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Weak Semi-Local Functions in Ideal Topological Spaces

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Abstract. In this study, we define the concept of weak semi-local functions by using semi-open sets and semi-closure operators in ideal topological spaces. We also introduce properties of weak semi-local functions and investigate the relationship between weak semi-local functions and predefined operators (local functions, semi-local functions, semi-closure local functions and local closure functions).

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1. Introduction

Let (X, σ) be a topological space and the interior of any subset M of X is denoted by $Int(M)$ and the closure of M denoted by $Cl(M)$. Levine [7] defined semi-open sets in topological spaces and he investigated the properties of semi-open sets. Semi-open sets are defined as: Let (X, σ) be a topological space, any subset M of X is called semi-open; if there exists any open set V of X such that $V \subseteq M \subseteq Cl(V)$. Also, Levine gave a theorem that equivalents to the definition of semi-open set in [7], a subset M in topological space (X, σ) is semi-open set if and only if $M \subseteq Cl(Int(M))$. The collection of all semi-open subsets of X is denoted by $SO(X)$.The family of semi-open neighborhoods of any point x is denoted by $SO(X, x)$. Mathematically, $SO(X, x) = \{U \in SO(X) : x \in U\}$. The complement of a semi-open set is called semi-closed set [2]. Equivalenty, a subset M is semi-closed if and only if $Int(Cl(M)) \subseteq M$. The intersection of all semi-closed sets of X containing of $M \subseteq X$ is the semi-closure of M and is denoted by $sCl(M)$. The union of all semi-open subsets of M is called the semi-interior of M and is denoted by $sInt(M)$. If M is any subset in a topological space, $Int(M) \subseteq sInt(M)$. So, every open subset of any topological space is semi-open set.

Velicko defined the concept of θ-open set in [9]. A subset M is said to be θ-open in topological spaces, if every point in M has an open neigbourhood whose closure is contained in M. The θ-interior of M in topological spaces is the union of all θ-open subsets of M and is denoted by $Int_\theta(M)$. The complement of a θ-open set is said to be θ-closed. A point $x \in X$ is said to be in θ-closure of a subset $M \subseteq X$ [9], if for every open neigbourhood U of x, $Cl(U) \cap M \neq \emptyset$. θ-closure set of a subset $M \subseteq X$ denoted by $Cl_\theta(M)$ and $Cl_\theta(M) = \{x \in X : Cl(U) \cap M \neq \emptyset,$ for every $U \in \sigma_{(x)}\}$, where $\sigma_{(x)} = \{U \in \sigma : x \in U\}$.

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Ideal and local function were first defined and gave properties by Kuratowski [6]. Vaidyanathaswamy obtained a new topology with the aid of local function and Kuratowski closure operator [8]. Jankovic and Hamlet introduced more properties of concept of ideal in topological spaces [4]. Khan and Noiri defined a semi-local function $M_*(\Lambda, \sigma)$ by using semi-open sets in [5]. Also,in [1] an approximation of the local function has been done by Al-Omari and Noiri, with the help of closure operator of topological spaces. Later, Islam and Modak introduced another approximation of local function with the help of semi-closure operator of topological spaces and it is called semi-closure local function [3].

Definition 1.1 ([6]). Let Λ be any non-empty class of subsets of X. If Λ satisfies the following axioms, Λ is called an ideal on X.

(1) $M \in \Lambda$ and $N \subseteq M$ implies $N \in \Lambda$ (heredity property)
(2) $M, N \in \Lambda$ implies $M \cup N \in \Lambda$ (finite additivity property)

Let (X, σ) be a topological space and an ideal Λ on X. Then (X, σ, Λ) is called an ideal topological space.It can also be said that if there is both a topological structure σ and an ideal Λ on the X, (X, σ, Λ) is called ideal topological space. If an ideal on X does not contain X, it is called proper ideal. $\varnothing$ and $P(X)$ (powerset of X) are ideals on X. All ideals on X forms a class of sets that is a poset (partially ordered set) according to the subset relation. So, $\varnothing$ and $P(X)$ are called minimal ideal and maximal ideal, respectively.

Definition 1.2 ([6]). Let M be any subset of an ideal topological space (X, σ, Λ). Then,

$$M^*(\Lambda, \sigma) = \{x \in X : (U \cap M) \notin \Lambda \text{ for every } U \in \sigma_{(x)}\}$$

is said to be the local function of M with respect to an ideal Λ and a topology σ on X.

