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Vector-Valued Weighted Sobolev Spaces with Variable Exponent

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Abstract. Our aim is to introduce the vector-valued weighted variable exponent Lebesgue spaces. We discuss two different type of Hölder inequalities in this spaces. We will also show that every elements of vector-valued weighted variable exponent Lebesgue spaces are locally integrable. Hence we can define vector-valued weighted variable exponent Sobolev spaces. Finally under some conditions we will investigate some basic properties of vector-valued weighted variable exponent Sobolev spaces.

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1. Introduction

Spaces of weakly differentiable functions, so called Sobolev spaces, play an important role in modern Analysis. Since their discovery by Sergei Sobolev in the 1930's they have become the base for the study of many subjects such as partial differentiable equations and calculus of variations. Vector-valued Lebesgue and Sobolev spaces are now widely used in analysis, abstract evolution equations and in the theory of integral operators [1, 2, 11, 13, 14]. Also, the use of theory of vector-valued Sobolev spaces can be applied for solutions of some elliptic partial differential equations, new embedding results for weighted Sobolev spaces. The variable exponent Lebesgue space $L^{p(.)}(\mathbb{R}^n)$ and Sobolev space $W^{k,p(.)}(\mathbb{R}^n)$ were introduced by Kováčik and Rákosník [12] in 1991. Since 1991, variable exponent Lebesgue, Sobolev, Besov, Triebel-Lizorkin, Lorentz, amalgam and Morrey spaces, have attracted many attentions (see [6, 8, 12]). Vector-valued variable exponent Bochner-Lebesgue spaces $L^{p(.)}(\mathbb{R}^n, E)$ defined by Cheng and Xu [5] in 2013. They proved dual space, the reflexivity, uniformly convexity and uniformly smoothness of $L^{p(.)}(\mathbb{R}^n, E)$. Furthermore, they gave some properties of the Banach valued Bochner-Sobolev spaces with variable exponent. In this study, we focus on vector-valued weighted variable exponent Lebesgue $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ and Sobolev spaces $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$, and discuss some basic properties, such as completeness, reflexive and uniformly convex.

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2. Definition and Preliminary Results

Definition 2.1. For a measurable function $p : \mathbb{R}^n \to [1, \infty)$ (called a variable exponent on $\mathbb{R}^n$), we put

$$p^- = \underset{x\in\mathbb{R}^n}{\text{essinf}}\, p(x), \qquad p^+ = \underset{x\in\mathbb{R}^n}{\text{esssup}}\, p(x).$$

The variable exponent Lebesgue spaces $L^{p(.)}(\mathbb{R}^n)$ consist of all measurable functions f such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$, equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}(\frac{f}{\lambda}) \leq 1\right\},$$

where

$$\varrho_{p(.)}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx.$$

If $p^+ < \infty$, then $f \in L^{p(.)}(\mathbb{R}^n)$ iff $\varrho_{p(.)}(f) < \infty$. The space $\left(L^{p(.)}(\mathbb{R}^n), \|.\|_{p(.)}\right)$ is a Banach space. If $p(.) = p$ is a constant function, then the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$ [6, 8, 12]. In this paper we assume that $p^+ < \infty$.

A positive, measurable and locally integrable function $\vartheta : \mathbb{R}^n \to (0, \infty)$ is called a weight function. The weighted modular is defined by

$$\varrho_{p(.),\vartheta}(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, \vartheta(x) dx.$$

The weighted variable exponent Lebesgue space $L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ consists of all measurable functions f on $\mathbb{R}^n$ for which $\|f\|_{p(.),\vartheta} = \left\|f\vartheta^{\frac{1}{p(.)}}\right\|_{p(.)} < \infty$. The relations between the modular $\varrho_{p(.),\vartheta}(.)$ and $\|.\|_{p(.),\vartheta}$ are in the following:

$$\begin{aligned}
\min\left\{\varrho_{p(.),\vartheta}(f)^{\frac{1}{p^-}}, \varrho_{p(.),\vartheta}(f)^{\frac{1}{p^+}}\right\} &\leq \|f\|_{p(.),\vartheta} \leq \max\left\{\varrho_{p(.),\vartheta}(f)^{\frac{1}{p^-}}, \varrho_{p(.),\vartheta}(f)^{\frac{1}{p^+}}\right\} \\
\min\left\{\|f\|^{p^+}_{p(.),\vartheta}, \|f\|^{p^-}_{p(.),\vartheta}\right\} &\leq \varrho_{p(.),\vartheta}(f) \leq \max\left\{\|f\|^{p^+}_{p(.),\vartheta}, \|f\|^{p^-}_{p(.),\vartheta}\right\}
\end{aligned}$$

[3]. Moreover, if $0 < C \leq \vartheta$, then we have $L^{p(.)}_{\vartheta}(\mathbb{R}^n) \hookrightarrow L^{p(.)}(\mathbb{R}^n)$, since one easily sees that

$$C \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx \leq \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, \vartheta(x) dx$$

and $C \|f\|_{p(.)} \leq \|f\|_{p(.),\vartheta}$.

