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Determinants and Permanents of Hessenberg Matrices with Fibonacci-Like Sequences

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Abstract. In this paper, we consider Hessenberg matrices and Fibonacci-Like sequences that is defined by the recurrence relation $T_n = T_{n-1} + T_{n-2}$, $n \geq 2$ and $T_0 = m$, $T_1 = m$ where m is a fixed positive integer. We define two $n \times n$ Hessenberg matrices with applications to the Fibonacci-Like sequences and investigate the determinantal and permanental properties. We obtain that the determinants and permanents of these Hessenberg matrices are equal to the nth term of Fibonacci-Like sequences.

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1. Introduction

Sequences of integer number such as the Fibonacci, Lucas, Pell, Jacobsthal and Padovan sequences are well-known second order recurrence sequences. These sequences of integer number contribute significantly to mathematics, especially to the field of number theory, as Koshy observed [6, 7] . In particular, the Fibonacci sequence is considered as one of the major sequences among the well-known sequences of integer number. For $n \geq 2$, Fibonacci sequence and Lucas sequence are defined by the recurrence relations as follows

$$F_n = F_{n-1} + F_{n-2}, \quad F_0 = 0, F_1 = 1$$
$$L_n = L_{n-1} + L_{n-2}, \quad L_0 = 2, L_1 = 1$$

where F_n is the nth Fibonacci number and L_n is the nth Lucas number. Many authors have been defined Fibonacci pattern based sequences which are popularized and known as Fibonacci-Like sequences and investigated the properties of these sequences (see, for example, [13, 16, 18]).

We now give generalized Fibonacci-Like sequences associated with Fibonacci sequences can be generalized as follows.

Definition 1.1. [18] Generalized Fibonacci-Like sequence $\{T_n\}$ is defined by recurrence relation

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$$T_n = T_{n-1} + T_{n-2}, \quad n \geq 2 \tag{1.1}$$

with initial conditions $T_0 = m$, $T_1 = m$ where m being a fixed positive integer.

The first terms of the sequence $\{T_n\}$ are $m, m, 2m, 3m, 5m, 8m, 13m, 21m, \ldots$

Definition 1.2. [13] Generalized Fibonacci-Like sequence $\{S_n\}$ is defined by recurrence relation

$$S_n = S_{n-1} + S_{n-2}, \quad n \geq 2 \tag{1.2}$$

with initial conditions $S_0 = 2$ and $S_1 = 2$.

The first terms of the sequence $\{S_n\}$ are $2, 2, 4, 6, 10, 16, 26, 42, \ldots$

The corresponding characteristic equation of the equations (1.1) and (1.2) is $x^2 - x - 1 = 0$ and its roots are $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then the Binet's formulas for generalized Fibonacci-Like sequences $\{T_n\}$ and $\{S_n\}$ are given respectively by

$$\begin{aligned} T_n &= m\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right), \\ S_n &= 2\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right). \end{aligned}$$

Also, the roots α and β verify the relations

$$\begin{aligned} \alpha + \beta &= 1, \\ \alpha - \beta &= \sqrt{5}, \\ \alpha\beta &= -1. \end{aligned}$$

On the other hand, there are many relationships between the number theory and matrix theory. In particular, the properties of sequences of integer number are deduced directly from elemantary matrix algebra. In matrix algebra, computations of determinants and permanents have a great importance in many branches of mathematics. Also, determinants and permanents have many applications in physics, chemistry, electrical engineering, and so on. There are a lot of relationships between determinantal and permanentel representations of matrices and these sequences of integer number (see, for example [2, 8, 11]).

For example, Minc [9] defined a $n \times n$ $(0,1)$-matrix $F(n,k)$ and showed that the permanents of $F(n,k)$ is equal to the generalized order-k Fibonacci numbers. Öcal et al. [11] gave some determinantal and permanental representations of k-generalized Fibonacci and Lucas numbers. Yılmaz and Bozkurt [19] derived some relationships between Pell and Perrin sequences and permanents and determinants of a type of Hessenberg matrices. Some authors investigated the relationships between the Hessenberg matrices and the Fibonacci type numbers (see, for example, [3–5, 15, 17]).

Let $A_n = (a_{ij})$ be a $n \times n$ matrix and S_n is a symetric group of permutations over the set $\{1, 2, \ldots n\}$. The determinant of A matrix defined by

$$detA = \sum_{\alpha \in S_n} sgn(\alpha) \prod_{i=1}^{n} \alpha_{i\alpha(i)}$$

where the sum ranges over all the permutations of the integers $1, 2, ..., n$ ([12]).

