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Deferred Statistical Convergence of Double Sequences in Intuitionistic Fuzzy Normed Linear Spaces

Ömer Kişi[1,*], Erhan Güler[1]

[1]*Department of Mathematics, Faculty of Science, Bartın University, 74100 Kutlubey, Bartın, Turkey.*

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Abstract. In this study, the intuitionistic fuzzy deferred statistical convergence of double sequences in the intuitionistic fuzzy normed space is defined by considering deferred density given in 2016 by Küçükaslan and M. Yılmaztürk. Besides the main properties of this new method, it is compared with intuitionistic fuzzy statistical convergence of double sequences and itself under different restrictions on the method. Some special cases of the obtained results coincide with known results in literature.

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1. Introduction and Background

Fuzzy set theory began with the work of Zadeh [24] in 1965 as an alternative approach to the decision making problems in engineering. Since then, many researchers have been interested in this new subject and many of them have tried to establish whether analogues of classical theories are true or not in the fuzzy case.

In the last two decades, fuzzy logic finds application in different areas of science such as nonlinear dynamic system, control of chaos, quantum physics, etc. It has also many applications in different brances of mathematics; metric and topological spaces and approximation theory, etc.

The notion of statistical convergence of real valued sequences was first defined by Fast and Steinhaus in 1951 [5] and [23]. Later on, statistical convergence turned out to be one of the most active areas of research in summability theory after the works of [7, 21]. Also, statistical convergence in topology was studied by Maio and Kočinac in 2018 [15].

In the last two decades, fuzzy logic finds application in different areas of science such as nonlinear dynamic system [9], control of chaos [6], quantum physics [14], etc. It has also many applications in different brances of mathematics; metric and topological spaces ([4, 8, 10, 13]).

*Corresponding Author
Email addresses: okisi@bartin.edu.tr (Ö. Kişi), eguler@bartin.edu.tr (E. Güler)

In 1932, R. P. Agnew [1] defined the deferred Cesaro mean $D_{p,q}$ of a sequence $x = (x_n)$ by

$$\left(D_{p,q}x\right)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k,$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of positive natural numbers satisfying

$$p(n) < q(n) \text{ and } q(n) \to \infty. \tag{1.1}$$

A sequence $x = (x_n)$ is called

1. deferred Cesaro convergent to L if

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} (x_k - L) \to 0;$$

2. strongly deferred Cesaro convergent to L if

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} |x_k - L| \to 0.$$

We denote $\lim_{n\to\infty} x_n = L(DS[p,q])$.

Let K be an arbitrary subset of $\mathbb{N}$ and

$$K_{p,q}(n) = \{p(n) < k \le q(n), k \in K\}$$

be an associated set of K for the arbitrary sequences $p(n)$ and $q(n)$ satisfying (1.1).

Let K be an arbitrary subset of $\mathbb{N}$. If the following limit

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|K_{p,q}(n)\right|$$

exists, then $\delta_{p,q}(K)$ is called deferred density of the subset K.

A sequence $x = (x_k)$ is said to be deferred statistically convergent to $l \in \mathbb{R}$ if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} |\{k, p(n) < k \le q(n) : |x_k - L| \ge \varepsilon\}| = 0.$$

It is denoted by $\lim_{n\to\infty} x_n = L(D[p,q])$.

The theory of intuitionistic fuzzy sets was introduced by Atanassov in ([2, 3]) as a generalization of fuzzy sets theory, and it has been extensively used in decision-making problems. The concept of intuitionistic fuzzy metric space was introduced in [16].

A triangular norm (t-norm) is a continuous mapping $* : [0,1]\times[0,1] \to [0,1]$, such that $(S, *)$ is an Abelian monoid with unit one and $c * d \le a * b$ if $c \le a$ and $d \le b$, for all $a, b, c, d \in [0,1]$.

A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$, is said to be a continuous t-conorm if it satisfies the following conditions:

1. $\diamond$ is associative and commutative,
2. $\diamond$ is continuous,
3. $a\diamond 0 = a$, for all $a \in [0,1]$,
4. $c\diamond d \le a\diamond b$ if $c \le a$ and $d \le b$, for all $a, b, c, d \in [0,1]$.

Using the continuous t-norm and t-conorm, Saadati and Park [20] have recently introduced the concept of intuitionistic fuzzy normed space, as follows.