Theorem 2.2. *Let* $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ *and* $\vartheta^* = \vartheta^{1-q(.)}$. *Then for* $f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ *and* $g \in L^{q(.)}_{\vartheta^*}(\mathbb{R}^n)$, *we have* $fg \in L^1(\mathbb{R}^n)$ *and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \leq C \|f\|_{L^{p(.)}_{\vartheta}(\mathbb{R}^n)} \|g\|_{L^{q(.)}_{\vartheta^*}(\mathbb{R}^n)},$$

where $\vartheta^* = \vartheta^{1-q(.)}$.

Proof. By the Hölder inequality for variable exponent Lebesgue spaces, we get

$$\begin{aligned}
\int_{\mathbb{R}^n} |f(x)g(x)|\, dx &= \int_{\mathbb{R}^n} |f(x)g(x)|\, \vartheta(x)^{\frac{1}{p(x)} - \frac{1}{p(x)}} dx \\
&\leq C \left\|f\vartheta^{\frac{1}{p(.)}}\right\|_{p(.)} \left\|g\vartheta^{-\frac{1}{p(.)}}\right\|_{q(.)}
\end{aligned}$$

for some $C > 0$. That is the desired result. □

So the dual space of $L^{p(.)}_{\vartheta}(\mathbb{R}^n)$ is $L^{q(.)}_{\vartheta^*}(\mathbb{R}^n)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ and $\vartheta^* = \vartheta^{1-q(.)}$.

Let $(E, \|.\|_E)$ be a Banach space and E^* its dual space.

Definition 2.3 ([9]). A function $f : \mathbb{R}^n \to E$ is Bochner (or strongly) measurable if there exists a sequence $\{f_n\}$ of simple functions $f_n : \mathbb{R}^n \to E$ such that $f_n(x) \overset{E}{\to} f(x)$ as $n \to \infty$ for almost all $x \in \mathbb{R}^n$.

Definition 2.4 ([9]). A measurable function $f : \mathbb{R}^n \to E$ is called Bochner integrable if there exists a sequence of simple functions $\{f_n\}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^n} \|f_n - f\|_E \, dx = 0$$

for almost all $x \in \mathbb{R}^n$.

Theorem 2.5 (Bochner's Theorem [9]). *A measurable function $f : \mathbb{R}^n \to E$ is Bochner integrable if and only if $\int_{\mathbb{R}^n} \|f\|_E \, dx < \infty$, that is, $\|f\|_E$ is Lebesgue integrable.*

Definition 2.6 ([5, 9]). Let (Ω, Σ, μ) be a measure space. Then a function $F : \Sigma \to E$ is called a vector measure, if for all sequences (A_n) of pairwise disjoint members of Σ such that $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ and $F\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} F(A_n)$, where the series converges in the norm topology of E.

Let $F : \Sigma \to E$ be a vector measure. The variation of F is the function $\|F\| : \Sigma \to [0, \infty]$ defined by

$$\|F\|(A) = \sup_{\pi} \sum_{B\in\pi}^{\infty} \|F(B)\|_E ,$$

where the supremum is taken over all finite disjoint partitions π of A. If $\|F\|(\Omega) < \infty$, then F is called a measure of bounded variation.

Definition 2.7 ([5, 9]). A Banach space E has the Radon-Nikodym property (RNP) with respect to (Ω, Σ, μ) if for each vector measure $F : \Sigma \to E$ of bounded variation, which is absolutely continuous with respect to μ, there exists a function $g \in L^1(\Omega, E)$ such that

$$F(A) = \int_A g d\mu$$

for all $A \in \Sigma$.

Definition 2.8. Let ϑ be a weight function and $1 < p^- \le p(x) \le p^+ < \infty$. The weighted variable exponent Bochner-Lebesgue space $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ stands for all (equivalence classes of) E-valued Bochner integrable functions f on $\mathbb{R}^n$ such that

$$L^{p(.)}_{\vartheta}(\mathbb{R}^n, E) = \left\{ f : \|f\|_{p(.),\vartheta,E} < \infty \right\},$$

where

$$\|f\|_{p(.),\vartheta,E} = \left\| f\vartheta^{\frac{1}{p(.)}} \right\|_{p(.),E} = \inf\left\{ \lambda > 0 : \varrho_{p(.),\vartheta,E}(\frac{f}{\lambda}) \le 1 \right\}$$

and

$$\varrho_{p(.),\vartheta,E}(f) = \int_{\mathbb{R}^n} \|f(x)\|_E^{p(x)} \vartheta(x) dx.$$

The following properties proved by Cheng and Xu [5];

(i) $f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n, E) \Leftrightarrow \|f(.)\|_E^{p(.)} \in L^1_{\vartheta}(\mathbb{R}^n) \Leftrightarrow \|f(.)\|_E \in L^{p(.)}_{\vartheta}(\mathbb{R}^n)$.

(ii) $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is a generalization of the $L^p_{\vartheta}(\mathbb{R}^n, E)$ spaces.

