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Conservation Laws for a Model with both Cubic and Quadratic Nonlinearity

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Abstract

In this paper, the conservation laws for a model with both quadratic and cubic nonlinearity

$$m_t = bu_x + \frac{1}{2}a\left[\left(u^2 - u_x^2\right)m\right]_x + \frac{1}{2}c\left(2m \cdot u_x + m_x \cdot u\right); \quad m = u - u_{xx}$$

are considered for the six cases of coefficients. By using a variational derivative approach, conservation laws were constructed. The computations to derive multipliers and conservation law fluxes are conducted by using a Maple-based package which is called GeM.

1. Introduction

In this paper, we consider the conservation laws for the model

$$m_t = bu_x + \frac{1}{2}a\left[\left(u^2 - u_x^2\right)m\right]_x + \frac{1}{2}c\left(2m \cdot u_x + m_x \cdot u\right); \quad m = u - u_{xx}, \tag{1.1}$$

where a, b and c are arbitrary constants. Eq. (1.1) models the one-way propagation of a fluid that lies on a horizontal flat bottom.
Conservation laws, indicating that a certain measurable property (as mass, momentum or charge) of an isolated physical system does not change as the system evolves over time, are of fundamental importance in nonlinear science. The study of the conservation laws of the KdV equation was a milestone in the exploration of some techniques that include Miura transformation, Lax pair, inverse scattering transform, bi-Hamiltonian structures, for solving evolutionary equations [1]. Conservation laws have several applications in the field of differential equations. For example, Lax [2] proved global existence theorems by using conservation laws, DiPerna [3] used extra conservation laws for the decay of shock waves, and stability problems were considered by Benjamin. In [4], they were used for studying cracks and dislocations in elasticity (for more information see [5]). The existence of solitons is also closely related to the existence of an infinite number of conservation laws of partial differential equations and is a predictor of complete integrability.
There are many powerful methods used to find conservation laws such as Laplace's direct technique [6], Noether's theorem [7], the characteristic form (also known as multiplier or integrating factor) given by Steudel [8]. In this paper, we use the multiplier approach among these techniques to derive conservation laws and conserved quantities corresponding to the six different cases of coeffficients of Eq. (1.1). The multiplier approach will be explained in detail in the next section.
The emergence of symbolic computational packages provides great satisfaction in the performance of complex and tedious calculations. Over the past decades, researchers have focused on developing symbolic computational packages working with either *Maple* or *Mathematica* which are based on different approaches to conservation laws. Many computational packages have recently been developed, and we can classify these packages on the environment in which they work in two parts:

1. Packages which are based on *Mathematica* [9] environment: Goktas and Hereman developed **condens.m** [9], Adams and Hereman developed **TransPDEDensity.m** [10], and Poole and Hereman developed **ConservationLawsMD.m** [11].
2. Packages which are based on *Maple* environment: Cheviakov developed **GeM** [12, 13], Anderson and Cheb-Terrab developed **Vessiot suite** [14] , Rocha Filho and Figueiredo developed **SADE** [15].

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GeM package [12] will be used in the present paper to find the conservation laws of Eq. (1.1) in the six different cases of coeffficients. **GeM** package is developed to find the conservation laws and symmetries of differential equations. There exists a determining system for obtaining multipliers (and hence conservation laws) for any partial differential equation. To obtain symmetries, this package, firstly obtain an overdetermined system of determining equations, afterwards this system is simplified by a Rif package routines, and then a *Maple* command gives all symmetry generators of differential equations. For the conservation laws, **GeM** package firstly obtain a determining system for multipliers, afterwards the obtained system is simplified by Rif package to get multipliers. Once the multipliers are obtained, the fluxes are constructed by the direct method, homotopy methods or scaling symmetry formula.
Eq. (1.1) is studied in [16] where they mainly interested in peakon, weak kink ank kink-peakon interactional solutions. To show that Eq. (1.1) is completely integrable, they present the Lax representation, bi-Hamiltonian structure and infinitely many conservation laws for Eq. (1.1). In [16] the conservation laws are obtained explicitly only for $b=0, a\neq 0, c\neq 0$ case.
According to the different cases of coefficients, Eq. (1.1) reduces to the following six cases:

1. Case $(b\neq 0, a\neq 0, c\neq 0)$:
$m_t = bu_x + \frac{1}{2}a\left[\left(u^2-u_x^2\right)m\right]_x + \frac{1}{2}c\,(2m\cdot u_x + m_x\cdot u)$, which is a linear combination of CH and mCH or generalized CH equation, see (Qiao, Xia, and Li [e-print arXiv:1205.2028v3 (2012)]).
2. Case $(b\neq 0, a=0, c=-2)$:
$m_t = bu_x - (2mu_x + m_x u)$,wich is a quadratic nonlinear equation.
3. Case $(b\neq 0, a=-2, c=0)$:
$m_t = bu_x - \left[\left(u^2-u_x^2\right)m\right]_x$, which is a cubic nonlinear equation.
4. Case $(b=0, a\neq 0, c\neq 0)$:
$m_t = \frac{1}{2}a\left[m\left(u^2-u_x^2\right)\right]_x + \frac{1}{2}c\,(2mu_x + m_x u)$, which known as FQXL model.
5. Case $(b=0, a=-2, c=0)$:
$m_t = -(2mu_x + m_x u)$, which is known as Camassa -Holm equation (CH).
6. Case $(b=0, a=0, c=-2)$:
$m_t = -\left[\left(u^2-u_x^2\right)m\right]_x$, which known as modified Camassa-Holm equation (mCH).

In the present paper, the conservation laws of the above six cases of the coefficients are computed explicitly using GeM Maple routines which are based on multiplier method. The multipliers are used to make the system being studied get a divergence form, then by equating this divergence to zero one can obtain a conservation law. For the convenience, these are explained in detail in Section 2. The computations are performed in Section 3, and the results are summarized in the last section.

2. Basic concepts on the method proposed

To compute conserved densities and fluxes, we use a multiplier approach based on the fact that the Euler operator eliminates a total divergence. Let u be dependent variable and t, x be independent variables.

1. Consider an *nth*-order partial differential equation
$$G(t,x,u,u_t,u_x,u_{tt},u_{tx},\dots)=0. \tag{2.1}$$
2. The standard Euler operator E_u is defined as
$$E_u = \frac{\delta}{\delta u} = \frac{\partial}{\partial u} - \mathrm{D}_t\frac{\partial}{\partial u_t} - \mathrm{D}_x\frac{\partial}{\partial u_x} + \mathrm{D}_t^2\frac{\partial}{\partial u_{tt}} + \mathrm{D}_x^2\frac{\partial}{\partial u_{xx}} - \dots.$$
where D_t and D_x are the total differentiation operators which are given by:
$$\begin{aligned}\mathrm{D}_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{tt}\frac{\partial}{\partial u_t} + u_{tx}\frac{\partial}{\partial u_x} + \dots,\\ \mathrm{D}_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x} + u_{xt}\frac{\partial}{\partial u_t} + \dots.\end{aligned}$$
The Euler operator tests whether an expression is a total derivative without using any integration by parts [9].
3. A vector $T=(T^t,T^x)$ is defined as the conserved vector of (2.1) if $\mathrm{D}_tT^t + \mathrm{D}_xT^x = 0$ holds for all solutions of (2.1), where T^t is *conserved density* and T^x is associated *flux*. The divergence expression $\mathrm{D}_tT^t + \mathrm{D}_xT^x = 0$ is called the local conservation law for (2.1).
4. A multiplier λ of (2.1) is a function on the solution space which satisfies
$$\mathrm{D}_tT^t + \mathrm{D}_xT^x = \lambda G \tag{2.2}$$
for any function $u(x,t)$ [17, 18]. The multipliers may be chosen to depend on both the variables (independent and dependent) and derivatives up to a certain order.
5. The multipliers may be determined by taking the variational derivative of (2.2)
$$\frac{\delta}{\delta u}(\lambda G) = 0, \tag{2.3}$$
where $\delta/\delta s$ is the Euler operator defined as above. The conserved vectors can be derived using (2.2) after computing the multipliers from (2.3).