Definition 1.1 ([22])**.** The 5-tuple $(X, \mu, \nu, *, \diamond)$ is said to be intuitionistic fuzzy normed linear space (in short, IFNLS) if X is a linear space, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, and μ and ν are fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x,t) + \nu(x,t) \le 1$,
(b) $\mu(x,t) > 0$,
(c) $\mu(x,t) = 1$ if and only if $x = 0$,
(d) $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \ne 0$,

(e) $\mu(x,t) * \mu(y,s) \leq \mu(x+y,t+s)$,
(f) $\mu(x,t) : (0,\infty) \to [0,1]$ is continuous in t,
(g) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
(h) $\nu(x,t) < 1$,
(i) $\nu(x,t) = 0$ if and only if $x = 0$,
(j) $\nu(\alpha x,t) = \nu\left(x, \frac{t}{|\alpha|}\right)$ for each $\alpha \neq 0$,
(k) $\nu(x,t) \diamond \nu(y,s) \geq \nu(x+y,t+s)$,
(l) $\nu(x,t) : (0,\infty) \to [0,1]$ is continuous in t,
(m) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.
In this case (μ,ν) is called an intuitionistic fuzzy norm.

Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - \xi, t) > 1 - \varepsilon$ and $\nu(x_k - \xi, t) < \varepsilon$ for all $k \geq k_0$. It is denoted by $(\mu,\nu) - \lim x = \xi$.

Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be Cauchy sequence with respect to the intuitionistic fuzzy norm (μ,ν) if, for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\mu(x_k - x_m, t) > 1 - \varepsilon$ and $\nu(x_k - x_m, t) < \varepsilon$ for all $k, m \geq k_0$.

Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS. A sequence $x = (x_k)$ in X is a bounded sequence if there exists a positive real number M such that $\mu(x_k, t) > 1 - M$ and $\nu(x_k, t) < M$ for all $k \in \mathbb{N}$.

Take an IFNS $(X,\mu,\nu,*,\diamond)$. A sequence (x_k) is said to be statistically convergent with respect to IFN (μ,ν) (briefly, FSC-IFN), if there is a number $\xi \in X$ such that for every $\varepsilon > 0$ and $t > 0$, the set

$$K_\varepsilon := \{k \leq n : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon\}$$

has natural density zero., i.e., $d(K_\varepsilon) = 0$. That is,

$$\lim_{n\to\infty} \frac{1}{n} |k \leq n : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon| = 0.$$

In this case, we write $x_k \to \xi(S(\mu,\nu))$.

In the paper [11], the statistical convergence of sequences in IFNS is studied. Later in ([18,19]), lacunary statistical convergence and λ-statistical convergence of sequences in IFNS are defined and some interesting results are given, respectively.

Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS and $x = (x_k)$ be a sequence of X. Then, a sequence $x = (x_k)$ is said to be $D_{p,q}$-summable to ξ with respect to the intuitionistic fuzzy norm (μ,ν) if every $\varepsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \mu(x_k - \xi, t) > 1 - \varepsilon \quad \text{and} \quad \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \nu(x_k - \xi, t) < \varepsilon$$

hold for all $n > n_0$.

Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS and $x = (x_k)$ be a sequence of X. Then, a sequence $x = (x_k)$ is said to be deferred statistically convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if every $\varepsilon \in (0,1)$ and $t > 0$,

$$\delta_p^q(k \in \mathbb{N} : \mu(x_k - \xi, t) \leq 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) \geq \varepsilon) = 0,$$

or equivalently

$$\delta_p^q(k \in \mathbb{N} : \mu(x_k - \xi, t) > 1 - \varepsilon \text{ or } \nu(x_k - \xi, t) < \varepsilon) = 1.$$

In this case, we write

$$x_{kl} \to \xi\left(D_p^q S(\mu,\nu)\right).$$

In [17], the intuitionistic fuzzy deferred statistical convergence in the intuitionistic fuzzy normed space was defined by considering deferred density given in [12].

2. Main Results

Definition 2.1. Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS and $x = (x_{kl})$ be a double sequence of X. Then, the double sequence x is said to be convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ,ν) if every $\varepsilon \in (0,1)$ and $t > 0$, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\mu(x_{kl} - \xi, t) > 1 - \varepsilon \text{ and } \nu(x_{kl} - \xi, t) < \varepsilon$$

hold for all $n > n_0$, $m > m_0$. We denote this by $(\mu, \nu) - \lim x = \xi$ or $x_{kl} \overset{(\mu,\nu)}{\to} \xi$.

