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A Truncated Bell Series Approach to Solve Systems of Generalized Delay Differential Equations with Variable Coefficients

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Abstract. In this study, a matrix method based on collocation points and Bell polynomials are improved to obtain the approximate solutions of systems of high-order generalized delay differential equations with variable coefficients. The presented technique reduces the solution of the mentioned delay system under the initial conditions to the solution of a matrix equation with the unknown Bell coefficients. Thereby, the approximate solution is obtained in terms of Bell polynomials. In addition, some examples along with residual error analysis are performed to illustrate the efficiency of the method; the obtained results are scrutinized and interpreted.

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Keywords: Bell polynomials and series, collocation points and matrix method, system of delay differential equations.

1. Introduction

The systems of differential, difference, differential-difference and delay differential equations and their solutions play an important role in explaining many different phenomena and particularly, arise in industrial applications and in studies based on biology, economy, electro dynamics, physics and chemistry. Since these type systems are usually difficult to solve analytically, a numerical method is needed. In recent years for solving these equation,numerical methods have been developed.For example, Adomian decomposition method [14], Differential transformation method [1], Haar functions method [10], homotopy analysis method [18], via Laplace Transformation [15], Taylor collocation method [8], Chelyshkov collocation method [12].
In this study, we introduce a novel collocation method based on Bell polynomials for solving the system of linear delay differential equations in the form

$$\sum_{k=0}^{m}\sum_{j=1}^{J} P_{ij}^{k}(x)y_j^{(k)}(\alpha_{jk}x+\beta_{jk}) = g_i(x),\ \ i=1,2,\dots,J,\ \ 0\le a\le x\le b \tag{1.1}$$

under the mixed conditions

$$y_j^{(k)}(a)=\lambda_{jk};\ \ j=1,2,...,J,\ \ k=0,1,...,m-1. \tag{1.2}$$

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where $y_j^{(0)}(x) = y_j(x),\ j = 1, 2, ..., J$ are unknown functions; $P_{ij}^k(x)$ and $g_i(x)$are continuous functions on $[a, b]$ and $\lambda_{jk},\ \alpha_{jk}$ *and* β_{jk} is real constant coefficients.
Our aim is to obtain an approximate solution of (1.1) in the following Bell polynomial form

$$y_j(x) \cong y_{j,N}(x) = \sum_{n=0}^{N} a_{jn} B_n(x) \tag{1.3}$$

where a_{Jn} , $n = 0, 1, \cdots N$ are unknown Bell coefficients and $B_n(x)$,$n = 0, 1, \cdots,\ N$ are Bell polynomial defined by

$$B_n(x) = \sum_{k=0}^{n} S(n, k)\ x^k \tag{1.4}$$

where

$$S(n, k) = \sum_{j=0}^{k} \frac{(-1)^{k-j}}{k!} \binom{k}{j} j^n$$

is stirling numbers of the second kind [2–4, 16].

2. Fundamental Matrix Relations

In this section, we convert the equations (1.1)-(1.3) to the matrix forms.Firstly we will convert Bell polynomials defined in Eq. (1.4) into matrix form

$$\mathbf{B}(\mathrm{x}) = \mathbf{X}(\mathrm{x})\,\mathbf{S} \tag{2.1}$$

where

$$\mathbf{B}(\mathrm{x}) = [\mathrm{B}_0(\mathrm{x})\ \ \mathrm{B}_1(\mathrm{x}) \ldots \mathrm{B}_\mathrm{N}(\mathrm{x})] \quad , \mathbf{X}(\mathrm{x}) = \left[1\ \ \mathrm{x}\ \ \mathrm{x}^2 \ldots \mathrm{x}^\mathrm{N}\right]$$

and

$$\mathbf{S}= \begin{bmatrix} \mathrm{S}(0,0) & \mathrm{S}(1,0) & \mathrm{S}(2,0) & \cdots & \mathrm{S}(\mathrm{N},0) \\ 0 & \mathrm{S}(1,1) & \mathrm{S}(2,1) & \cdots & \mathrm{S}(\mathrm{N},1) \\ 0 & 0 & \mathrm{S}(2,2) & \cdots & \mathrm{S}(\mathrm{N},2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{S}(\mathrm{N},\mathrm{N}) \end{bmatrix}.$$