(iii) If $E = \mathbb{R}$ or $\mathbb{C}$, then $L^{p(.)}_{\vartheta}(\mathbb{R}^n, \mathbb{R}) = L^{p(.)}_{\vartheta}(\mathbb{R}^n)$.

Theorem 2.9. *$L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is a Banach space with respect to $\|.\|_{p(.),\vartheta,E}$.*

Proof. Let (u_j) be a Cauchy sequence in $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$. Then, $\left(u_j\vartheta^{\frac{1}{p(.)}}\right)$ is a Cauchy sequence in the Banach space $L^{p(.)}(\mathbb{R}^n, E)$ in [7] due to

$$\left\|u_j - u_{j'}\right\|_{p(.),\vartheta,E} = \left\|\left(u_j - u_{j'}\right)\vartheta^{\frac{1}{p(.)}}\right\|_{p(.),E} \to 0,$$

so it converges to some u in $L^{p(.)}(\mathbb{R}^n, E)$. Consequently, (u_j) converges to $u\vartheta^{-\frac{1}{p(.)}}$ in $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$. □

Theorem 2.10 (Hölder's Inequality, scalar-valued case)**.** *Let* $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ *and* $\vartheta^* = \vartheta^{1-q(.)}$*. Then for* $f \in L_{\vartheta}^{p(.)}(\mathbb{R}^n, E)$ *and* $g \in L_{\vartheta^*}^{q(.)}(\mathbb{R}^n, \mathbb{R})$ *we have* $fg \in L^1(\mathbb{R}^n, E)$ *and Hölder inequality implies*

$$\|fg\|_{1,E} \le C \|f\|_{p(.),\vartheta,E} \|g\|_{q(.),\vartheta^*}$$

for some $C > 0$*.*

Proof. By the Hölder inequality for variable exponent Lebesgue spaces, we get

$$\begin{aligned} \int_{\mathbb{R}^n} \|f(x)g(x)\|_E \, dx &= \int_{\mathbb{R}^n} \|f(x)\|_E \, |g(x)| \, dx \\ &= \int_{\mathbb{R}^n} \|f(x)\|_E \, |g(x)| \, \vartheta(x)^{\frac{1}{p(x)} - \frac{1}{p(x)}} dx \\ &\le C \left\| f\vartheta^{\frac{1}{p(.)}} \right\|_{p(.),E} \left\| g\vartheta^{-\frac{1}{p(.)}} \right\|_{q(.)} \end{aligned}$$

for some $C > 0$. The proof is completed. □

The following Lemma for variable exponent case can be used to prove the Theorem 2.12.

Lemma 2.11. *If* $p > 1, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then for any positive real numbers* r *and* s *we have*

$$rs \le \frac{r^p}{p} + \frac{s^q}{q}.$$

Proof. Define a function k by $k(t) = \frac{t^p}{p} + \frac{t^{-q}}{q}$ for all $t > 0$. Then the derivative of k is $k'(t) = t^{p-1} - t^{-q-1}$. Now $k'(1) = 0$, so k has a critical point at $t = 1$. Furthermore, it is clear that if $t > 1$ then $k'(t) > 0$, whereas if $0 < t < 1$ then $k'(t) < 0$. Thus k has an absolute minimum $t = 1$. But $k(1) = 1$, so for every $t > 0$ we have $1 \le \frac{t^p}{p} + \frac{t^{-q}}{q}$. Setting $t = r^{\frac{1}{q}} / s^{\frac{1}{p}}$ we obtain $1 \le \frac{r^{\frac{p}{q}}}{ps} + \frac{s^{\frac{q}{p}}}{qr}$, so that $rs \le \frac{r^p}{p} + \frac{s^q}{q}$. □

Theorem 2.12 (Hölder's Inequality, dual-valued case)**.** *Let* $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ *and* $\vartheta^* = \vartheta^{1-q(.)}$*. Then for* $f \in L_{\vartheta}^{p(.)}(\mathbb{R}^n, E)$ *and* $g \in L_{\vartheta^*}^{q(.)}(\mathbb{R}^n, E^*)$ *the dual pair* $< f(.), g(.) > \in L^1(\mathbb{R}^n, \mathbb{R})$ *and Hölder inequality implies*

$$\|< f, g >\|_{1,\mathbb{R}} \le C \|f\|_{p(.),\vartheta,E} \|g\|_{q(.),\vartheta^*,E^*}$$

for some $C > 0$*, where* E^* *has the Radon-Nikodym Property (RNP).*

Proof. Let $g \in L_{\vartheta^*}^{q(.)}(\mathbb{R}^n, E^*)$ and let (g_n) be a sequence of simple functions in $L_{\vartheta^*}^{q(.)}(\mathbb{R}^n, E^*)$ converging to g a.e. Suppose $f \in L_{\vartheta}^{p(.)}(\mathbb{R}^n, E)$ and define $< f, g > (w) = g(w)(f(w))$ for $w \in \mathbb{R}^n$. Certainly $< f, g_n >$ is measurable for each n, and it is only slightly less evident that $\lim_n < f, g_n > = < f, g >$ a.e. Consequently, $< f, g >$ is measurable. Moreover, the absolute value of the product $< f, g >$ can be estimated by $\|f\|_E \|g\|_{E^*}$. So we have