Definition 2.2. Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS and $x = (x_{kl})$ be a sequence of X. Then, a double sequence $x = (x_{kl})$ is said to be statistically convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if every $\varepsilon \in (0, 1)$ and $t > 0$,

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - \xi, t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_{kl} - \xi, t\right) \geq \varepsilon\right) = 0,$$

or equivalently

$$\delta\left((k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - \xi, t\right) > 1 - \varepsilon \text{ or } \nu\left(x_{kl} - \xi, t\right) < \varepsilon\right) = 1.$$

In this case, we write

$$x_{kl} \to \xi\left(S\left(\mu, \nu\right)\right).$$

Definition 2.3. Let $x = (x_{kl})$ be a double sequence and $\beta(n) = q(n) - p(n)$, $\gamma(m) = r(m) - t(m)$, and let $\{p(n)\}$, $\{q(n)\}$, $\{r(m)\}$ and $\{t(m)\}$ be sequences of nonnegative integers satisfying the conditions

$$p(n) < t(m), \ t(m) < r(m) \text{ and}$$

$$\lim_{n\to\infty} q(n) = \infty, \ \lim_{m\to\infty} r(m) = \infty.$$

Then deferred Cesaro mean $D_{\beta,\gamma}$ of the double sequence x is defined by

$$\left(D_{\beta,\gamma} x\right)_{nm} = \frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} x_{kl}.$$

Definition 2.4. Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS and $x = (x_{kl})$ be a double sequence of X. Then, the double sequence $x = (x_{ki})$ is said to be $D_{\beta,\gamma}$-summable to ξ with respect to the intuitionistic fuzzy norm (μ, ν) if every $\varepsilon \in (0, 1)$ and $t > 0$, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \mu\left(x_{kl} - \xi, t\right) > 1 - \varepsilon \text{ and } \frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \nu\left(x_{kl} - \xi, t\right) < \varepsilon$$

hold for all $n > n_0$, $m > m_0$.

Definition 2.5. Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS and $x = (x_{kl})$ be a double sequence of X. Then, the double sequence $x = (x_{kl})$ is said to be deferred statistically convergent to $\xi \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if every $\varepsilon \in (0, 1)$ and $t > 0$,

$$\delta_\beta^\gamma\left((k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - \xi, t\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_{kl} - \xi, t\right) \geq \varepsilon\right) = 0,$$

or equivalently

$$\delta_\beta^\gamma\left((k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - \xi, t\right) > 1 - \varepsilon \text{ or } \nu\left(x_{kl} - \xi, t\right) < \varepsilon\right) = 1.$$

In this case, we write

$$x_{kl} \to \xi\left(D_\beta^\gamma S\left(\mu, \nu\right)\right).$$

2.1. $D_\beta^\gamma S\left(\mu, \nu\right)$**-convergence in IFNS.** In this part, we are going to give the main results about $D_\beta^\gamma S\left(\mu, \nu\right)$.

Theorem 2.6. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS and $x = (x_{kl})$ be a sequence of X. Then, $x_{kl} \overset{(\mu,\nu)}{\to} \xi$, implies $x_{kl} \overset{(\mu,\nu)}{\to} \xi\left(D_\beta^\gamma\left(\mu, \nu\right)\right)$.*

Proof. For every $\varepsilon \in (0, 1)$ and $t > 0$, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\mu\left(x_{kl} - \xi, t\right) > 1 - \varepsilon \text{ and } \nu\left(x_{kl} - \xi, t\right) < \varepsilon \tag{2.1}$$

hold for all $n > n_0$, $m > m_0$. If the inequalities in (2.1) are assumed from to $q(n)$ and $t(m) + 1$ to $r(m)$, then the following inequality is obtained:

$$\sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \mu\left(x_{kl} - \xi, t\right) > (1 - \varepsilon)\beta(n)\gamma(m) \text{ and } \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \nu\left(x_{kl} - \xi, t\right) < \varepsilon\beta(n)\gamma(m) \tag{2.2}$$

If both sides of (2.2) are divided by $\beta(n)\gamma(m)$, then the desired result is obtained. □

Theorem 2.7. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS and $x=(x_{kl})$ be a sequence of X. Then, $x_{kl}\overset{(\mu,\nu)}{\rightarrow}\xi$, implies $x_{kl}\overset{(\mu,\nu)}{\rightarrow}\xi\left(D_{\beta}^{\gamma}S\,(\mu,\nu)\right)$.*

Proof. By hypothesis, for every $\varepsilon\in(0,1)$ and $t>0$, there exist $n_0,m_0\in\mathbb{N}$ such that

$$\mu(x_{kl}-\xi,t)>1-\varepsilon \text{ and } \nu(x_{kl}-\xi,t)<\varepsilon$$

hold for all $n>n_0$, $m>m_0$.

