y_0x_2-4x_1y_2+4x_2y_1$, $b_{23}=x_0y_1-x_1y_0-3x_0y_2+3y_0x_2+3x_1y_2-3x_2y_1$,
$b_{33}=-2x_0y_1+2x_1y_0+x_0y_2-y_0x_2$, $b_{43}=x_0y_1-x_1y_0$

Proof. Since $B(t)=\dfrac{\alpha'(t)\Lambda\alpha''(t)}{\left\|\alpha'(t)\Lambda\alpha''(t)\right\|}$, using the following determinant as $\alpha'(t)\Lambda\alpha''(t)$

$$\alpha'(t)\Lambda\alpha''(t) = 6\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\left(\begin{bmatrix} 2y_0z_2 - 2z_0y_2 - 4y_1z_2 + 4y_2z_1 \\ y_0z_1 - y_1z_0 - 3y_0z_2 + 3z_0y_2 + 3y_1z_2 - 3y_2z_1 \\ -2y_0z_1 + 2y_1z_0 + y_0z_2 - z_0y_2 \\ y_0z_1 - y_1z_0 \end{bmatrix}\begin{bmatrix} -2x_0z_2 + 2x_2z_0 + 4x_1z_2 - 4x_2z_1 \\ -x_0z_1 + x_1z_0 + 3x_0z_2 - 3x_2z_0 - 3x_1z_2 + 3x_2z_1 \\ 2x_0z_1 - 2x_1z_0 - x_0z_2 + x_2z_0 \\ -x_0z_1 + x_1z_0 \end{bmatrix}\begin{bmatrix} 2x_0y_2 - 2y_0x_2 - 4x_1y_2 + 4x_2y_1 \\ x_0y_1 - x_1y_0 - 3x_0y_2 + 3y_0x_2 + 3x_1y_2 - 3x_2y_1 \\ -2x_0y_1 + 2x_1y_0 + x_0y_2 - y_0x_2 \\ x_0y_1 - x_1y_0 \end{bmatrix}\right)$$

If we replace b_{ij},

$$\alpha'\Lambda\alpha'' = 6\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix}$$

and we can write it in matrix product form, this complete the proof.

Theorem 2.8 Normal vecror field of a cubic Bezier curve is

$$N(t) = \frac{6\begin{bmatrix} t^5 & t^4 & t^3 & t^2 & t & 1 \end{bmatrix}}{\eta m}\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \\ n_{51} & n_{52} & n_{53} \\ n_{51} & n_{52} & n_{53} \end{bmatrix}$$

where; $\left\|\alpha'(t)\Lambda\alpha''(t)\right\| = m$

$n_{11} = b_{12}d_{13} - b_{13}d_{12}$, $n_{21} = b_{12}d_{23} - b_{13}d_{22} + b_{22}d_{13} - b_{23}d_{12}$,

$n_{31} = b_{12}d_{33} - b_{13}d_{32} + b_{22}d_{23} - b_{23}d_{22} + b_{32}d_{13} - d_{12}b_{33}$

$n_{41} = b_{22}d_{33} - b_{23}d_{32} + b_{32}d_{23} - d_{12}b_{43} - b_{33}d_{22} + b_{42}d$,

$n_{51} = b_{32}d_{33} - b_{33}d_{32} + b_{42}d_{23} - d_{22}b_{43}$, $n_{61} = b_{42}d_{33} - b_{43}d_{32}$, $n_{12} = -b_{11}d_{13} + b_{13}d_{11}$,

$n_{22} = -b_{11}d_{23} - b_{21}d_{13} + b_{13}d_{21} + d_{11}b_{23}$, $n_{32} = -b_{11}d_{33} - b_{21}d_{23} + b_{13}d_{31} - b_{31}d_{13} + d_{11}b_{33} + b_{23}d_{21}$,