$$\begin{aligned} \int_{\mathbb{R}^n} |< f(.), g(.) >| \, dx &\le \int_{\mathbb{R}^n} \|f\|_E \|g\|_{E^*} \, dx \\ &\le C \|f\|_{p(.),\vartheta,E} \|g\|_{q(.),\vartheta^*,E^*} \end{aligned}$$

by the Hölder inequality. □

Corollary 2.13. *Let* $g \in L_{\vartheta^*}^{q(.)}(\mathbb{R}^n, E^*)$*. Then the functional* $\varphi_g : L_{\vartheta}^{p(.)}(\mathbb{R}^n, E) \to \mathbb{C}$*, which is defined by*

$$\varphi_g(f) = \int_{\mathbb{R}^n} < f(.), g(.) > dx,$$

is linear and continuous. Hence φ_g *is a member of* $\left(L_{\vartheta}^{p(.)}(\mathbb{R}^n, E)\right)^*$ *whose norm is not greater than* $\|g\|_{q(.),\vartheta^*,E^*}$*, and we have the embedding* $L_{\vartheta^*}^{q(.)}(\mathbb{R}^n, E^*) \hookrightarrow \left(L_{\vartheta}^{p(.)}(\mathbb{R}^n, E)\right)^*$*. Further for all* $g \in L_{\vartheta^*}^{q(.)}(\mathbb{R}^n, E^*)$ *it holds that* $\|\varphi_g\|_{\left(L_{\vartheta}^{p(.)}(\mathbb{R}^n,E)\right)^*} \le C \|g\|_{q(.),\vartheta^*,E^*}$ *hence this embedding is continuous. The reverse inequality* $\|\varphi_g\|_{\left(L_{\vartheta}^{p(.)}(\mathbb{R}^n,E)\right)^*} \ge C \|g\|_{q(.),\vartheta^*,E^*}$ *was proved by the following theorem.*

Theorem 2.14 ([5]). *If E^* has the Radon-Nikodym Property (RNP), then the mapping $g \mapsto \varphi_g$, $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$, $L^{q(.)}_{\vartheta^*}(\mathbb{R}, E^*) \to L^{p(.)}_{\vartheta}(\mathbb{R}, E)^*$ which is defined by*

$$< \varphi_g, f >= \int_{\mathbb{R}^n} < g, f > dx$$

for any $f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is a linear isomorphism and

$$\|g\|_{q(.),\vartheta^*,E^*} \leq \left\|\varphi_g\right\|_{\left(L^{p(.)}_{\vartheta}(\mathbb{R}^n,E)\right)^*} \leq 2\|g\|_{q(.),\vartheta^*,E^*},$$

where $\vartheta^* = \vartheta^{1-q(.)}$. Hence, the dual space $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)^*$ is isometrically isomorphic to $L^{q(.)}_{\vartheta^*}(\mathbb{R}^n, E^*)$, where E^* has RNP.

Corollary 2.15. *(i) If E is reflexive, then E^* is also reflexive.*

(ii) Every reflexive space has the Radon-Nikodym property.

(iii) If E is reflexive and $1 < p^- \leq p^+ < \infty$, then $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is reflexive.

(iv) Let E be a Banach space such that E^* has the Radon-Nikodym property, then $L^{p(.)}_{\vartheta}(\mathbb{R}, E)^* \cong L^{q(.)}_{\vartheta^*}(\mathbb{R}, E^*)$, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$.

(v) If E is a uniformly convex Banach space and $1 < p^- \leq p^+ < \infty$, then $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is also a uniformly convex [5].

The space $L^1_{loc}(\mathbb{R}^n, E)$ consists of all (classes of) all E-valued measurable functions f such that $f\chi_K \in L^1(\mathbb{R}^n, E)$ for any compact subset $K \subset \mathbb{R}^n$. It is a topological vector space with the family of seminorms $f \mapsto \|f\chi_K\|_{1,E}$.

Proposition 2.16. *Let ϑ be a weight function and $1 < p^- \leq p(.) \leq p^+ < \infty$. If $\vartheta^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\mathbb{R}^n)$, then $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E) \hookrightarrow L^1_{loc}(\mathbb{R}^n, E)$.*