This guarantees that the cardinality of the set

$$\{(k,l)\in\mathbb{N}\times\mathbb{N}:\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}$$

is finite. So, immediately we see that

$$\delta_{\beta}^{\gamma}((k,l)\in\mathbb{N}\times\mathbb{N}:\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon)=0.$$

This gives the proof. □

Remark 2.8. The converse of Theorem 2 is not true, in general.

Let us consider the IFNLS $(\mathbb{R},\mu_0,\nu_0,*,\diamond)$ and

$$x_{kl}=\begin{cases}1, & k=m^2, l=n^2\\ 0, & \text{otherwise.}\end{cases}$$

For any $\varepsilon>0$ and $t>0$, consider the following set:

$$K_{\beta}^{\gamma}(\varepsilon,t)=\left\{\begin{array}{c}p(n)+1\le k\le q(n), t(m)+1\le l\le r(m):\\ \mu_0(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu_0(x_{kl}-\xi,t)\ge\varepsilon\end{array}\right\}.$$

It is clear that

$$K_{\beta}^{\gamma}(\varepsilon,t)=\left\{\begin{array}{c}p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\\ |x_{kl}|\ge\frac{\varepsilon t}{1-\varepsilon} \text{ or } \nu_0(x_{kl}-\xi,t)\ge\varepsilon\end{array}\right\}$$

$$=\left\{\begin{array}{c}p(n)+1\le k\le q(n),\ t(m)+1\le l\le r(m):\\ k=m^2, l=n^2,\ m,n\in\mathbb{N}\end{array}\right\}$$

and we have

$$\delta_{\beta}^{\gamma}\left(K_{\beta}^{\gamma}(\varepsilon,t)\right)\le\lim_{n,m\to\infty}\frac{\sqrt{\beta(n)}\sqrt{\gamma(m)}}{\beta(n)\gamma(m)}=0.$$

The last inequality gives that $x_{kl}\to 0\left(D_{\beta}^{\gamma}S\,(\mu_0,\nu_0)\right)$. By Lemma 4.10 in [20], the sequence (x_{kl}) is not (μ_0,ν_0) convergent to zero because it is not convergent to zero in $(\mathbb{R},|.|)$.

From the definition, we can give the following result without proof.

Lemma 2.9. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS, and $x=(x_{kl})$ be a double sequence in X. Then, for every $\varepsilon>0$ and $t>0$, the following statements are equivalent.*

(a) $x_{kl}\to\xi\left(D_{\beta}^{\gamma}S\,(\mu,\nu)\right)$;
(b) $\delta_{\beta}^{\gamma}(\{(k,l)\in\mathbb{N}\times\mathbb{N}:\mu(x_{kl}-\xi,t)\le 1-\varepsilon\})=\delta_{\beta}^{\gamma}(\{(k,l)\in\mathbb{N}\times\mathbb{N}:\nu(x_{kl}-\xi,t)\ge\varepsilon\})=0$;
(c) $\delta_{\beta}^{\gamma}(\{(k,l)\in\mathbb{N}\times\mathbb{N}:\widetilde{\mu}(x_{kl}-\xi,t)\ge\varepsilon\})=\delta_{\beta}^{\gamma}((k,l)\in\mathbb{N}\times\mathbb{N}:\nu(x_{kl}-\xi,t)\ge\varepsilon)=0$;
(d) $\delta_{\beta}^{\gamma}(\{(k,l)\in\mathbb{N}\times\mathbb{N}:\widetilde{\mu}(x_{kl}-\xi,t)<\varepsilon\})=\delta_{\beta}^{\gamma}((k,l)\in\mathbb{N}\times\mathbb{N}:\nu(x_{kl}-\xi,t)<\varepsilon)=1$;
(e) $\widetilde{\mu}(x_{kl}-\xi,t)\to 0\left(D_{\beta}^{\gamma}S\right)$ *and* $\nu(x_{kl}-\xi,t)\to 0\left(D_{\beta}^{\gamma}S\right)$.