3. Multipliers and conservation laws for the six cases via GeM Maple Routines

In this section, we use a multiplier approach technique for deriving conservation laws and conserved quantities corresponding to the six cases of Eq. (1.1) via Maple-based GeM package.
We start with the sixth case $(b=0, a=0, c=-2)$. Consider the following model with both cubic and quadratic nonlinearity:

$$m_t = bu_x + \frac{1}{2}a\left[\left(u^2-u_x^2\right)m\right]_x + \frac{1}{2}c\left(2m\cdot u_x + m_x\cdot u\right); m = u - u_{xx}, \tag{3.1}$$

By using the following Maple command

```
> restart : with(PDEtools) : declare(m(t,x),u(t,x)) :
> pde := diff(m(t,x),t) − b ∗ diff(u(t,x),x) − (1/2) ∗ a ∗ (diff((u(t,x)^2 − (diff(u(t,x),x))^2) ∗ m(t,x),x)) − (1/2) ∗ c ∗ (2 ∗ m ∗ (diff(u(t,x),x)) + u ∗ (diff(m(t,x),x))) = 0 :
> m(t,x) = u(t,x) − (diff(u(t,x),x,x)) :
> CH := eval(pde,{b = 0,a = 0,c = −2,m(t,x) = u(t,x) − (diff(u(t,x),x,x))});
```

we can rewrite equation (3.1) as follows:

$$CH := u_t - u_{txx} + 2\left(u-u_{xx}\right)u_x + u\left(u_x - u_{xxx}\right) = 0 \tag{3.2}$$

where $u = u(t,x)$. We will explain the case $(b=0, a=0, c=-2)$ in detail along with GeM Maple routines given in [12, 13]. Dependent and independent variables and Eq. (3.2) can be defined in GeM by the following commands:

```
> read("H:/gem32_12.mpl") :
> with(linalg) : With(GeM);
> gem_decl_vars(indeps = [t,x],deps = [u(t,x)], freeconst = [b,a,c]);
> gem_decl_eqs([diff(u(t,x),t) − (diff(u(t,x),x,x,t)) + ((2 ∗ (u(t,x) − (diff(u(t,x),x,x)))) ∗ (diff(u(t,x),x)) + (diff(u(t,x),x) − (diff(u(t,x),x,x,x))) ∗ u(t,x)) = 0],solve_for = [diff(u(t,x),t)]);
```

Let us take the multipliers as $\lambda = \lambda\left(t,x,u,u_t,u_x,u_{xx},u_{xxx}\right)$. The Maple routines to be used in GeM for which the multipliers will be obtained from are

```
> det_CH := gem_conslaw_det_eqs([t,x,u(t,x),diff(u(t,x),t),diff(u(t,x),x),diff(u(t,x),x,x),diff(u(t,x),x,x,x)]);
> CL_CH_mult := gem_conslaw_multipliers();
> simpli_CH := DEtools[rifsimp](det_CH,CL_CH_mult,mindim = 1);
```

For the simplified form of multipliers, the determining equations are

$$\lambda_{uu} = \frac{3\lambda_{u_{xx}}}{2u-2u_{xx}}, \lambda_{uu_{xx}} = -\frac{3\lambda_{u_{xx}}}{2u-2u_{xx}}, \lambda_{u_{xx}u_{xx}} = \frac{3\lambda_{u_{xx}}}{2u-2u_{xx}}, \lambda_t = 0, \lambda_x = 0, \lambda_{u_t} = 0, \lambda_{u_x} = 0, \lambda_{u_{xxx}} = 0 \tag{3.3}$$