Proof. Suppose that $f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ and let $K \subset \mathbb{R}^n$ be any compact set. For $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$, by using Hölder's inequality for variable exponent Lebesgue spaces [12], then there exists a $A_K > 0$ such that

$$\begin{aligned}
\|f\|_{L^1_{loc}(\mathbb{R}^n,E)} &= \|f\|_{1,K,E} = \int_K \|f(x)\|_E\, dx \\
&= \int_{\mathbb{R}^n} \|f(x)\|_E \chi_K(x)\vartheta(x)^{\frac{1}{p(x)}-\frac{1}{p(x)}}\, dx \\
&\leq A_K \left\|f\vartheta^{\frac{1}{p(.)}}\right\|_{p(.),E} \left\|\chi_K\vartheta^{-\frac{1}{p(.)}}\right\|_{q(.)} \\
&\leq A_K \|f\|_{p(.),\vartheta,E} \left\|\chi_K\vartheta^{-\frac{1}{p(.)}}\right\|_{q(.)} \qquad (2.1)
\end{aligned}$$

by Hölder's inequality for scalar-valued case (Theorem 2.12). It is known that $\left\|\chi_K\vartheta^{-\frac{1}{p(.)}}\right\|_{q(.)} < \infty$ if and only if $\varrho_{q(.)}(\chi_K\vartheta^{-\frac{1}{p(.)}}) < \infty$ for $q^+ < \infty$. Since $\vartheta^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\mathbb{R}^n)$, then we have

$$\varrho_{q(.)}(\chi_K\vartheta^{-\frac{1}{p(.)}}) = \int_{\mathbb{R}^n} \left|\chi_K(x)\vartheta(x)^{-\frac{1}{p(.)}}\right| dx = \int_K \vartheta(x)^{-\frac{1}{p(x)-1}}\, dx = B_K < \infty. \qquad (2.2)$$

If we use (2.1) and (2.2), then the proof is completed. □

Remark 2.17. Let $1 < p^- \leq p(x) \leq p^+ < \infty$ and $\vartheta^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\mathbb{R}^n)$. Then every function in $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ has distributional derivatives by Proposition 2.16.

3. Vector-Valued Weighted Variable Sobolev Spaces

Let $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}^n_0$ be a multi-index. Its length is defined as $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. For another vector $z \in \mathbb{R}^n$ we define $z^\alpha := z_1^{\alpha_1}...z_n^{\alpha_n}$. as the multiplicity of α. Multi-indexes can be partially ordered via $\alpha \leq \beta \Leftrightarrow \alpha_k \leq \beta_k$ for all k. Let $D_k := \frac{\partial}{\partial x_k}$, then for a multi-index α we have

$$D^\alpha = D_1^{\alpha}...D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1}...\partial z_n^{\alpha_n}}.$$

Definition 3.1. Let $C_0^\infty(\mathbb{R}^n, E)$ (or $D(\mathbb{R}^n, E)$, test functions) denote the collection of E-valued infinitely differentiable functions on $\mathbb{R}^n$ with compact support in $\mathbb{R}^n$, that is,

$$C_0^\infty(\mathbb{R}^n, E) = \{\varphi \in C^\infty(\mathbb{R}^n, E) : \ \operatorname{supp}\varphi \text{ compact in } \mathbb{R}^n\}.$$

The space $C_0^\infty(\mathbb{R}^n, E)$ is topologized in the following way: a sequence $(\varphi_j) \subset C_0^\infty(\mathbb{R}^n, E)$ is said to be convergent in $C_0^\infty(\mathbb{R}^n, E)$ to $\varphi \in C_0^\infty(\mathbb{R}^n, E)$, $\varphi_j \underset{D}{\rightarrow} \varphi$, if and only if there is a compact set $K \subset \mathbb{R}^n$ such that

$$\operatorname{supp}\varphi_j \subset K, \ j \in \mathbb{N}, \ \operatorname{supp}\varphi \subset K, \tag{3.1}$$

and

$$D^\alpha \varphi_j \rightrightarrows D^\alpha \varphi \text{ (uniformly) for all } \alpha \in \mathbb{N}_0^n \tag{3.2}$$

on K.

Definition 3.2. $D'(\mathbb{R}^n, E)$ denote the collection of E-valued linear continuous functionals T over $D(\mathbb{R}^n, E)$, that is,

$$T : D(\mathbb{R}^n, E) \rightarrow E, T : \varphi \mapsto T(\varphi), \ \varphi \in D(\mathbb{R}^n, E),$$

$$T(\lambda_1\varphi_1 + \lambda_2\varphi_2) = \lambda_1 T(\varphi_1) + \lambda_2 T(\varphi_2), \ \lambda_1, \lambda_2 \in \mathbb{C}; \ \varphi_1, \varphi_2 \in D(\mathbb{R}^n, E),$$

and

$$T(\varphi_j) \rightarrow T(\varphi) \text{ for } j \rightarrow \infty \text{ whenever } \varphi_j \underset{D}{\rightarrow} \varphi, \tag{3.3}$$

according to (3.1) and (3.2). $T \in D'(\mathbb{R}^n, E)$ is called a distribution.