Theorem 2.10. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS, and $x=(x_{kl})$ be a double sequence in X. Then, the $D_{\beta}^{\gamma}S\,(\mu,\nu)$ limit of (x_{kl}) is unique.*

Proof. Suppose that there exist two distinct elements $\xi_1, \xi_2 \in X$ such that $x_{kl} \to \xi_1 \left(D_\beta^\gamma S(\mu, \nu)\right)$ and $x_{kl} \to \xi_2 \left(D_\beta^\gamma S(\mu, \nu)\right)$. Given $\varepsilon \in (0, 1)$, choose $r > 0$ such that $(1-r) * (1-r) > 1 - \varepsilon$ and $r \diamond r < \varepsilon$. Then, from the assumption for every $t > 0$, we have

$$\delta_\beta^\gamma\left(K_{\mu,1}(\varepsilon, t)\right) = \delta_\beta^\gamma\left(K_{\nu,1}(\varepsilon, t)\right) = 0 \text{ and } \delta_\beta^\gamma\left(K_{\mu,2}(\varepsilon, t)\right) = \delta_\beta^\gamma\left(K_{\nu,2}(\varepsilon, t)\right) = 0,$$

where

$$K_{\mu,1}(\varepsilon, t) = \{p(n) + 1 \le k \le q(n),\ t(m) + 1 \le l \le r(m) : \mu(x_{kl} - \xi_1, t) \le 1 - \varepsilon\},$$

$$K_{\mu,2}(\varepsilon, t) = \{p(n) + 1 \le k \le q(n),\ t(m) + 1 \le l \le r(m) : \mu(x_{kl} - \xi_2, t) \le 1 - \varepsilon\},$$

and

$$K_{\nu,1}(\varepsilon, t) = \{p(n) + 1 \le k \le q(n),\ t(m) + 1 \le l \le r(m) : \nu(x_{kl} - \xi_1, t) \ge \varepsilon\},$$

$$K_{\nu,2}(\varepsilon, t) = \{p(n) + 1 \le k \le q(n),\ t(m) + 1 \le l \le r(m) : \nu(x_{kl} - \xi_2, t) \ge \varepsilon\}.$$

If we denote the set

$$K_{\mu,\nu}(\varepsilon, t) = \left(K_{\mu,1}(\varepsilon, t) \cup K_{\mu,2}(\varepsilon, t)\right) \cap \left(K_{\nu,1}(\varepsilon, t) \cup K_{\nu,2}(\varepsilon, t)\right).$$

then it is enough to show that $\delta_\beta^\gamma\left(K_{\mu,\nu}(\varepsilon, t)\right) = 0$, which implies that $\delta_\beta^\gamma\left(\mathbb{N} - K_{\mu,\nu}(\varepsilon, t)\right) = 1$.

Now, let $k \in \mathbb{N} - K_{\mu,\nu}(\varepsilon, t)$, then there are two cases:

$$k \in \mathbb{N} - \left\{K_{\mu,1}(\varepsilon, t) \cup K_{\mu,2}(\varepsilon, t)\right\} \text{ or } k \in \mathbb{N} - \left\{K_{\nu,1}(\varepsilon, t) \cup K_{\nu,2}(\varepsilon, t)\right\}.$$

Firstly, assume that $k \in \mathbb{N} - \left\{K_{\mu,1}(\varepsilon, t) \cup K_{\mu,2}(\varepsilon, t)\right\}$. From Definition 1.1($e$), we have

$$\begin{aligned} \mu(\xi_1 - \xi_2, t) &\ge \mu\left(x_{kl} - \xi_1, \frac{t}{2}\right) * \mu\left(x_{kl} - \xi_2, \frac{t}{2}\right) \\ &> (1-r) * (1-r) \end{aligned}$$

and

$$\mu(\xi_1 - \xi_2, t) > 1 - \varepsilon \tag{2.3}$$

holds. Since ε is arbitrary in (2.3), then $\mu(\xi_1 - \xi_2) > 1$ holds. Hence, Definition 1.1(c), it follows that $\xi_1 = \xi_2$.

Secondly, assume that $k \in \mathbb{N} - \{K_{\nu,1}(\varepsilon, t) \cup K_{\nu,2}(\varepsilon, t)\}$. From the assumption and Definition 1.1(k), we have

$$\begin{aligned} \nu(\xi_1 - \xi_2, t) &< \nu\left(x_{kl} - \xi_1, \frac{t}{2}\right) \diamond \nu\left(x_{kl} - \xi_2, \frac{t}{2}\right) \\ &< r \diamond r \end{aligned}$$

By using the fact $r \diamond r < \varepsilon$,

$$\nu(\xi_1 - \xi_2, t) < \varepsilon \tag{2.4}$$

is obtained. Since $\varepsilon > 0$ is arbitrary in (2.4), then $\xi_1 = \xi_2$ is obtained. Therefore, the limit of the sequence is unique. □

Theorem 2.11. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS. Then a double sequence $x = (x_{kl})$ in X. If a sequence (x_{kl}) is $D_\beta^\gamma S(\mu, \nu)$-convergent, then it is $D_\beta^\gamma S(\mu, \nu)$-Cauchy sequence in IFNS.*