Also, the approximate solutions $y_j(x)$ in (1.3) can be expressed as

$$\boldsymbol{y_j}(\boldsymbol{x}) = \mathbf{B}(\boldsymbol{x})\,\mathbf{A_j}\ ;\ \mathrm{j} = 1, 2, ..., \mathrm{J} \tag{2.2}$$

where

$$\mathbf{A_j} = \begin{bmatrix} a_{j0} & a_{j1} & \ldots & a_{jN} \end{bmatrix}^T.$$

By using (2.1) and (2.2), we obtain the relation

$$y_j(x) = \mathbf{X}(\mathrm{x})\,\mathbf{S}\mathbf{A_j}.$$

On the other hand, it is cleary seen [17] that the relation between the matrix $\mathbf{X}(\mathrm{x})$ and its kth derivative $\mathbf{X}^{(k)}(\mathrm{x})$ is

$$\mathbf{X}^{(k)}(x) = \mathbf{X}(x)\,\mathbf{M}^k \tag{2.3}$$

where

$$\mathbf{M}= \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},\ \mathbf{M}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Thus, from the relations (2.2) and (2.3), we obtain the matrix relations

$$y_j^{(k)}(x) = \mathbf{X}(x)\,\mathbf{M}^k\mathbf{S}\mathbf{A_j}\ ; \mathrm{j} = 1, 2, ..., \mathrm{J}. \tag{2.4}$$

Similarly, if we put $x \to \alpha_{jk}x + \beta_{jk}$ into (2.4), we obtain the matrix relation

$$y_j^{(k)}(\alpha_{jk}x + \beta_{jk}) = \mathbf{X}\left(\alpha_{jk}x + \beta_{jk}\right)\mathbf{M}^k\mathbf{S}\mathbf{A_j}\mathbf{=}\mathbf{X}\,(\mathrm{x})\left(\alpha_{jk}, \beta_{jk}\right)\mathbf{M}^k\mathbf{SA}. \tag{2.5}$$

If $\alpha_{jk} \neq 0$ $and \beta_{jk} \neq 0$, [6]

$$\boldsymbol{\mu}\left(\alpha_{jk}, \beta_{jk}\right) = \begin{bmatrix} \binom{0}{0}(\alpha_{\mathrm{jk}})^0(\beta_{\mathrm{jk}})^0 & \binom{1}{0}(\alpha_{\mathrm{jk}})^0(\beta_{\mathrm{jk}})^1 & \binom{2}{0}(\alpha_{\mathrm{jk}})^0(\beta_{\mathrm{jk}})^2 & \cdots & \binom{\mathrm{N}}{0}(\alpha_{\mathrm{jk}})^0(\beta_{\mathrm{jk}})^{\mathrm{N}} \\ 0 & \binom{1}{1}(\alpha_{\mathrm{jk}})^1(\beta_{\mathrm{jk}})^0 & \binom{2}{1}(\alpha_{\mathrm{jk}})^1(\beta_{\mathrm{jk}})^1 & \cdots & \binom{\mathrm{N}}{1}(\alpha_{\mathrm{jk}})^1(\beta_{\mathrm{jk}})^{\mathrm{N}-1} \\ 0 & 0 & \binom{2}{2}(\alpha_{\mathrm{jk}})^2(\beta_{\mathrm{jk}})^0 & \cdots & \binom{\mathrm{N}}{2}(\alpha_{\mathrm{jk}})^2(\beta_{\mathrm{jk}})^{\mathrm{N}-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{\mathrm{N}}{\mathrm{N}}(\alpha_{\mathrm{jk}})^{\mathrm{N}}(\beta_{\mathrm{jk}})^0 \end{bmatrix}.$$