Corresponding to every $u \in L^1_{loc}(\mathbb{R}^n, E)$ (all local integrable functions valued in E over $\mathbb{R}^n$) there is a distribution $T_u \in D'(\mathbb{R}^n, E)$ defined by

$$T_u(\varphi) =< T_u, \varphi >= \int_{\mathbb{R}^n} u(x)\varphi(x)dx, \varphi \in D(\mathbb{R}^n, \mathbb{R}). \tag{3.4}$$

(3.4) generates a one-to-one correspondence

$$u \in L^1_{loc}(\mathbb{R}^n, E) \Longleftrightarrow T_u \in D'(\mathbb{R}^n, E).$$

Now we will show that $T_u : D(\mathbb{R}^n, E) \rightarrow E$ is continuous. For $\varphi \in D(\mathbb{R}^n, \mathbb{R})$, we have

$$\begin{aligned} \|T_u(\varphi)\|_E &\leq \int_{\mathbb{R}^n} \|u(x)\varphi(x)\|_E \, dx = \int_{\mathbb{R}^n} \|u(x)\|_E \, |\varphi(x)| \, dx \\ &\leq \sup_{x\in K} |\varphi(x)| \int_K \|u(x)\|_E \, dx < \infty, \end{aligned}$$

where $\operatorname{supp}\varphi \subset K$ and $K \subset \mathbb{R}^n$ is compact. Moreover, by (3.3) the proof is completed.

Remark 3.3. The chain of inclusions is obtained by the following way

$$D(\mathbb{R}^n, E) \subset C^\infty(\mathbb{R}^n, E) \subset L^{p(.)}_{\vartheta,loc}(\mathbb{R}^n, E) \subset L^1_{loc}(\mathbb{R}^n, E) \subset D'(\mathbb{R}^n, E).$$

Definition 3.4. Let $\alpha \in \mathbb{N}_0^n$ and $T \in D'(\mathbb{R}^n, E)$. Then the distributional derivative $D^\alpha T \in D'(\mathbb{R}^n, E)$ is given by

$$(D^\alpha T)(\varphi) = (-1)^{|\alpha|} T(D^\alpha \varphi), \varphi \in D(\mathbb{R}^n, \mathbb{R}).$$

We now define the weak derivative of a locally integrable function. Let $u \in L^1_{loc}(\mathbb{R}^n, E)$. There may or may not exist a function $\upsilon_\alpha \in L^1_{loc}(\mathbb{R}^n, E)$ such that $T_{\upsilon_\alpha} = D^\alpha T_u$ in $D'(\mathbb{R}^n, E)$. If such a υ_α exists, it is unique up to sets of measure zero and it is called the weak derivative of u and is denoted by $D^\alpha u$. Thus $D^\alpha u = \upsilon_\alpha$ in the weak (distributional) sense provided $\upsilon_\alpha \in L^1_{loc}(\mathbb{R}^n, E)$ satisfies

$$\int_{\mathbb{R}^n} u(x) D^\alpha \varphi(x) dx = (-1)^{|\alpha|} \int_{\mathbb{R}^n} \upsilon_\alpha(x)\varphi(x)dx$$

for every $\varphi \in D(\mathbb{R}^n, \mathbb{R})$.

Let $1 < p^- \le p(.) \le p^+ < \infty$, $\vartheta^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\mathbb{R}^n)$ and $k \in \mathbb{N}$. We define the vector-valued weighted variable Sobolev spaces $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$ by

$$W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E) = \left\{ f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n, E) : D^{\alpha} f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n, E), 0 \le |\alpha| \le k \right\}$$

equipped with the norm

$$\|f\|_{k,p(.),\vartheta,E} = \sum_{0\le|\alpha|\le k} \|D^{\alpha} f\|_{p(.),\vartheta,E} .$$

Clearly, $W^{0,p(.)}_{\vartheta}(\mathbb{R}^n, E) = L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$. For any k, the continuous embedding $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E) \hookrightarrow L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is valid. It can be shown that $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n)$ is a reflexive Banach space. Throughout this paper, we will always assume that $1 < p^- \le p(x) \le p^+ < \infty$ and $\vartheta^{-\frac{1}{p(.)-1}} \in L^1_{loc}(\mathbb{R}^n)$.

The space $W^{1,p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is defined by

$$W^{1,p(.)}_{\vartheta}(\mathbb{R}^n, E) = \left\{ f \in L^{p(.)}_{\vartheta}(\mathbb{R}^n, E) : |\nabla f| \in L^{p(.)}_{\vartheta}(\mathbb{R}^n, E) \right\}.$$

The function $\varrho_{1,p(.),\vartheta,E} : W^{1,p(.)}_{\vartheta}(\mathbb{R}^n, E) \to [0, \infty)$ is defined as $\varrho_{1,p(.),\vartheta,E}(f) = \varrho_{p(.),\vartheta,E}(f) + \varrho_{p(.),\vartheta,E}(\nabla f)$. The norm $\|f\|_{1,p(.),\vartheta,E} = \|f\|_{p(.),\vartheta,E} + \|\nabla f\|_{p(.),\vartheta,E}$.

Now, we give some basic properties of $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$.