Proof. Assume that the sequence Let $x = (x_{kl})$ is $D_\beta^\gamma S(\mu, \nu)$-convergent to $\xi \in X$. Let us choose $s > 0$ so that $(1-\varepsilon) * (1-\varepsilon) > 1 - s$ and $\varepsilon \diamond \varepsilon < s$ hold for any $\varepsilon > 0$. Then, for any $t > 0$, we have,

$$\delta_\beta^\gamma\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - \xi, \frac{t}{2}\right) \le 1 - \varepsilon \text{ or } \nu\left(x_{kl} - \xi, \frac{t}{2}\right) \ge \varepsilon\right\}\right) = 0$$

and this implies that

$$\delta_\beta^\gamma\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - \xi, \frac{t}{2}\right) > 1 - \varepsilon \text{ or } \nu\left(x_{kl} - \xi, \frac{t}{2}\right) < \varepsilon\right\}\right) = 0.$$

Let $(m,n) \in \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - \xi, \frac{t}{2}\right) \le 1 - \varepsilon \text{ or } \nu\left(x_{kl} - \xi, \frac{t}{2}\right) \ge \varepsilon\right\}$ be an arbitrary element.

Let us denote

$$B(\varepsilon, t) := \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - x_{mn}, \frac{t}{2}\right) \le 1 - \varepsilon \text{ or } \nu\left(x_{kl} - x_{mn}, \frac{t}{2}\right) \ge \varepsilon\right\}.$$

It is sufficient to show that

$$B(\varepsilon, t) \subset \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - x_{mn}, \frac{t}{2}\right) \le 1 - \varepsilon \text{ or } \nu\left(x_{kl} - x_{mn}, \frac{t}{2}\right) \ge \varepsilon\right\}.$$

Let $(k,l) \in B(\varepsilon, t) - \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \mu\left(x_{kl} - x_{mn}, \frac{t}{2}\right) \leq 1 - \varepsilon \text{ or } \nu\left(x_{kl} - x_{mn}, \frac{t}{2}\right) \geq \varepsilon\right\}$.

Then, we have $\mu(x_{kl} - x_{mn}, t) \leq 1 - \varepsilon$ and $\mu\left(x_{kl} - \xi, \frac{t}{2}\right) \leq 1 - \varepsilon$, in particular $\mu\left(x_{mn} - \xi, \frac{t}{2}\right) > 1 - \varepsilon$. Hence,

$$1 - s \geq \mu(x_{kl} - x_{mn}, t) \geq \mu\left(x_{kl} - \xi, \tfrac{t}{2}\right) * \mu\left(x_{mn} - \xi, \tfrac{t}{2}\right)$$

$$> (1 - \varepsilon) * (1 - \varepsilon) > 1 - s,$$

which is impossible. On the other hand, $\nu(x_{kl} - x_{mn}, t) \geq s$ and $\nu\left(x_{kl} - \xi, \frac{t}{2}\right) < \varepsilon$, in particular $\nu\left(x_{mn} - \xi, \frac{t}{2}\right) < \varepsilon$. Then,

$$s \geq \nu(x_{kl} - x_{mn}, t) \leq \nu\left(x_{kl} - \xi, \frac{t}{2}\right) * \nu\left(x_{mn} - \xi, \frac{t}{2}\right) < \varepsilon \diamond \varepsilon < s,$$

which is impossible. This proves our claim. □

2.2. **Comparison of $D_\beta^\gamma(\mu,\nu)$ and $D_\beta^\gamma S(\mu,\nu)$.** In this subsection we deal with the relation between $D_\beta^\gamma(\mu,\nu)$ and $D_\beta^\gamma S(\mu,\nu)$ in an intuitionistic fuzzy normed space.

Theorem 2.12. *Let $(X, \mu, \nu, *, \diamond)$ be an IFNLS, and $x = (x_{kl})$ be a double sequence in X. Then, $x_{kl} \to \xi\left(D_\beta^\gamma(\mu,\nu)\right)$, implies $x_{kl} \to \xi\left(D_\beta^\gamma S(\mu,\nu)\right)$.*

Proof. Assume $x_{kl} \to \xi\left(D_\beta^\gamma(\mu,\nu)\right)$. That is, for any $\varepsilon > 0$ and $t > 0$, there exist $n_0, m_0 \in \mathbb{N}$ such that