3. Bell Matrix Collocation Method

Firstly, the system (1.1) by using (2.4) and (2.5) for i,j =1,2,. . . ,J can be written in the following matrix form

$$\sum_{k=0}^{m} \mathbf{P}_k\mathbf{Y}^{(k)}(\alpha_k x + \beta_k) = \mathbf{G}(\mathrm{x}) \tag{3.1}$$

where

$$\mathbf{Y}^{(k)}(\alpha_k x + \beta_k) = \begin{bmatrix} y_1^{(k)}(\alpha_{1k}x + \beta_{1k}) \\ y_2^{(k)}(\alpha_{2k}x + \beta_{2k}) \\ \vdots \\ y_{jk}^{(k)}(\alpha_{jk}x + \beta_{jk}) \end{bmatrix} = \begin{bmatrix} \mathbf{X}\,(\mathrm{x})\,\boldsymbol{\mu}\,(\alpha_{1\mathrm{k}}, \beta_{1\mathrm{k}})\,\mathbf{M}^{\mathrm{k}}\mathbf{SA}_1 \\ \mathbf{X}\,(\mathrm{x})\,\boldsymbol{\mu}\,(\alpha_{2\mathrm{k}}, \beta_{2\mathrm{k}})\,\mathbf{M}^{\mathrm{k}}\mathbf{SA}_2 \\ \vdots \\ \mathbf{X}\,(\mathrm{x})\,\boldsymbol{\mu}\left(\alpha_{\mathrm{jk}}, \beta_{\mathrm{jk}}\right)\mathbf{M}^{\mathrm{k}}\mathbf{SA}_{\mathrm{J}} \end{bmatrix} = \overline{\mathbf{X}}\,(x)\,\overline{\boldsymbol{\mu}}\,(\alpha_k, \beta_k)\,(\overline{\mathbf{M}})^k\overline{\mathbf{S}}\mathbf{A},$$

$$\overline{\boldsymbol{\mu}}\,(\alpha_k, \beta_k) = \begin{bmatrix} \boldsymbol{\mu}\,(\alpha_k, \beta_k) & 0 & \cdots & 0 \\ 0 & \boldsymbol{\mu}\,(\alpha_k, \beta_k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \boldsymbol{\mu}\,(\alpha_k, \beta_k) \end{bmatrix}, \ \overline{\mathbf{M}}^k = \begin{bmatrix} \mathbf{M}^k & 0 & \cdots & 0 \\ 0 & \mathbf{M}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{M}^k \end{bmatrix},$$

$$\overline{\mathbf{X}}\,(x) = \begin{bmatrix} X\,(x) & 0 & \cdots & 0 \\ 0 & X\,(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X\,(x) \end{bmatrix}, \ \overline{\mathbf{S}} = \begin{bmatrix} \mathbf{S} & 0 & \cdots & 0 \\ 0 & \mathbf{S} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{S} \end{bmatrix},$$

$$P_k(x) = \begin{bmatrix} P_{11}^k(x) & P_{12}^k(x) & \ldots & P_{1J}^k(x) \\ P_{21}^k(x) & P_{22}^k(x) & \ldots & P_{2J}^k(x) \\ \vdots & \vdots & \ddots & \vdots \\ P_{J1}^k(x) & P_{J2}^k(x) & \ldots & P_{JJ}^k(x) \end{bmatrix}, \ G(x) = \begin{bmatrix} g_1(x) \\ g_2(x) \\ \vdots \\ g_J(x) \end{bmatrix} A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_J \end{bmatrix}.$$