It can be denoted by $sgn(\alpha) = \pm 1$ the signature of α, equal to $+1$ if α is the product an even number of transposition and -1 otherwise. The permanent of A matrix is defined by

$$perA = \sum_{\alpha \in S_n} \prod_{i=1}^{n} \alpha_{i\alpha(i)}$$

where the summation extends over all permutations α of the symmetric group S_n ([9]). The permanent of A matrix is analogous to the determinant, where all of the signs used in the Laplace expansion of minors are positive.

Let $A = (a_{ij})$ be a $m \times n$ matrix with row vectors $r_1, r_2, \ldots, r_m$. We call A is contractible on column k, if column k contains exactly two nonzero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{(ij:k)}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the contraction of A on column k relative to rows i and j. If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = (A^T_{(ij:k)})^T$ is called the contraction of A on row k relative to columns i and j.

Brualdi and Gibson [1] proved the following result about the permanent of a matrix.

Lemma 1.3. *[1] Let X be a nonnegative integral matrix of order n for $n > 1$ and let Y be a contraction of X. Then,*

$$perX = perY \tag{1.3}$$

A matrix is said to be lower Hessenberg [10] if all entries above the superdiagonal are zero and transposition of a lower Hessenberg matrix is called as upper Hessenberg matrix. A $n \times n$ matrix $A_n = (a_{ij})$ is called lower Hessenberg matrix if $a_{ij} = 0$ when $j - i > 1$, i.e.,

$$A_n = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 & 0 & \ldots & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & \ldots & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} & a_{n-1,4} & \ldots & a_{n-1,n} \\ a_{n,1} & a_{n,2} & a_{n,3} & a_{n,4} & \ldots & a_{n,n} \end{pmatrix}$$

Theorem 1.4. *[2] Let A_n be a $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $det(A_0) = 1$. Then*

$$det(A_1) = a_{11}$$

and for $n \geq 2$

$$det(A_n) = a_{n,n}det(A_{n-1}) + \sum_{r=1}^{n-1}\left[(-1)^{n-r}a_{n,r}\prod_{j=r}^{n-1}a_{j,j+1}det\,(A_{r-1})\right]. \tag{1.4}$$

A matrix that is both upper and lower Hessenberg is called as tridiagonal. For example, Strang [14] presents a family of tridiagonal matrices given by

$$M_n = \begin{pmatrix} 3 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 3 \end{pmatrix}$$

where $M(n)$ is a $n \times n$ Hessenberg matrix and its determinant is the Fibonacci number F_{2n+2}. Lee defined the matrix L_n as follows

$$L_n = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{pmatrix}$$

and showed that

$$per(L_n) = L_n$$

where L_n is the nth usual Lucas number [8],

In this paper, we consider Hessenberg matrices and Fibonacci-Like sequences $\{T_n\}$ and $\{S_n\}$. The purpose of this paper is to use the method to evaluate the determinants and permanents of the Hessenberg matrices, we get more identities and Hessenberg matrices about terms of generalized Fibonacci-Like sequence $\{T_n\}$ and $\{S_n\}$, these Hessenberg matrices only have three or four diagonals are non zero elements, other elements are all zeroes.

2. Main Results

In this paper, we define type lower Hessenberg matrices whose entries are the Fibonacci-Like sequences as a different way to obtain the nth term of the Fibonacci-Like sequences and show that the determinants and permanents of these type matrices are equal to the nth term of the Fibonacci-Like sequences $\{T_n\}$ and $\{S_n\}$.

2.1. The Determinantal Representations of Sequences $\{T_n\}$ and $\{S_n\}$.

Definition 2.1. The $n \times n$ Hessenberg matrix $K_n(m) = (k_{i,j})$ is defined by

$$K_n(m) = \begin{pmatrix} 2m & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ m & 1 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix} \tag{2.1}$$

with $k_{i,i} = 1, k_{i,i+1} = -1, k_{i+1,i} = 1$ for $1 \leq i \leq n$ where $k_{11} = 2m, k_{21} = m$ and 0 otherwise.

Theorem 2.2. *Let the matrix $K_n(m)$ be as in equation (2.1). Then for $n \geq 1$*

$$detK_n(m) = T_{n+1}$$

where T_n is the nth term of generalized Fibonacci-Like sequence $\{T_n\}$.