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \widetilde{\mu}(x_{kl} - \xi, t) > 1 - \varepsilon \text{ and } \frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \nu(x_{kl} - \xi, t) < \varepsilon$$

hold for all $n > n_0$, $m > m_0$. From the simple calculation we have following facts:

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \widetilde{\mu}(x_{kl} - \xi, t)$$

$$\geq \frac{|\{p(n)+1 \leq k \leq q(n), t(m)+1 \leq l \leq r(m) : \widetilde{\mu}(x_{kl} - \xi, t)\}|}{\beta(n)\gamma(m)} =: A_\beta^\gamma(n,m)$$

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1 \\ l=t(m)+1}}^{q(n),r(m)} \nu(x_{kl} - \xi, t)$$

$$\geq \frac{|\{p(n)+1 \leq k \leq q(n), t(m)+1 \leq l \leq r(m) : \widetilde{\mu}(x_{kl} - \xi, t)\}|}{\beta(n)\gamma(m)} =: B_\beta^\gamma(n,m).$$

As a consequence of $x_{kl} \to \xi\left(D_\beta^\gamma(\mu,\nu)\right)$ and the above inequalities, we have $\delta_\beta^\gamma\left(A_\beta^\gamma(n,m)\right) = 0$ and $\delta_\beta^\gamma\left(B_\beta^\gamma(n,m)\right) = 0$. Therefore,

$$\delta_\beta^\gamma(\{(n,m) : \widetilde{\mu}(x_{kl} - \xi, t) \geq \varepsilon \text{ or } \nu(x_{kl} - \xi, t) \geq \varepsilon\}) = 0.$$

It means that $x_{kl} \to \xi\left(D_\beta^\gamma S(\mu,\nu)\right)$. □

Remark 2.13. Generally, the inverse of last theorem is not true. For example, let $q(n)$ and $r(m)$ be monotone increasing sequences of positive integers and h_1, h_2 be fixed numbers. Let us define a double sequence for $n, m = 1, 2, ...$

$$x_{kl} = \begin{cases} k^2 l^2, & \begin{cases} \left\|\sqrt{q(n)}\right\| - h_1 < k \leq \left\|\sqrt{q(n)}\right\| \\ \left\|\sqrt{r(m)}\right\| - h_2 < l \leq \left\|\sqrt{r(m)}\right\| \end{cases} \\ 0, & \text{otherwise.} \end{cases}.$$

If we consider D_β^γ for the sequence $p(n)$, $t(m)$ satisfying

$$0 < p(n) \leq \left\|\sqrt{q(n)}\right\| - h_1, \; 0 < t(m) \leq \left\|\sqrt{r(m)}\right\| - h_2$$

then we have

$$\frac{|\{p(n)+1\leq k\leq q(n), t(m)+1\leq l\leq r(m): \mu_0(x_{kl},t)\geq 1-\varepsilon \text{ or } \nu_0(x_{kl},t)\geq\varepsilon\}|}{\beta(n)\gamma(m)}$$

$$= \frac{\left|\left\{p(n)+1\leq k\leq q(n), t(m)+1\leq l\leq r(m): |x_{kl}|\geq \frac{\varepsilon t}{1-\varepsilon}\right\}\right|}{\beta(n)\gamma(m)} = \frac{h_1 h_2}{\beta(n)\gamma(m)},$$

which implies that $x_{kl} \to \xi\left(D_\beta^\gamma S\,(\mu,\nu)\right)$. Also, we have the following inequality:

$$\lim_{n,m\to\infty} \frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} \nu(x_{kl},t) \geq h_1 h_2.$$

From here, since $h_1 \neq 0$ and $h_2 \neq 0$, $x_{kl} \nrightarrow \xi\left(D_\beta^\gamma(\mu,\nu)\right)$.

Theorem 2.14. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS, and $x=(x_{kl})$ be a bounded sequence in X. Then, the convergence $x_{kl} \to \xi\left(D_\beta^\gamma S\,(\mu,\nu)\right)$ implies that $x_{kl} \to \xi\left(D_\beta^\gamma(\mu,\nu)\right)$.*

Proof. Suppose that (x_{kl}) is a bounded sequence and $x_{kl} \to \xi\left(D_\beta^\gamma S\,(\mu,\nu)\right)$. Under the assumption on (x_{kl}) there exists a positive real number M, such that $\mu(x_{kl}-\xi,t) > 1-M$ and $\nu(x_{kl}-\xi,t) < M$ hold for all k,l.