The following Maple command is used to solve the system (3.3)

```
> multipliers_CH_sol := pdsolve(simpli_CH[Solved]);
```

which yields

$$\lambda\left(t,x,u,u_t,u_x,u_{xx},u_{xxx}\right) = \_C2u + \_C3 + \frac{\_C1}{\sqrt{-u+u_{xx}}}.$$

Here $\_C1, \_C2$ and $\_C3$ are arbitrary constants. There arise three linearly independent conservation laws from the following multipliers

$$\lambda^{(1)} = 1, \lambda^{(2)} = u, \lambda^{(3)} = \frac{1}{\sqrt{-u+u_{xx}}} \tag{3.4}$$

The next task is the construction of conservation laws from the multipliers given in (3.4). The *direct method* is used to compute the flux expressions in the Maple command

```
> gem_get_CL_fluxes(multipliers_CH_sol);
```

.
We have the conservation law fluxes which are presented in the following table for the multipliers (3.4) :

Case	Multiplier	Fluxes
$b=0$, $a=0$, $c=-2$	$\lambda^{(1)}=1$	$T^t = u - u_{xx}$ $T^x = \frac{3}{2}u^2 - uu_{xx} - \frac{1}{2}u_x^2$
	$\lambda^{(2)}=u$	$T^t = \frac{1}{2}u^2 - uu_{xx} - \frac{1}{2}u_x^2$ $T^x = u^3 - u_{xx}u^2 + u_tu_x$
	$\lambda^{(3)} = \frac{1}{\sqrt{-u+u_{xx}}}$	$T^t = -2\sqrt{-u+u_{xx}}$ $T^x = -2\sqrt{-u+u_{xx}}u$

The homotopy formulas will be employed for the fluxes since the multipliers do not contain arbitrary constants. The following Maple command is used for *first homotopy formula*

```
> gem_get_CL_fluxes(multipliers_CH_sol,method = "Homotopy1");
```

to get conservation law fluxes

Case	Multiplier	Fluxes
$b=0$ $a=0$ $c=-2$	$\lambda^{(1)}=1$	$T^t=u-u_{xx}$ $T^x=\frac{3}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$
	$\lambda^{(2)}=u$	$T^t=\frac{u(u-u_{xx})}{2}$ $T^x=u^3-u_{xx}u^2-\frac{1}{2}u_{tx}u+\frac{1}{2}u_xu_t$
	$\lambda^{(3)}=\frac{1}{\sqrt{-u+u_{xx}}}$	$T^t=-2\sqrt{-2u+2u_{xx}}$ $T^x=-2u\sqrt{-2u+2u_{xx}}$

For *second homotopy* formula, the Maple command

```
> gem_get_CL_fluxes(multipliers_CH_sol, method = "Homotopy2");
```

yields the expressions for conservation law fluxes which are presented in the following table :

Case	Multiplier	Fluxes
$b=0$ $a=0$ $c=-2$	$\lambda^{(1)}=1$	$T^t=u-\frac{u_{xx}}{3}$ $T^x=\frac{3}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2-\frac{2}{3}u_{tx}$
	$\lambda^{(2)}=u$	$T^t=\frac{1}{2}u^2-\frac{1}{3}uu_{xx}+\frac{1}{6}u_x^2$ $T^x=u^3-u_{xx}u^2-\frac{2}{3}uu_{tx}+\frac{1}{3}u_tu_x$
	$\lambda^{(3)}=\frac{1}{\sqrt{-u+u_{xx}}}$	$T^t=-\frac{2\sqrt{-u+u_{xx}}\left(u^3+\left(-\frac{13u_{xx}}{6}-\frac{u_{xxxx}}{6}\right)u^2+\left(\frac{3u_{xx}^2}{2}+\frac{u_{xx}u_{xxxx}}{6}-\frac{5(u_x-\frac{3u_{xxx}}{5})(u_x-u_{xxx})}{12}\right)u+\frac{(-2u_{xx}^2+u_x(u_x-u_{xxx}))u_{xx}}{6}\right)}{(u-u_{xx})^3}$ $T^x=-\frac{1}{(u-u_{xx})^3}\left(2\sqrt{-u+u_{xx}}\left(u^4-3u^3u_{xx}+\left(3u_{xx}^2-\frac{5u_{tx}}{6}+\frac{u_{txxx}}{6}\right)u^2-\frac{u_{xx}(4u_{xx}u_{tx}+(u_x+u_{xx})u_t-2u_{txx}u_x)}{6}\right)\right)$