Proof. To prove $detK_n(m) = T_{n+1}$, we use the mathematical induction on n. Using the equation (1.4) we have

$n = 1, detK_1(m) = k_{11} = 2m = T_2$

$$\begin{aligned} n = 2, detK_2(m) &= k_{22}detK_1(m) + \textstyle\sum_{r=1}^{1}\left[(-1)^{2-r}k_{2,r}\prod_{j=r}^{1}k_{j,j+1}detK_{r-1}(m)\right] \\ &= (1)(2m) + (-1)k_{21}k_{12}detK_0(m) \\ &= (1)(2m) + (-1)m(-1)(1) \\ &= 3m \\ &= T_3 \end{aligned}$$

where $detK_0(m) = 1$.We assume that it is true for $n \in Z^+$, namely

$$detK_n(m) = T_{n+1}, detK_{n-1}(m) = T_n, detK_{n-2}(m) = T_{n-1}, \dots$$

and we show that it is true for $n + 1$. Using induction's hypothesis we obtain

$$
\begin{aligned}
detK_{n+1}(m) &= k_{n+1,n+1}detK_n(m) + \sum_{i=1}^{n}\left[(-1)^{n+1-r}k_{n+1,r}\prod_{j=r}^{n}k_{j,j+1}detK_{r-1}(m)\right] \\
&= (1)detK_n(m) + \sum_{r=1}^{n-1}\left[(-1)^{n+1-r}k_{n+1,r}\prod_{j=r}^{n}k_{j,j+1}detK_{r-1}(m)\right] + (-1)k_{n+1,n}k_{n,n+1}detK_{n-1}(m) \\
&= (1)detK_n(m) + [(-1)(1)(-1)\det K_{n-1}(m) \\
&= (1)detK_n(m) + detK_{n-1}(m) \\
&= T_{n+1} + T_n \\
&= T_{n+2}.
\end{aligned}
$$

So the proof is completed. □

Note that for $m = 2$ in Definition 2.1, we have the Hessenberg matrices of Fibonacci-Like sequences $\{S_n\}$. So we can write following corollory.

Corollary 2.3. *Let the matrix $K_n(2) = (k_{ij})$ be as in equation (2.1). Then for $n \geq 1$*

$$detK_n(2) = S_{n+1}$$

where S_n is the nth term of generalized Fibonacci-Like sequence $\{S_n\}$.

2.2. The Permanental Representations of Sequences $\{T_n\}$ and $\{S_n\}$.

Definition 2.4. The $n \times n$ Hessenberg matrix $P_n(m) = (p_{i,j})$ is defined by

$$
P_n(m) = \begin{pmatrix}
2m & 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
m & 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1
\end{pmatrix} \tag{2.2}
$$

with $p_{i,i} = 1$, $p_{i,i+1} = p_{i+1,i} = 1$ for $1 \leq i \leq n$ where $p_{11} = 2m$, $p_{21} = m$ and 0 otherwise.

Theorem 2.5. *Let matrix $P_n(m)$ be as in equation (2.2). Then for $n \geq 1$*

$$perP_n(m) = perP_n^{(n-2)}(m) = T_{n+1}$$

where T_n is the nth term of generalized Fibonacci-Like sequence $\{T_n\}$.

Proof. Using Definition 2.4, it can be contracted on first column. Let $P_n^r(m)$ be rth contraction of $P_n(m)$, $1 \leq r \leq n-2$. The matrix $P_n(m)$ can be contracted on column 1, so that

$$
P_n^1(m) = \begin{pmatrix}
3m & 2m & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1
\end{pmatrix}
$$

$$= \begin{pmatrix} T_3 & T_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

where $T_3 = 3m$ and $T_2 = 2m$. Since the matrix $P_n^1(m)$ also can be contracted on column 1,

$$P_n^2(m) = \begin{pmatrix} 5m & 3m & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} T_4 & T_3 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

where $T_4 = 5m$ and $T_3 = 3m$. Continuing with this process, we have the rth contraction of the matrix $P_n(m)$ as follows

$$P_n^r(m) = \begin{pmatrix} T_{r+2} & T_{r+1} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 \end{pmatrix}$$

for $3 \le r \le n-4$. Hence

$$P_n^{n-3}(m) = \begin{pmatrix} T_{n-1} & T_{n-2} & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

which by contraction of $P_n^{n-3}(m)$ on column 1, we obtain

$$P_n^{n-2}(m) = \begin{pmatrix} T_{n-2} + T_{n-1} & T_{n-1} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} T_n & T_{n-1} \\ 1 & 1 \end{pmatrix}$$

by using equation (1.1). From the equation (1.3), we have

$$perP_n(m) = perP_n^{(n-2)}(m) = T_n + T_{n-1} = T_{n+1}$$

So the proof is completed. □

Note that for $m = 2$ in Definition 2.4, we have the Hessenberg matrices of Fibonacci-Like sequences $\{S_n\}$. So we can write following corollory.

Corollary 2.6. *Let the matrix $P_n(2) = (p_{ij})$ be as in equation (2.2). Then for $n \geq 1$*

$$perP_n(2) = perP_n^{(n-2)}(2) = S_{n+1}$$

where S_n is the nth term of generalized Fibonacci-Like sequence $\{S_n\}$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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