The collocation points x_t are defined by [11]

$$x_t = a + \frac{b-a}{N}t\ ,\quad t = 0, 1, ..., N. \tag{3.2}$$

and by using the points (3.2), it is obtained the system of the matrix equations

$$\sum_{k=0}^{m} \mathbf{P}_k(x_t)\overline{\mathbf{X}}\,(x_t)\,\overline{\boldsymbol{\mu}}^*\,(\alpha_k, \beta_k)\,(\overline{\mathbf{M}})^k\overline{\mathbf{S}}\mathbf{A}\mathbf{=}\mathbf{G}(x_t) \tag{3.3}$$

where

$$\overline{\mathbf{P}_k}=\begin{bmatrix}\mathbf{P}_k(x_0) & 0 & \cdots & 0\\ 0 & \mathbf{P}_k(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{P}_k(x_N)\end{bmatrix},\ \overline{\boldsymbol{\mu}}^*(\alpha_k,\beta_k)=\begin{bmatrix}\overline{\boldsymbol{\mu}}(\alpha_k,\beta_k)\\ \overline{\boldsymbol{\mu}}(\alpha_k,\beta_k)\\ \vdots\\ \overline{\boldsymbol{\mu}}(\alpha_k,\beta_k)\end{bmatrix},$$

$$\overline{\mathbf{X}}=\begin{bmatrix}\overline{\mathbf{X}}(x_0) & 0 & \cdots & 0\\ 0 & \overline{\mathbf{X}}(x_1) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \overline{\mathbf{X}}(x_N)\end{bmatrix},\ \mathbf{G}=\begin{bmatrix}\mathbf{G}(x_0)\\ \mathbf{G}(x_1)\\ \vdots\\ \mathbf{G}(x_N)\end{bmatrix}.$$

The fundamental matrix Eq. (3.3) for (1.1) corresponds to a system of $k(N+1)$ algebraic equation for the $k(N+1)$ unknown Bell coefficients

$$\mathbf{WA}=\mathbf{G}\ or\ \ [\mathbf{W};\mathbf{G}] \tag{3.4}$$

where

$$\mathbf{W}=\left\{\ \sum_{k=0}^{m}\mathbf{P}_k(x_t)\overline{\mathbf{X}}(x_t)\overline{\boldsymbol{\mu}}^*(\alpha_k,\beta_k)(\overline{\mathbf{M}})^k\overline{\mathbf{S}}\right\}.$$

By using the relations (2.4), we get the matrix form of the conditions (1.2) for$\mathrm{j}=1,2,...,\mathrm{J},\quad \mathrm{k}=0,1,...,\mathrm{m}-1$ as follows:

$$\begin{bmatrix}y_1^{(k)}(a)\\ y_2^{(k)}(a)\\ \vdots\\ y_{jk}^{(k)}(a)\end{bmatrix}=\begin{bmatrix}\mathbf{X}(\mathrm{a})\mathbf{M}^{\mathrm{k}}\mathbf{S} & 0 & \cdots & 0\\ 0 & \mathbf{X}(\mathrm{a})\mathbf{M}^{\mathrm{k}}\mathbf{S} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{X}(\mathrm{a})\mathbf{M}^{\mathrm{k}}\mathbf{S}\end{bmatrix}\begin{bmatrix}\mathbf{A}_1\\ \mathbf{A}_2\\ \vdots\\ \mathbf{A}_{\mathrm{J}}\end{bmatrix}=\begin{bmatrix}\lambda_{1k}\\ \lambda_{2k}\\ \vdots\\ \lambda_{Jk}\end{bmatrix}$$

or briefly

$$\mathbf{U}_{\mathrm{k}}\mathbf{A}=\lambda_k\ \ or\ \ [\mathbf{U};\lambda_k]\ ,\ \ j=0,\ 1\ ,\ \ldots\ ,\ m-1. \tag{3.5}$$

Therefore, the rows of the matrix (3.5) are replaced by last rows of the matrix (3.4), we obtain the new augmented matrix

$$\widetilde{\mathbf{W}}\mathbf{A}=\widetilde{\mathbf{G}}\ \ or\ \left[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}\right]. \tag{3.6}$$