Definition 1.3 ([5]). Let M be any subset of an ideal topological space (X, σ, Λ). Then,

$$M_*(\Lambda, \sigma) = \{x \in X : (U \cap M) \notin \Lambda \text{ for every } U \in SO(X, x)\}$$

is said to be the semi-local function of M with respect to an ideal Λ and a topology σ on X.

Lemma 1.4 ([5]). *Let M be any subset of an ideal topological space (X, σ, Λ). Then,*

$$M_*(\Lambda, \sigma) \subseteq M^*(\Lambda, \sigma)$$

for every subset M of X.

Definition 1.5 ([1]). Let M be any subset of an ideal topological space (X, σ, Λ). Then,

$$\Gamma(M)(\Lambda, \sigma) = \{x \in X : (Cl(U) \cap M) \notin \Lambda \text{ for every } U \in \sigma_{(x)}\}$$

is said to be the local closure function of M with respect to an ideal Λ and a topology σ on X.

Definition 1.6 ([3]). Let M be any subset of an ideal topological space (X, σ, Λ). Then,

$$\gamma(M)(\Lambda, \sigma) = \{x \in X : (sCl(U) \cap M) \notin \Lambda \text{ for every } U \in \sigma_{(x)}\}$$

is said to be the semi-closure local function of M with respect to an ideal Λ and a topology σ on X.

Lemma 1.7 ([3]). *Let (X, σ, Λ) be an ideal topological space and for a subset M of X. Then,*

$$M^*(\Lambda, \sigma) \subseteq \gamma(M)(\Lambda, \sigma) \subseteq \Gamma(M)(\Lambda, \sigma)$$

for every subset M of X.

2. Weak Semi-Local Function

Definition 2.1. Let M be a subset of an ideal topological space (X, σ, Λ). We define the following operator:

$$\xi(M)(\Lambda, \sigma) = \{x \in X : (sCl(U) \cap M) \notin \Lambda \text{ for every } U \in SO(X, x)\},$$

is called the weak semi-local function of M with respect to an ideal Λ and a topology σ on X. If there is no confusion, we sometimes write $\xi(M)$ instead of $\xi(M)(\Lambda, \sigma)$.

We will give examples and a lemma showing the relationship between weak semi-local functions and other operators.

Lemma 2.2. *Let (X, σ, Λ) be an ideal topological space, let M be a subset of X. Then following property is hold:*

$$M_*(\Lambda, \sigma) \subseteq \xi(M)(\Lambda, \sigma) \subseteq \gamma(M)(\Lambda, \sigma) \subseteq \Gamma(M)(\Lambda, \sigma).$$

Proof. Let $x \in M_*(\Lambda, \sigma)$. Then, $(U \cap M) \notin \Lambda$ for every $U \in SO(X, x)$. Since $(U \cap M) \subseteq (sCl(U) \cap M)$ and the definition of ideal, $(sCl(U) \cap M) \notin \Lambda$. Hence, $x \in \xi(M)(\Lambda, \sigma)$ and $M_*(\Lambda, \sigma) \subseteq \xi(M)(\Lambda, \sigma)$. Again, let $x \in \xi(M)(\Lambda, \sigma)$. Then, $(sCl(U) \cap M) \notin \Lambda$ for every $U \in SO(X, x)$. Since $\sigma_{(x)} \subseteq SO(X, x)$, $x \in \gamma(M)(\Lambda, \sigma)$. Consequently, $\xi(M)(\Lambda, \sigma) \subseteq \gamma(M)(\Lambda, \sigma)$. Moreover, $\xi(M)(\Lambda, \sigma) \subseteq \gamma(M)(\Lambda, \sigma) \subseteq \Gamma(M)(\Lambda, \sigma)$ from Lemma 1.7. □

Example 2.3. Let $X = \{x, y, z, t\}$ $\sigma = \{X, \varnothing, \{x\}, \{z\}, \{x, z\}, \{x, t\}, \{x, y, z\}, \{x, z, t\}\}$, with $\Lambda = \{\varnothing, \{y\}\}$. If $M = \{y, t\}$, then

$$M_*(\Lambda, \sigma) = \{t\},$$

$$\xi(M)(\Lambda, \sigma) = \{x, t\},$$

$$\gamma(M)(\Lambda, \sigma) = \{x, y, t\}$$

and additionally, $M^*(\Lambda, \sigma) = \{t\} \subseteq \xi(M)(\Lambda, \sigma)$.