Therefore, the following inequality

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} \widetilde{\mu}(x_{kl}-\xi,t)$$

$$= \frac{1}{\beta(n)\gamma(m)}\left\{\left(\sum_{\substack{k=p(n)+1,l=t(m)+1\\ \widetilde{\mu}(x_{kl}-\xi,t)<\varepsilon}}^{q(n),r(m)} + \sum_{\substack{k=p(n)+1,l=t(m)+1\\ \widetilde{\mu}(x_{kl}-\xi,t)\geq\varepsilon}}^{q(n),r(m)}\right)\widetilde{\mu}(x_{kl}-\xi,t)\right\}$$

$$< \varepsilon\frac{1}{\beta(n)\gamma(m)}\left|\{p(n)+1\leq k\leq q(n), t(m)+1\leq l\leq r(m): \widetilde{\mu}(x_{kl}-\xi,t)<\varepsilon\}\right|$$

$$+M.\frac{1}{\beta(n)\gamma(m)}\left|\{p(n)+1\leq k\leq q(n), t(m)+1\leq l\leq r(m): \widetilde{\mu}(x_{kl}-\xi,t)\geq\varepsilon\}\right|$$

holds. If we take limit by considering $x_{kl} \to \xi\left(D_\beta^\gamma S\,(\mu,\nu)\right)$, then it is obtained that

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} \widetilde{\mu}(x_{kl}-\xi,t) > \varepsilon.$$

This gives

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} \widetilde{\mu}(x_{kl}-\xi,t) < 1-\varepsilon. \tag{2.5}$$

Also, the following inequality

$$\frac{1}{\beta(n)\gamma(m)} \sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} \nu(x_{kl}-\xi,t)$$

$$= \frac{1}{\beta(n)\gamma(m)}\left\{\left(\sum_{\substack{k=p(n)+1,l=t(m)+1\\ \nu(x_{kl}-\xi,t)<\varepsilon}}^{q(n),r(m)} + \sum_{\substack{k=p(n)+1,l=t(m)+1\\ \nu(x_{kl}-\xi,t)\geq\varepsilon}}^{q(n),r(m)}\right)\nu(x_{kl}-\xi,t)\right\}$$

$$< \varepsilon\frac{1}{\beta(n)\gamma(m)}\left|\{p(n)+1\leq k\leq q(n), t(m)+1\leq l\leq r(m): \nu(x_{kl}-\xi,t)<\varepsilon\}\right|$$

$$+M.\frac{1}{\beta(n)\gamma(m)}\left|\{p(n)+1\leq k\leq q(n), t(m)+1\leq l\leq r(m): \nu(x_{kl}-\xi,t)\geq\varepsilon\}\right|$$

holds. This gives that

$$\frac{1}{\beta(n)\gamma(m)}\sum_{\substack{k=p(n)+1\\ l=t(m)+1}}^{q(n),r(m)} \nu(x_{kl}-\xi,t) < \varepsilon. \tag{2.6}$$

So, by considering (2.5) and (2.6), the proof of the theorem is completed. □

Theorem 2.15. *Let $(X,\mu,\nu,*,\diamond)$ be an IFNLS, $x=(x_{kl})$ be a bounded sequence in X, and the sequences $\left\{\frac{p(n)}{\beta(n)}\right\}$, $\left\{\frac{t(m)}{\gamma(m)}\right\}$ be bounded. Then, $x_{kl}\to\xi(S(\mu,\nu))$ implies $x_{kl}\to\xi\left(D_\beta^\gamma S(\mu,\nu)\right)$.*

Proof. Since $\lim_{n\to\infty}\beta(n)=\infty$ and $x_{kl}\to\xi(S(\mu,\nu))$, then

$$\lim_{n\to\infty}\frac{1}{\beta(n)}\left|\{k\le\beta(n):\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}\right|=0$$

holds. Also, the following inclusion

$$\{p(n)+1\le k\le q(n):\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}$$

$$\subset\{k\le q(n):\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}$$

and the inequality

$$\left|\{p(n)+1\le k\le q(n):\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}\right|$$

$$\le\left|\{k\le q(n):\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}\right|$$

hold. So, we get

$$\frac{1}{\beta(n)}\left|\{p(n)+1\le k\le q(n):\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}\right|$$

$$\le L(n)\,.\frac{1}{q(n)}\left|\{k\le q(n):\mu(x_{kl}-\xi,t)\le 1-\varepsilon \text{ or } \nu(x_{kl}-\xi,t)\ge\varepsilon\}\right|,$$

where $L(n)\,.\frac{\beta(n)}{q(n)}$. By taking limit when $n\to\infty$, we get $x_{kl}\to\xi\left(D_\beta^\gamma S(\mu,\nu)\right)$. □

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