If $rank(\widetilde{\mathbf{W}})=rank\left[\widetilde{\mathbf{W}}\ ;\ \widetilde{\mathbf{G}}\right]=k(N+1)$,then we can write

$$\mathbf{A}=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}\,.$$

Thus the matrix **A** is uniquely determined and the Eq. (1.1) under the coefficient equation (1.2) has unique solution.This solution is given by truncated Bell series

$$y_j(x)\ \cong y_{j,N}(x)=\sum_{n=0}^{N}a_{jn}B_n(x)\,.$$

4. Residual Error Analysis

We can easily check the accuracy of the obtained solutions as follows. Since the truncated Bell series (1.3) is approximate solution of the system (1.1), using the residual correction method [5, 7, 9, 13].
Firstly, the residual function of the method can be defined as

$$R_{iN}(x)=L[y_{iN}(x)]-g_i(x)\ i=1,2,...,k \tag{4.1}$$

where $L\left[y_{iN}(x)\right]\cong g_i(x)$and $y_{iN}(x)$, $\mathrm{i}=0,1,2,...,\mathrm{k}$ are the Bell polynomial solutions (1.3) of the problems (1.1) - (1.2). Then $y_{jN}(x)$correspond the problem

$$\left\{\begin{array}{l}\sum_{k=0}^{m}\sum_{j=1}^{J}P_{ij}^k(x)y_j^{(k)}(\alpha_{jk}x+\beta_{jk})=g_i(x)+R_{iN}(x),\ \mathrm{i}=1,2,...,\mathrm{k}\\ y_j^{(k)}(a)=\lambda_{jk},\ \mathrm{j}=1,2,...,\mathrm{J},\ \mathrm{k}=0,1,...,\mathrm{m}-1\end{array}\right\}.$$

Furhermore ,the exact solution $y_j(x)$ and the approximate solution $y_{jN}(x)$ are called, the error function $e_{jN}(x)$ is calculated by the following form

$$e_{jN}(x) = y_j(x) - y_{jN}(x). \tag{4.2}$$

From Eqs. (1.1), (1.2), (4.1) and (4.2), we obtain the system of the error differential equations

$$L[e_{iN}(x)] = L[y_i(x)] - L[y_{iN}(x)] = -R_{iN}(x)$$

and the error problem

$$\left\{ \begin{array}{c} \sum_{k=0}^{m}\sum_{j=1}^{J} P_{ij}^{k}(x) {e_{jN}}^{(k)}(\alpha_{jk}x+\beta_{jk}) = -R_{iN}(x) \quad i=1,2,...,k \\ {e_{jN}}^{(k)}(a)=0\ ;\ j=1,2,...,J \ and\ k=0,1,...,m-1 \end{array} \right\}.$$

If $e_{jN}(x) \to 0$ when N is sufficiently large enough,then the error decreases.

5. Numerical Examples

Example 1: First, we consider the system of linear delay-differential equations

$$\left\{ \begin{array}{c} y_1^{(2)} + xy_1(x-1) + xy_2(x) = -2 + 2x^2 - x^3 \\ y_2^{(2)} + 2xy_2(x-1) + 2xy_1(x) = 4x^2 - 2x^3 \end{array} \right.$$

and the initial conditions $y_1(0) = 0$, $y_2(0) = 1$, $y_1'(0) = 1$ and $y_2'(0) = 1$with the exact solutions are $y_1(x) = x - x^2$ and $y_2(x) = x + 1$. For $N = 2$, the approximate solutions $y_i(x)$ by the truncated Bell series

$$y_{j,2}(x) = \sum_{n=0}^{2} a_{jn} B_n(x)\ ,\ j = 1,2$$

where $k = 2, J = 2, g_1(x) = -2 + 2x^2 - x^3, g_2(x) 4x^2 - 2x^3, P_{11}^0 = x, P_{12}^0 = x, P_{11}^2 = 1, P_{21}^0 = 2x, P_{22}^0 = 2x,$
$P_{22}^2 = 1, \alpha_{10} = 1, \beta_{10} = -1\ and\ \ \alpha_{20} = 1, \beta_{20} = -1.$
By using (3.2) the collocation points for $N = 2$ is calculated as