Example 2.4. Let $X = \{x, y, z, t\}$, $\sigma = \{X, \varnothing, \{t\}, \{x, z\}, \{x, z, t\}\}$ with $\Lambda = \{\varnothing, \{z\}\}$. If $M = \{x, z\}$, then

$$\xi(M)(\Lambda, \sigma) = \{x, z\},$$

$$M^*(\Lambda, \sigma) = \{x, y, z\}$$

so, $\xi(M)(\Lambda, \sigma) \subseteq M^*(\Lambda, \sigma)$.

Corollary 2.5. *Local function and weak semi-local function can not be compared to each other according to the subset relation, when considering Example 2.3 and Example 2.4.*

Proposition 2.6. *Let Λ and Δ be two ideals on a set X. Also, let M and N be two subsets in a topological space (X, σ). Then the following properties are hold:*

(1) *If $M \subseteq N$, then $\xi(M) \subseteq \xi(N)$*
(2) *If $\Lambda \subseteq \Delta$, then $\xi(M)(\Delta) \subseteq \xi(M)(\Lambda)$*
(3) *$\xi(M) = sCl(\xi(M)) \subseteq Cl_\theta(M)$*
(4) *If $M \in \Lambda$, then $\xi(M) = \varnothing$*

Proof. (1) $M \subseteq N$ and suppose that $x \notin \xi(N)$. There exist a subset $U \in SO(X, x)$ such that $(sCl(U) \cap N) \in \Lambda$. Because of the heredity of the ideal, $(sCl(U) \cap M) \in \Lambda$ and $x \notin \xi(M)$. This proves that $\xi(M) \subseteq \xi(N)$.
(2) Let $x \in \xi(M)(\Delta)$. Then $(sCl(U) \cap M) \notin \Delta$ for every $U \in SO(X, x)$. Since $\Lambda \subseteq \Delta$, $(sCl(U) \cap M) \notin \Lambda$ and hence $x \in \xi(M)(\Lambda)$. Therefore, we have $\xi(M)(\Delta) \subseteq \xi(M)(\Lambda)$.
(3) It is obvious that $\xi(M) \subseteq sCl(\xi(M))$. We only prove that $sCl(\xi(M)) \subseteq \xi(M)$ .Let $x \in sCl(\xi(M))$. Then $\xi(M) \cap U \neq \varnothing$ for every $U \in SO(X, x)$.Therefore, there exist some $y \in \xi(M) \cap U$ and $U \in SO(X, y)$. Since $y \in \xi(M)$, we have $(sCl(U) \cap M) \notin \Lambda$ and $x \in \xi(M)$ from Definition 2.1. Hence, $sCl(\xi(M)) \subseteq \xi(M)$. Consequently, $\xi(M) = sCl(\xi(M))$. Again, let $x \in \xi(M) = sCl(\xi(M))$. $(sCl(U) \cap M) \notin \Lambda$ for every $U \in SO(X, x)$. Since $\sigma_{(x)} \subseteq SO(X, x)$ and $sCl(U) \subseteq Cl(U)$, $(Cl(U) \cap M) \notin \Lambda$ for every $U \in \sigma_{(x)}$. So, $x \in Cl_\theta(M)$.
(4) Let $M \in \Lambda$. Since $X \in SO(X, x)$ for every $x \in X$, $M \cap X = M \in \Lambda$ and hence, $\xi(M) = \varnothing$. □

Lemma 2.7. *Let M be any subset of an ideal topological space (X, σ, Λ). Then*

(1) *If $W \in \sigma_\theta$, then $W \cap \xi(M) = W \cap \xi(W \cap M) \subseteq \xi(W \cap M)$.*
(2) *If $U \in \sigma$, then $U \cap \xi(M) = U \cap \xi(sCl(U) \cap M) \subseteq \xi(sCl(U) \cap M)$.*

Proof. (1) Let $x \in W \cap \xi(M)$. This implies $x \in W$ and $x \in \xi(M)$. Since $W \in \sigma_\theta$, there exist a subset $L \in \sigma$ such that $x \in L \subseteq sCl(L) \subseteq Cl(L) \subseteq W$. Consider the set $V \in SO(X, x)$. Since the intersection of an open and a semi-open set is semi-open, $V \cap L \in SO(X, x)$. $sCl(V \cap L) \cap M \notin \Lambda$, since $x \in \xi(M)$. Therefore, $sCl(V \cap L) \cap M \subseteq sCl(V) \cap (sCl(L) \cap M) \subseteq sCl(V) \cap (W \cap M) \notin \Lambda$ and hence $x \in \xi(W \cap M)$. This show that $W \cap \xi(M) \subseteq \xi(W \cap M)$ and $W \cap \xi(M) \subseteq W \cap \xi(W \cap M)$. Morever, from Proposition 2.6 (1) $\xi(W \cap M) \subseteq \xi(M)$ and $W \cap \xi(W \cap M) \subseteq W \cap \xi(M)$. Consequently, $W \cap \xi(M) = W \cap \xi(W \cap M)$.