Now, by repeating the previous processes for the cases (1,2,3,4 and 5), we find multipliers and conserved vectors using the *direct method* and *first homotopy method*, which are given in Tables 1, 2, respectively.

Case	Multiplier	Fluxes
$b\neq 0$ $a\neq 0$ $c\neq 0$	$\lambda^{(1)}=1$	$T^t=u-u_{xx}$ $T^x=-\frac{au^3}{2}+\frac{(2au_{xx}-3c)u^2}{4}+\frac{\left(2au_x^2+2cu_{xx}-4b\right)u}{4}-\frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$
	$\lambda^{(2)}=u$	$T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$ $T^x=-\frac{3au^4}{8}+\frac{(4au_{xx}-4c)u^3}{8}+\frac{\left(2au_x^2+4cu_{xx}-4b\right)u^2}{8}-\frac{u_x^2u_{xx}au}{2}+\frac{u_x\left(u_x^3a+8u_t\right)}{8}$
	$\lambda^{(3)}=\frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}}$	$T^t=-2\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$ $T^x=\left(au^2-au_x^2+cu\right)\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$
$b\neq 0$ $a=0$ $c=-2.$	$\lambda^{(1)}=1$	$T^t=u-u_{xx}$ $T^x=\frac{3u^2}{2}+\frac{(-2b-2u_{xx})u}{2}-\frac{u_x^2}{2}$
	$\lambda^{(2)}=u$	$T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$ $T^x=u^3+\frac{(-b-2u_{xx})u^2}{2}+u_tu_x$
	$\lambda^{(3)}=\frac{2}{\sqrt{2b-4u+4u_{xx}}}$	$T^t=-\sqrt{2b-4u+4u_{xx}}$ $T^x=-\sqrt{2b-4u+4u_{xx}}u$
$b\neq 0$ $a=-2$ $c=0$	$\lambda^{(1)}=1$	$T^t=u-u_{xx}$ $T^x=u^3-u_{xx}u^2+\left(-u_x^2-b\right)u+u_x^2u_{xx}$
	$\lambda^{(2)}=u$	$T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$ $T^x=\frac{3u^4}{4}-u_{xx}u^3+\frac{\left(-2u_x^2-2b\right)u^2}{4}+u_x^2u_{xx}u-\frac{u_x^4}{4}+u_tu_x$
	$\lambda^{(3)}=\frac{-u+u_{xx}}{\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}$	$T^t=-\frac{\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}{2}$ $T^x=-\frac{\left(u^2-u_x^2\right)\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}{2}$
$b=0$ $a\neq 0.$ $c\neq 0$	$\lambda^{(1)}=1$	$T^t=u-u_{xx}$ $T^x=-\frac{au^3}{2}+\frac{(2au_{xx}-3c)u^2}{4}+\frac{\left(2au_x^2+2cu_{xx}\right)u}{4}-\frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$
	$\lambda^{(2)}=u$	$T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$ $T^x=-\frac{3au^4}{8}+\frac{(4au_{xx}-4c)u^3}{8}+\frac{\left(2au_x^2+4cu_{xx}\right)u^2}{8}-\frac{u_x^2u_{xx}au}{2}+\frac{u_x\left(2u_x^3a+8u_t\right)}{8}$
	$\lambda^{(3)}=\frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})((u-u_{xx})a+c)}}$	$T^t=-2\sqrt{(u-u_{xx})((u-u_{xx})a+c)}$ $T^x=\sqrt{(u-u_{xx})((u-u_{xx})a+c)}\left(au^2-au_x^2+cu\right)$
$b=0$ $a=-2$ $c=0$	$\lambda^{(1)}=1$	$T^t=u-u_{xx}$ $T^x=(u-u_x)(u+u_x)(u-u_{xx})$
	$\lambda^{(2)}=u$	$T^t=\frac{1}{2}u^2-uu_{xx}-\frac{1}{2}u_x^2$ $T^x=-\frac{u_x^4}{4}-\frac{u(u-2u_{xx})u_x^2}{2}+u_tu_x+\frac{3u^4}{4}-u_{xx}u^3$
	$\lambda^{(3)}=\frac{1}{(u-u_{xx})^2}$	$T^t=-\frac{1}{u-u_{xx}}$ $T^x=\frac{3u^2-4uu_{xx}+u_x^2}{u-u_{xx}}$