$$\{x_0 = 0, 1/2, x_1 = 1\}$$

and from the (3.3) fundamental matrix equation is

$$\left\{\mathbf{P}_0\overline{\mathbf{X}}\overline{\boldsymbol{\mu}}^*(\alpha_k,\beta_k)(\overline{\mathbf{M}})^{\mathbf{0}}\overline{\mathbf{S}}+\mathbf{P}_2\overline{\mathbf{X}}(\overline{\mathbf{M}})^{\mathbf{2}}\overline{\mathbf{S}}\right\}\mathbf{A}=\mathbf{G}$$

where

$$\mathbf{P}_0(x) = \begin{bmatrix} x & x \\ 2x & 2x \end{bmatrix},\ \mathbf{P}_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_0(0) & 0 & 0 \\ 0 & \mathbf{P}_0(1/2) & 0 \\ 0 & 0 & \mathbf{P}_0(1) \end{bmatrix},\ \mathbf{P}_1 = \begin{bmatrix} \mathbf{P}_1(0) & 0 & 0 \\ 0 & \mathbf{P}_1(1/2) & 0 \\ 0 & 0 & \mathbf{P}_1(1) \end{bmatrix},$$

$$\overline{\mathbf{X}}(x) = \begin{bmatrix} \mathbf{X}(x) & 0 \\ 0 & \mathbf{X}(x) \end{bmatrix},\ \overline{\mathbf{X}} = \begin{bmatrix} \overline{\mathbf{X}}(0) & 0 & 0 \\ 0 & \overline{\mathbf{X}}(1/2) & 0 \\ 0 & 0 & \overline{\mathbf{X}}(1) \end{bmatrix},$$

$$\overline{\mathbf{M}} = \begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix},\ \mathbf{M}= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix},\ \overline{\mathbf{S}} = \begin{bmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{S} \end{bmatrix},\ \mathbf{S}= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\overline{\boldsymbol{\mu}}^*(\alpha_k,\beta_k) = \begin{bmatrix} \overline{\boldsymbol{\mu}}(\alpha_k,\beta_k) \\ \overline{\boldsymbol{\mu}}(\alpha_k,\beta_k) \\ \overline{\boldsymbol{\mu}}(\alpha_k,\beta_k) \end{bmatrix},\ \overline{\boldsymbol{\mu}}(\alpha_k,\beta_k) = \begin{bmatrix} \overline{\boldsymbol{\mu}}(\alpha_{1k},\beta_{1k}) & 0 \\ 0 & \overline{\boldsymbol{\mu}}(\alpha_{2k},\beta_{2k}) \end{bmatrix},$$

$$\overline{\boldsymbol{\mu}}(\alpha_{1k},\beta_{1k}) = \overline{\boldsymbol{\mu}}(\alpha_{2k},\beta_{2k}) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{G}=\begin{bmatrix} \boldsymbol{g}(0) \\ \boldsymbol{g}(1/2) \\ \boldsymbol{g}(1) \end{bmatrix},\ \boldsymbol{g}(0)=\begin{bmatrix} -2 \\ 0 \end{bmatrix},\ \boldsymbol{g}(1/2)=\begin{bmatrix} -13/8 \\ 3/4 \end{bmatrix},\ \boldsymbol{g}(1)=\begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\boldsymbol{A}=\begin{bmatrix} \boldsymbol{A_1} \\ \boldsymbol{A_2} \end{bmatrix},\ \boldsymbol{A_1}=\begin{bmatrix} a_{10} & a_{11} \end{bmatrix}^{\boldsymbol{T}},\ \boldsymbol{A_2}=\begin{bmatrix} a_{20} & a_{21} \end{bmatrix}^{\boldsymbol{T}}.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$[\mathbf{W}\ ;\ \mathbf{G}]=\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & ; & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 1/2 & -1/4 & 15/8 & 1/2 & -1/4 & -1/8 & ; & -13/8 \\ 1 & -1/2 & -1/4 & 1 & -1/2 & 7/4 & ; & 3/4 \\ 1 & 0 & 2 & 1 & 0 & 0 & ; & 0 \\ 2 & 0 & 0 & 2 & 0 & 2 & ; & 2 \end{bmatrix}.$$