Proposition 3.5. *The space* $\left(W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E), \|.\|_{k,p(.),\vartheta,E}\right)$ *is a Banach space.*

Proof. Let $\left(u_j\right)$ be a Cauchy sequence in $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$. We show that there exists $u \in W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$ such that $u_j \to u$ in $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$ as $j \to \infty$. Then, $\left\{D^{\alpha} u_j\right\}$ is a Cauchy sequences in $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ for $0 \le |\alpha| \le k$. Since $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is a Banach space there exist functions u and u_{α} in $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ such that $u_j \to u$ and $D^{\alpha} u_j \to u_{\alpha}$ in $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E)$ as $j \to \infty$. Now we will show that $u_{\alpha} = D^{\alpha} u$ in the distributional sense on $\mathbb{R}^n$ for $0 \le |\alpha| \le k$. Since $L^{p(.)}_{\vartheta}(\mathbb{R}^n, E) \hookrightarrow L^1_{loc}(\mathbb{R}^n, E)$ by Proposition 2.16, then u_j determines a distribution $T_{u_j} \in D'(\mathbb{R}^n, E)$. For any $\varphi \in D(\mathbb{R}^n, \mathbb{R})$ we have

$$\begin{aligned} \left\|T_{u_j}(\varphi) - T_u(\varphi)\right\|_E &\le \int_{\mathbb{R}^n} \left\|u_j(x) - u(x)\right\|_E |\varphi(x)|\, dx \\ &\le C \left\|u_j - u\right\|_{p(.),\vartheta,E} \|\varphi\|_{q(.),\vartheta^*} \end{aligned}$$

for some $C > 0$ by Theorem 2.12, where $\frac{1}{p(.)} + \frac{1}{q(.)} = 1$ and $\vartheta^* = \vartheta^{1-q(.)}$. Hence $T_{u_j}(\varphi) \to T_u(\varphi)$ for every $\varphi \in D(\mathbb{R}^n, \mathbb{R})$ as $j \to \infty$. Similarly, $T_{D^{\alpha} u_j}(\varphi) \to T_{u_{\alpha}}(\varphi)$ for every $\varphi \in D(\mathbb{R}^n, \mathbb{R})$. It follows that

$$\begin{aligned} T_{u_{\alpha}}(\varphi) &= \lim_{j\to\infty} T_{D^{\alpha} u_j}(\varphi) = \lim_{j\to\infty} (-1)^{|\alpha|}\, T_{u_j}(D^{\alpha}\varphi) \\ &= (-1)^{|\alpha|}\, T_u(D^{\alpha}\varphi) \end{aligned}$$

for every $\varphi \in D(\mathbb{R}^n, \mathbb{R})$. Thus $u_{\alpha} = D^{\alpha} u$ in the distributional sense on $\mathbb{R}^n$ for $0 \le |\alpha| \le k$, whence $u \in W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$. Since $\lim_{j\to\infty} \left\|u_j - u\right\|_{k,p(.),\vartheta,E} = 0$, $W^{k,p(.)}_{\vartheta}(\mathbb{R}^n, E)$ is complete. □

We say that $\vartheta_1 \prec \vartheta_2$ if and only if there exists a $C > 0$ such that $\vartheta_1(x) \le C\vartheta_2(x)$ for all $x \in \mathbb{R}^n$. Two weight functions are called equivalent and written $\vartheta_1 \approx \vartheta_2$, if $\vartheta_1 \prec \vartheta_2$ and $\vartheta_2 \prec \vartheta_1$.

Proposition 3.6. *Let υ_1 and υ_2 be weight functions on $\mathbb{R}^n$. If $\upsilon_1 \prec \upsilon_2$, then the embedding $W^{k,p(.)}_{\vartheta_2}(\mathbb{R}^n, E) \hookrightarrow W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E)$ holds.*

Proof. Since $\upsilon_1 \prec \upsilon_2$, then there exists a $C > 0$ such that $\vartheta_1(x) \le C\vartheta_2(x)$ for all $x \in \mathbb{R}^n$. Hence we have $L^{p(.)}_{\vartheta_2}(\mathbb{R}^n, E) \hookrightarrow L^{p(.)}_{\vartheta_1}(\mathbb{R}^n, E)$ and $W^{k,p(.)}_{\vartheta_2}(\mathbb{R}^n, E) \hookrightarrow W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E)$. □

Corollary 3.7. *If $\vartheta_1 \approx \vartheta_2$, then $W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E) = W^{k,p(.)}_{\vartheta_2}(\mathbb{R}^n, E)$.*

Theorem 3.8. *Suppose that υ_1 and υ_2 are weight functions on $\mathbb{R}^n$ satisfying $\upsilon_1 \prec \upsilon_2$ and $k, t \in \mathbb{Z}^+$ with $k > t$. Then the embedding $W^{k,p(.)}_{\vartheta_2}(\mathbb{R}^n, E) \hookrightarrow W^{t,p(.)}_{\vartheta_1}(\mathbb{R}^n, E)$ holds.*

Proof. Let $f \in W^{k,p(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ be given. Then we can write $D^\alpha f \in L^{p(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ for $0 \le |\alpha| \le k$. Since $\upsilon_1 \prec \upsilon_2$, then $L^{p(.)}_{\vartheta_2}(\mathbb{R}^n, E) \hookrightarrow L^{p(.)}_{\vartheta_1}(\mathbb{R}^n, E)$ and there is a $C > 0$ such that

$$\|D^\alpha f\|_{p(.),\vartheta_1,E} \le C\,\|D^\alpha f\|_{p(.),\vartheta_2,E}\,.$$