(2) Suppose that $x \in U \cap \xi(M)$. In this case, $x \in U$ and $x \in \xi(M)$. Let V be any semi-open set containing x. So, $V \cap U \in SO(X, x)$. Since $x \in \xi(M)$, $sCl(V \cap U) \cap M \notin \Lambda$ and $sCl(V \cap U) \cap M \subseteq (sCl(V) \cap (sCl(U) \cap M)) \notin \Lambda$. Hence, $x \in \xi(sCl(U) \cap M)$ and $\xi(M) \subseteq \xi(sCl(U) \cap M)$. We can write $U \cap \xi(M) \subseteq U \cap \xi(sCl(U) \cap M)$. Since $sCl(U) \cap M \subseteq M$, $\xi(sCl(U) \cap M) \subseteq \xi(M)$ from Proposition 2.6 (1) and we can write $U \cap \xi(sCl(U) \cap M) \subseteq U \cap \xi(M)$ As a result, $U \cap \xi(M) = U \cap \xi(sCl(U) \cap M)$. □

Theorem 2.8. *Let M and N be two subsets of ideal topological space (X, σ, Λ). Then the following properties are hold:*

(1) $\xi(\varnothing) = \varnothing$
(2) $\xi(M \cup N) = \xi(M) \cup \xi(N)$

Proof. (1) This proof is obvious.

(2) Firstly, since $M \subseteq (M \cup N)$ and $N \subseteq (M \cup N)$, we have $\xi(M) \cup \xi(N) \subseteq \xi(M \cup N)$ from Proposition (2.6) (1). We only prove that $\xi(M \cup N) \subseteq \xi(M) \cup \xi(N)$. Let $x \in \xi(M \cup N)$. Then, $sCl(U) \cap (M \cup N) = (sCl(U) \cap M) \cup (sCl(U) \cap N) \notin \Lambda$ for every $U \in SO(X, x)$.Therefore, $(sCl(U) \cap M) \notin \Lambda$ or $(sCl(U) \cap N) \notin \Lambda$. This implies $x \in \xi(M)$ or $x \in \xi(N)$, hence $x \in (\xi(M) \cup \xi(N))$.Moreover, we have $\xi(M \cup N) \subseteq \xi(M) \cup \xi(N)$. Consequently, we obtain $\xi(M \cup N) = \xi(M) \cup \xi(N)$. □

Lemma 2.9. *Let M and N be two subsets of ideal topological space (X, σ, Λ). Then $\xi(M)\backslash\xi(N) = \xi(M\backslash N)\backslash\xi(N) \subseteq \xi(M\backslash N)$.*

Proof. Since $M = (M\backslash N) \cup (N \cap M)$, we have $\xi(M) = \xi(M\backslash N) \cup \xi(N \cap M)$ from Theorem 2.8 (2) .Hence

$$\begin{aligned} \xi(M)\backslash\xi(N) &= [\xi(M\backslash N) \cup \xi(N \cap M)] \cap (X\backslash\xi(N)) \\ &= [\xi(M\backslash N) \cap (X\backslash\xi(N))] \cup [\xi(N \cap M) \cap (X\backslash\xi(N))] \\ &= \xi(M\backslash N) \cap (X\backslash\xi(N)) \\ &= \xi(M\backslash N)\backslash\xi(N) \subseteq \xi(M\backslash N) \end{aligned}$$

□

Corollary 2.10. *Let M and N be two subsets of an ideal topological space (X, σ, Λ). If $N \in \Lambda$, then $\xi(M\backslash N) = \xi(M) = \xi(M \cup N)$.*

Proof. From Theorem 2.6 (4), $\xi(N) = \varnothing$ and from Lemma 2.9, $\xi(M\backslash N) = \xi(M)$. Since Theorem (2.8)(2) and Theorem 2.6 (4), $\xi(M \cup N) = \xi(M) \cup \xi(N) = \xi(M)$. □

We have seen above that weak semi-local functions provides almost every property of local functions.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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