From Eq. (3.5), the matrix form for initial conditions is computated as

$$[\mathbf{U};\lambda]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & ; & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & ; & 1 \end{bmatrix}.$$

Hence,the new augmented matrix based on conditions from system (4.1) can be obtained as follows

$$\left[\tilde{\mathbf{W}}\ ;\ \tilde{\mathbf{G}}\right]=\begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & ; & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & ; & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & ; & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & ; & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & ; & 1 \end{bmatrix}.$$

By solving this system, substituting the resulting unknown Bell coefficients matrix into Eq. (3.4) we obtain the exact solutions for $N=2$ as $y_1(x)=x-x^2$ and $y_2(x)=x+1$.

Example 2: Let us consider the system of linear delay-differential equations

$$\begin{cases} y_1^{(2)}+xe^{x-1}y_1'(x-1)+y_2(x)=e^x+e^{-x}-x \\ y_2^{(2)}+e^{-1-x}y_2'(x+1)+y_1(x)=e^x+e^{-x}+1 \end{cases}$$

and the initial conditions $y_1(0)=1$, $y_2(0)=1$, $y_1'(0)=-1$ and $y_2'(0)=1$ with the exact solutions are $y_1(x)=e^{-x}$, $y_2(x)=e^x$. From the (3.3) fundamental matrix equation is

$$\left\{\mathbf{P}_0\overline{\mathbf{X}}(\overline{\mathbf{M}})^{\mathbf{0}}\overline{\mathbf{S}}+\mathbf{P}_1\overline{\mathbf{X}}\overline{\boldsymbol{\mu}}^*(\alpha_k,\beta_k)(\overline{\mathbf{M}})^{\mathbf{1}}\overline{\mathbf{S}}+\mathbf{P}_2\overline{\mathbf{X}}(\overline{\mathbf{M}})^{\mathbf{2}}\overline{\mathbf{S}}\right\}\mathbf{A}\mathbf{=}\mathbf{G}.$$

Therefore, necessary operations are calculated, we obtain the approximate solution by the Bell polynomials of the problem for $i=1,2$ and $N=4,5$ *and* 6 respectively,

$$y_{1,4}(x)=1-x+0.5000x^2-0.1671x^3+0.0353x^4,$$
$$y_{2,4}(x)=1+x+0.4986x^2+0.1642x^3+0.0590x^4,$$

$$y_{1,5}(x)=1-x+0.5000x^2-0.1672x^3+0.0419x^4-0.0069x^5,$$
$$y_{2,5}(x)=1+x+0.4995x^2+0.1669x^3+0.0403x^4+0.0120x^5$$

and

$$y_{1,6}(x)=1-x+0.5000x^2-0.1668x^3+0.0419x^4-0.0083x^5+0.0011x^6,$$
$$y_{2,6}(x)=1+x+0.4999x^2+0.1665x^3+0.0419x^4+0.0080x^5+0.0020x^6.$$

Table 1. Comparison of the absolute errors of $y_1(x)$ for N= 4, 5,6.

x_i	$y(x) = e^{-x_i}$	$\lvert e_4(x_i)\rvert$	$\lvert e_5(x_i)\rvert$	$\lvert e_6(x_i)\rvert$
0	1	0	0	0
0.2	0.8187	1.1073e-05	3.5211e-06	6.9868e-07
0.4	0.6703	1.1077e-04	1.8862e-05	3.0924e-06
0.6	0.5488	3.3036e-04	3.3140e-05	4.2825e-06
0.8	0.4493	4.2528e-04	3.4116e-05	2.9028e-07
1	0.3679	3.2056e-04	7.9441e-05	2.0559e-05