Table 1: Multipliers and conserved vectors using direct method

Case	Multiplier	Fluxes
$b \neq 0$, $a \neq 0$, $c \neq 0$	$\lambda^{(1)} = 1$	$T^t = u - u_{xx}$ $T^x = -\frac{au^3}{2} + \frac{(2au_{xx}-3c)u^2}{4} + \frac{\left(2au_x^2+2cu_{xx}-4b\right)u}{4} - \frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$
	$\lambda^{(2)} = u$	$T^t = \frac{u(u-u_{xx})}{2}$ $T^x = -\frac{3au^4}{8} + \frac{(4au_{xx}-4c)u^3}{8} + \frac{\left(2au_x^2+4cu_{xx}-4b\right)u^2}{8} + \frac{\left(-4u_x^2u_{xx}a-4u_{tx}\right)u}{8} + \frac{u_x\left(u_x^3a+4u_t\right)}{8}$
	$\lambda^{(3)} = \frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}}$	$T^t = 2\sqrt{b} - 2\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$ $T^x = \left(au^2 - au_x^2 + cu\right)\sqrt{(u-u_{xx})^2a+cu-cu_{xx}+b}$
$b \neq 0$, $a = 0$, $c = -2.$	$\lambda^{(1)} = 1$	$T^t = u - u_{xx}$ $T^x = \frac{3u^2}{2} + \frac{(-2b-2u_{xx})u}{2} - \frac{u_x^2}{2}$
	$\lambda^{(2)} = u$	$T^t = \frac{u(u-u_{xx})}{2}$ $T^x = u^3 + \frac{(-b-2u_{xx})u^2}{2} - \frac{uu_{tx}}{2} + \frac{u_tu_x}{2}$
	$\lambda^{(3)} = \frac{2}{\sqrt{2b-4u+4u_{xx}}}$	$T^t = \sqrt{2}\sqrt{b} - \sqrt{2b-4u+4u_{xx}}$ $T^x = -\sqrt{2b-4u+4u_{xx}}u$
$b \neq 0$, $a = -2$, $c = 0$	$\lambda^{(1)} = 1$	$T^t = u - u_{xx}$ $T^x = u^3 - u_{xx}u^2 + \left(-u_x^2 - b\right)u + u_x^2u_{xx}$
	$\lambda^{(2)} = u$	$T^t = \frac{u(u-u_{xx})}{2}$ $T^x = \frac{3u^4}{4} - u_{xx}u^3 + \frac{\left(-2u_x^2-2b\right)u^2}{4} + \frac{\left(4u_x^2u_{xx}-2u_{tx}\right)u}{4} - \frac{u_x^4}{4} + \frac{u_tu_x}{2}$
	$\lambda^{(3)} = \frac{-u+u_{xx}}{\sqrt{2u^2-4uu_{xx}+2uu_{xx}^2-b}}$	$T^t = \frac{\sqrt{-b}}{2} - \frac{\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}{2}$ $T^x = \frac{(u-u_x)(u+u_x)\left(-2u^2+4uu_{xx}-2u_{xx}^2+b\right)}{2\sqrt{2u^2-4uu_{xx}+2u_{xx}^2-b}}$
$b = 0$, $a \neq 0.$, $c \neq 0$	$\lambda^{(1)} = 1$	$T^t = u - u_{xx}$ $T^x = -\frac{au^3}{2} + \frac{(2au_{xx}-3c)u^2}{4} + \frac{\left(2au_x^2+2cu_{xx}\right)u}{4} - \frac{\left(au_{xx}-\frac{c}{2}\right)u_x^2}{2}$