Using $k, t \in \mathbb{Z}^+$ with $k > t$, we have

$$\begin{aligned}\|D^\alpha f\|_{t,p(.),\vartheta_1,E} &\le \sum_{0\le|\alpha|\le t}\|D^\alpha f\|_{p(.),\vartheta_1,E} + \sum_{t+1\le|\alpha|\le k}\|D^\alpha f\|_{p(.),\vartheta_1,E}\\ &= C\,\|D^\alpha f\|_{k,p(.),\vartheta_2,E}\,.\end{aligned}$$

That is the desired result. □

Theorem 3.9. *Let $p_1(.), p_2(.)$ be variable exponents satisfying $p_1(.) \le p_2(.)$. Then the embedding $W^{k,p_2(.)}_{\vartheta}(\mathbb{R}^n, E) \hookrightarrow W^{k,p_1(.)}_{\vartheta}(\mathbb{R}^n, E)$ holds.*

Proof. Let $f \in W^{k,p_2(.)}_{\vartheta}(\mathbb{R}^n, E)$ be given. So $D^\alpha f \in L^{p_2(.)}_{\vartheta}(\mathbb{R}^n, E)$ for $0 \le |\alpha| \le k$. It is known that, if the condition $p_1(.) \le p_2(.)$ holds, then the embedding $L^{p_2(.)}_{\vartheta}(\mathbb{R}^n, E) \hookrightarrow L^{p_1(.)}_{\vartheta}(\mathbb{R}^n, E)$ is satisfied [7]. Similarly, it can be seen that

$$\|D^\alpha f\|_{p_1(.),\vartheta,E} \le C\,\|D^\alpha f\|_{p_2(.),\vartheta,E}\,.$$

This completes the proof. □

Theorem 3.10. *Let $p_1(.), p_2(.)$ be variable exponents satisfying $1 < p_2^- \le p_2(.) \le p_1(.) \le p_1^+ < \infty$ and $\left\|\frac{\vartheta_2}{\vartheta_1}\right\|_{\frac{p_1(.)}{p_1(.)-p_2(.)},\vartheta_1} < \infty$. Then the embedding $W^{k,p_1(.)}_{\vartheta_1}(\mathbb{R}^n, E) \hookrightarrow W^{k,p_2(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ holds.*

Proof. Suppose that $f \in W^{k,p_1(.)}_{\vartheta_1}(\mathbb{R}^n, E)$. It is known that $L^{p_1(.)}_{\vartheta_1}(\mathbb{R}^n, E) \hookrightarrow L^{p_2(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ with $\left\|\frac{\vartheta_2}{\vartheta_1}\right\|_{\frac{p_1(.)}{p_1(.)-p_2(.)},\vartheta_1} < \infty$ (Theorem 5.1, [10]). Hence we have the embedding $W^{k,p_1(.)}_{\vartheta_1}(\mathbb{R}^n, E) \hookrightarrow W^{k,p_2(.)}_{\vartheta_2}(\mathbb{R}^n, E)$. □

Theorem 3.11. *Let $p(.), q(.)$ be variable exponents on $\mathbb{R}^n$. If the inclusion $W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E) \subset W^{k,q(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ holds for the weights ϑ_1 and ϑ_2 if and only if the embedding $W^{k,p_2(.)}_{\vartheta_1}(\mathbb{R}^n, E) \hookrightarrow W^{k,p_1(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ is satisfied.*

Proof. The sufficient condition of the theorem is clear by the definition of continuous embedding. Now, assume that the inclusion $W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E) \subset W^{k,q(.)}_{\vartheta_2}(\mathbb{R}^n, E)$ is valid. Moreover, we define the sum norm $|||.||| = \|.\|_{k,p(.),\vartheta_1,E} + \|.\|_{k,p(.),\vartheta_2,E}$. It is easy to see that $\left(W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E), |||.|||\right)$ is a Banach space. If we define the unit function I from $\left(W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E), |||.|||\right)$ into $\left(W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E), \|.\|_{k,p(.),\vartheta_1,E}\right)$, then the function I is continuous. Because we can obtain the inequality $\|I(f)\|_{k,p(.),\vartheta_1,E} = \|f\|_{k,p(.),\vartheta_1,E} \le |||f|||$. By Banach's theorem I is a homeomorphism, see [4]. So the norms $|||.|||$ and $\|.\|_{k,p(.),\vartheta_1,E}$ are equivalent. Thus, for every $f \in W^{k,p(.)}_{\vartheta_1}(\mathbb{R}^n, E)$ there exists a $k > 0$ such that

$$|||f||| \le k\,\|f\|_{k,p(.),\vartheta_1,E}\,.$$

By the definition of the norm $|||.|||$ we have

$$\|.\|_{k,p(.),\vartheta_2,E} \le |||f||| \le k\,\|f\|_{k,p(.),\vartheta_1,E}\,.$$ □

Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this article.

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