Table 2. Comparison of the absolute errors of $y_2(x)$ for N= 4, 5,6.

x_i	$y(x) = e^{x_i}$	$\lvert e_4(x_i)\rvert$	$\lvert e_5(x_i)\rvert$	$\lvert e_6(x_i)\rvert$
0	1	0	0	0
0.2	1.2214	5.0758e-05	1.9238e-05	5.0302e-06
0.4	1.4918	2.9498e-05	6.8538e-05	2.1946e-05
0.6	1.8221	4.9080e-04	9.2400e-05	4.5168e-05
0.8	2.2255	0.0018	3.0912e-05	4.8960e-05
1	2.7183	0.0035	4.1817e-04	1.8172e-05

Figure 1. Numerical and Exact Solutions of $y_1(x)$ for N = 4,5,6

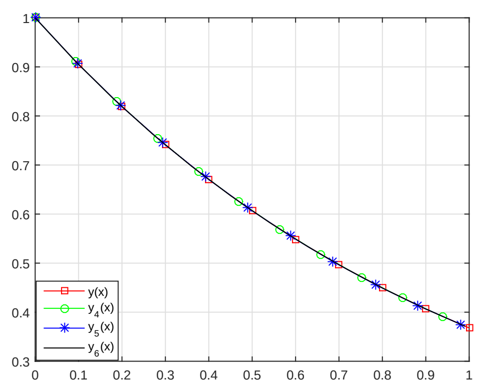


Figure 2. Numerical and Exact Solutions of $y_2(x)$ for N = 4,5,6

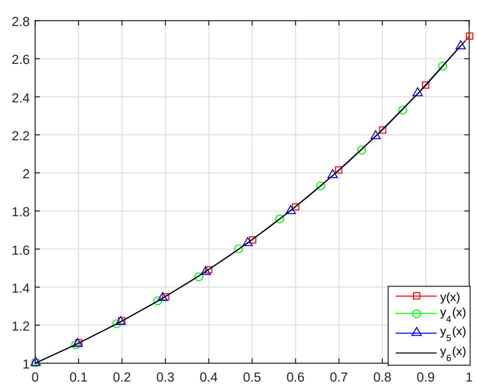


Figure 3. Residual Error Functions of $y_1(x)$ for N =4,5,6

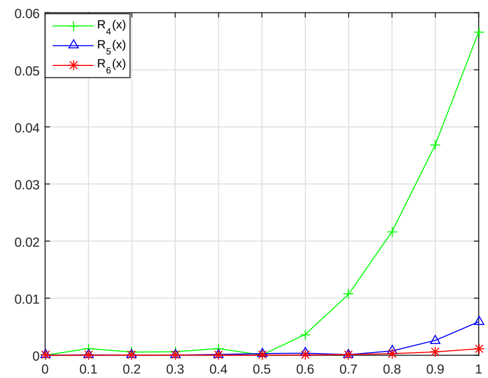
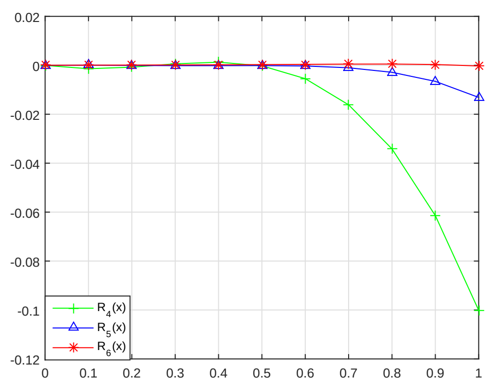


Figure 4. Numerical and Exact Solutions of $y_2(x)$ for N = 4,5,6



Conclusion

In this study, a new method was developed by using Bell polynomials for the solution of systems of linear delay-differential equations with variable coefficients. To illustrate the validity and applicability of this method, explanatory examples were solved, and an error analysis based on the residual function was performed to show the accuracy of the results. These comparisons and error estimates show that the proposed method is highly effective. We have calculated the solutions with the help of MATLAB.

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