$$n_{42}=-b_{21}d_{33}-b_{31}d_{23}+d_{11}b_{43}+b_{23}d_{31}-b_{41}d_{13}+d_{21}b_{33},$$
$$n_{52}=-b_{31}d_{33}-b_{41}d_{23}+d_{21}b_{43}+b_{33}d_{31},\; n_{62}=-b_{41}d_{33}+d_{31}b_{43},\; n_{13}=b_{11}d_{12}-b_{12}d_{11},$$
$$n_{23}=b_{11}d_{22}-b_{12}d_{21}+b_{21}d_{12}-b_{22}d_{11},\; n_{23}=b_{11}d_{32}-b_{12}d_{31}+b_{21}d_{22}-b_{22}d_{21}+b_{31}d_{12}-d_{11}b_{32},$$
$$n_{23}=b_{21}d_{32}-b_{22}d_{31}+b_{31}d_{22}-d_{11}b_{42}-b_{32}d_{21}+b_{41}d_{12},\; n_{23}=b_{31}d_{32}-b_{32}d_{31}+b_{41}d_{22}-d_{21}b_{42}$$
$$n_{63}=b_{41}d_{32}-b_{42}d_{31}.$$

Proof. Since $N(t)=B(t)\Lambda T(t)=\dfrac{(\alpha'\Lambda\alpha'')\Lambda\alpha'}{m\eta}$, we have

$$N(t)=\frac{6}{m\eta}\begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\\ b_{41} & b_{42} & b_{43}\end{bmatrix}\Lambda\begin{bmatrix}t^2 & t & 1\end{bmatrix}\begin{bmatrix}d_{11} & d_{12} & d_{13}\\ d_{21} & d_{22} & d_{23}\\ d_{31} & d_{32} & d_{33}\end{bmatrix}$$

$$N(t)=\frac{6}{m}\left(b_{41}+tb_{31}+t^2b_{21}+t^3b_{11}\quad b_{42}+tb_{32}+t^2b_{22}+t^3b_{12}\quad b_{43}+tb_{33}+t^2b_{23}+t^3b_{13}\right)$$
$$\Lambda\frac{1}{\eta}\left(d_{31}+td_{21}+t^2d_{11}\quad d_{32}+td_{22}+t^2d_{12}\quad d_{33}+td_{23}+t^2d_{13}\right)$$

Hence we have the proof as the result of the following determinant

$$N(t)=\frac{6}{m\eta}\begin{vmatrix} i & j & k\\ b_{41}+tb_{31}+t^2b_{21}+t^3b_{11} & b_{42}+tb_{32}+t^2b_{22}+t^3b_{12} & b_{43}+tb_{33}+t^2b_{23}+t^3b_{13}\\ d_{31}+td_{21}+t^2d_{11} & d_{32}+td_{22}+t^2d_{12} & d_{33}+td_{23}+t^2d_{13}\end{vmatrix}.$$

We have the matrices product replacing each components with n_{ij} we have the proof.

$$N(t)=\frac{6\begin{bmatrix}t^5 & t^4 & t^3 & t^2 & t & 1\end{bmatrix}}{\eta m}\begin{bmatrix}n_{11} & n_{12} & n_{13}\\ n_{21} & n_{22} & n_{23}\\ n_{31} & n_{32} & n_{33}\\ n_{41} & n_{42} & n_{43}\\ n_{51} & n_{52} & n_{53}\\ n_{51} & n_{52} & n_{53}\end{bmatrix}.$$

Theorem 2.9 First curvature of a cubic Bezier curve is

$$\kappa(t)=\frac{6}{\eta^3}\left(\left(b_{41}+tb_{31}+t^2b_{21}+t^3b_{11}\right)^2+\left(b_{42}+tb_{32}+t^2b_{22}+t^3b_{12}\right)^2+\left(b_{43}+tb_{33}+t^2b_{23}+t^3b_{13}\right)^2\right)^{\frac{1}{2}}.$$

Proof. Since the first curvature of a cubic Bezier curve is $\kappa(t)=\dfrac{\left\|\alpha'(t)\Lambda\alpha''(t)\right\|}{\left\|\alpha'(t)\right\|^3}$, first we have

$$\alpha'\Lambda\alpha''=6\begin{bmatrix}t^3 & t^2 & t & 1\end{bmatrix}\begin{bmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\\ b_{41} & b_{42} & b_{43}\end{bmatrix}$$

$$=\left(6b_{41}+6tb_{31}+6t^2b_{21}+6t^3b_{11},\quad 6b_{42}+6tb_{32}+6t^2b_{22}+6t^3b_{12},\quad 6b_{43}+6tb_{33}+6t^2b_{23}+6t^3b_{13}\right)$$