	$\lambda^{(2)} = u$	$T^t = \frac{u(u-u_{xx})}{2}$ $T^x = -\frac{3au^4}{8} + \frac{(4au_{xx}-4c)u^3}{8} + \frac{\left(2au_x^2+4cu_{xx}\right)u^2}{8} + \frac{\left(-4au_x^2u_{xx}-4u_{tx}\right)u}{8} + \frac{u_x\left(au_x^3+4u_t\right)}{8}$
	$\lambda^{(3)} = \frac{(-2u+2u_{xx})a-c}{\sqrt{(u-u_{xx})^2a+cu-cu_{xx}}}$	$T^t = -2\sqrt{(u-u_{xx})^2a+cu-cu_{xx}}$ $T^x = \left(au^2 - au_x^2 + cu\right)\sqrt{(u-u_{xx})^2a+cu-cu_{xx}}$
$b = 0$, $a = -2$, $c = 0$	$\lambda^{(1)} = 1$	$T^t = u - u_{xx}$ $T^x = u^3 - u_{xx}u^2 - u_x^2u + u_x^2u_{xx}$
	$\lambda^{(2)} = u$	$T^t = \frac{u(u-u_{xx})}{2}$ $T^x = \frac{3u^4}{4} - u_{xx}u^3 - \frac{u_x^2u^2}{2} + \frac{\left(8u_x^2u_{xx}-4u_{tx}\right)u}{8} + \frac{u_x\left(-2u_x^3+4u_t\right)}{8}$
	$\lambda^{(3)} = \frac{4u-4u_{xx}}{\sqrt{-2(u-u_{xx})^2}}$	$T^t = -2\sqrt{-2(u-u_{xx})^2}$ $T^x = \left(-2u^2 + 2u_x^2\right)\sqrt{-2(u-u_{xx})^2}$

Table 2: Multipliers and conserved vectors using the first homotopy formula

4. Conclusion

The conservation laws for Eq. (1.1) with both quadratic and cubic nonlinearity for the six cases of coefficients $((b \neq 0, a \neq 0, c \neq 0), (b \neq 0, a = 0, c = -2), (b \neq 0, a = -2, c = 0), (b = 0, a \neq 0, c \neq 0), (b = 0, a = -2, c = 0)$ and $(b = 0, a = 0, c = -2))$ are constructed via a Maple package called GeM. The conservation laws $\rho_t = F_x$ of Eq. (1.1) were obtained in [16]. But, they were given explicitly only for $b = 0, a \neq 0, c \neq 0$ case. In the present paper, the conservation laws of all the above six cases are computed explicitly. Three multipliers are obtained by defining the multipliers of the form $\lambda = \lambda(t, x, u, u_t, u_x, u_{xx}, u_{xxx})$ in GeM Maple routines. More multipliers may be computed in the case of including higher order derivatives in the multipliers. Direct method and homotopy formula are used to compute the fluxes for each cases. The fluxes obtained here can be used to construct the solutions of Eq. (1.1).

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