and it is easy to get that

$$\left\|\alpha'\Lambda\alpha''\right\|=m=6\left(\begin{bmatrix}b_{41}+tb_{31}+t^2b_{21}+t^3b_{11} & b_{42}+tb_{32}+t^2b_{22}+t^3b_{12} & b_{43}+tb_{33}+t^2b_{23}+t^3b_{13}\end{bmatrix}\begin{bmatrix}b_{41}+tb_{31}+t^2b_{21}+t^3b_{11}\\ b_{42}+tb_{32}+t^2b_{22}+t^3b_{12}\\ b_{43}+tb_{33}+t^2b_{23}+t^3b_{13}\end{bmatrix}\right)^{\frac{1}{2}}$$

$$=6\left(\left(b_{41}+tb_{31}+t^2b_{21}+t^3b_{11}\right)^2+\left(b_{42}+tb_{32}+t^2b_{22}+t^3b_{12}\right)^2+\left(b_{43}+tb_{33}+t^2b_{23}+t^3b_{13}\right)^2\right)^{\frac{1}{2}}$$

So the matrices product form completes the proof.

Theorem 2.10 Second curvature of a cubic Bezier curve is

$$\tau(t)=6\,\frac{x_0y_1z_2-x_0y_2z_1-x_1y_0z_2+x_1y_2z_0+x_2y_0z_1-x_2y_1z_0}{m^2}$$

Proof. Since $\tau(t)=\dfrac{\left\langle\alpha'(t)\Lambda\alpha''(t),\alpha'''(t)\right\rangle}{\left\|\alpha'(t)\Lambda\alpha''(t)\right\|^2}$ and

$$\left\langle \alpha'(t)\Lambda\alpha''(t),\alpha'''(t)\right\rangle = 6\begin{bmatrix} x_0(t^2-2t+1)-x_1(2t^2-2t)+t^2x_2 & y_0(t^2-2t+1)-y_1(2t^2-2t)+t^2y_2 & z_0(t^2-2t+1)-z_1(2t^2-2t)+t^2z_2 \\ (x_0-x_1)(t-1)-t(x_1-x_2) & (y_0-y_1)(t-1)-t(y_1-y_2) & (z_0-z_1)(t-1)-t(z_1-z_2) \\ x_0-2x_1+x_2 & y_0-2y_1+y_2 & z_0-2z_1+z_2 \end{bmatrix}$$

$$= 6(x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0)$$

Hence we have the proof.

Example 2.1 Find the cubic Bezier curve with control points $P_0 = (-1,3,2)$, $P_1 = (1,0,-1)$, $P_2 = (2,1,0)$, $P_3 = (3,-1,5)$

$$\alpha(t) = \sum_{I=0}^{3}\binom{3}{I} t^I (1-t)^{3-I} P_I$$

$$= (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t)P_2 + t^3P_3$$

$$= (1-t)^3(-1,3,2) + 3t(1-t)^2(1,0,-1) + 3t^2(1-t)(2,1,0) + t^3(3,-1,5)$$

$$\alpha(t) = (t^3 - 3t^2 + 6t - 1, -7t^3 + 12t^2 - 9t + 3, 12t^2 - 9t + 2)$$

Example 2.2 Find the control points of the cubic Bezier curve

$$\alpha(t) = (t^3 - 3t^2 + 6t - 1, -7t^3 + 12t^2 - 9t + 3, 12t^2 - 9t + 2)$$

we can write it in matrix form as in the following way

$$\alpha(t) = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -7 & 0 \\ -3 & 12 & 12 \\ 6 & -9 & -9 \\ -1 & 3 & 2 \end{bmatrix}$$

Since the equality the cubic matrix forms

$$\begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 1 & -7 & 0 \\ -3 & 12 & 12 \\ 6 & -9 & -9 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

using inverse matrix

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -7 & 0 \\ -3 & 12 & 12 \\ 6 & -9 & -9 \\ -1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -1 & 5 \end{bmatrix} = \begin{bmatrix} (x_0, y_0, z_0) \\ (x_1, y_1, z_1) \\ (x_2, y_2, z_2) \\ (x_3, y_3, z_3) \end{bmatrix}$$

we have the control points.

Example 2.3 Bézier curve with control points $P_0(0,0,0), P_1(1,0,0), P_2(0,1,0),$ and $P_3(0,0,1)$ has the following parametric form $\alpha(t) = (3t^3 + 3t^2 + 3t, -3t^3 + 3t^2, t^3)$

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