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A Numerical Approach for Solving the System of Differential Equations Related to the Spherical Curves in Euclidean 3-Space

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Abstract. In 1971, integral form of spherical curve in 3-dimensional Euclidean space was given in [3]. The explicit characterization of the spherical curves in n-dimensional Euclidean space was given in [12]. Morever, the position vector of spherical curves in Euclidean 3-space was determined in [10]. In the present work, a) it is given the system of differential equations of the spherical curves in 3-dimensional Euclidean space; b) it is shown that the numerical solutions of this system of differential equations are obtained in the truncated Taylor series form by using Taylor matris collocation method; c) an example together with error analysis are given to demonstrate the validity and applicability of present method.

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1. Introduction

Curves are used in many fields such as mechanics, kinematics and differential geometry. In differential geometry, the Frenet formulas express the kinematic properties of a particle along a continuous differentible curve in 3-dimensional Euclidean space, or the geometric properties of the curve itself irrespective of any motion. Therefore, in principle, every geometric problem about curves can be solved by means of the Frenet formulas. Important classes of Frenet curves are helices curves, spherical curves, and rectifying curves [4, 7, 13, 21].

A curve which lie upon a sphere is a called a spherical curve. In books on elementary differential geometry, the condition for a curve to be a spherical curve is usually given in the form

$$\frac{1}{\kappa}\tau + \left[\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right]' = 0 \tag{1.1}$$

where κ and τ are its first curvature function and second curvature function, respectively. Obviously, condition Eq. (1.1) has a meaning only if κ and τ are nowhere zero, and it is only under this precondition that Eq. (1.1) is a necessary and sufficient condition for a curve to be a spherical curve [8, 19]. In 1963, Wong developed a more comprehensive formula for a curve to be a spherical curve [20]. Then, Breuer and Gottlieb gave a solution for the differential equation which characterized the spherical curves in 3-dimensional Euclidean space and they obtained the equivalent of the radius of curvature of the curve in terms of its torsion [3]. Thereafter, in 1974, Özdamar and Hacısalihoğlu gave certain

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characterizations for the spherical curves in n-dimensional Euclidean space [15]. Afterwards, in 1989, Sezer gave a differential and an integral characterization for the spherical curves in 4-dimensional Euclidean space [16].

In recent years, Sezer et al. developed matrix collocation methods based on Taylor [1], Lucas [5], Hermite [2] and Bernstein [14] polynomials to find the approximate solutions of third order linear differential equations with variable coefficients characterizing spherical curve.

In this study, we present that the position vector of a curve lying on a sphere satisfies a linear system of differential equation with variable coefficient. Then, we obtain the approximate solutions of this system of differential equation by using Taylor matrix collocation method.

2. Preliminaries

In this section, we briefly introduce some fundamental concepts on differential geometry of the spherical curves in 3-dimensional Euclidean space.

In 3-dimensional Euclidean space scalar product is given by

$$\langle\, ,\rangle = dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is rectangular coordinate system in E^3. The norm of the vector $\vec{x} \in E^3$ is described by $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x}\rangle}$ [11, 14].

Definition 2.1 ([13]). Let $\alpha : I \to R^3$ be a unit speed curve, so $|\alpha'(s)| = 1$ for each s in I.

Definition 2.2 ([9]). Let $\alpha : I \to R^3$ be a unit speed curve with $\kappa(s) > 0$. Then

$$\vec{t} = \alpha', \qquad \vec{n} = \frac{\vec{t}'}{\kappa}, \qquad \vec{b} = \vec{t} \times \vec{n}$$

are called the unit tangent vector field, the principal normal vector field and the binormal vector field, respectively. The triple $(\vec{t}, \vec{n}, \vec{b})$ is called the Frenet frame field on α.

Theorem 2.3 ([13]). *$\alpha : I \to E^3$ is a unit speed curve with curvature $\kappa > 0$ and torsion τ, then*

$$\begin{aligned} \vec{t}' &= \kappa\vec{n} \\ \vec{n}' &= -\kappa\vec{t} + \tau\vec{b} \\ \vec{b}' &= -\tau\vec{n} \end{aligned} \quad . \tag{2.1}$$

3. Spherical Curves

Let $\alpha(s)$ be the position vector on a sphere with the origin center and radius r. We assume $\alpha(s)$ is suitably smooth curve. $(\vec{t}, \vec{n}, \vec{b})$ will denote as usual, the moving trihedral of the curve and κ, τ the curvature and torsion, recpectively. It is known (see, for instance [1, 2, 5, 14]) that $\alpha(s)$ satisfies a third order differential equation (in s) with coefficients involving κ and τ. However, as we shall see, a simplier equation is possible when α lies on a sphere centered at origin. Since $(\vec{t}, \vec{n}, \vec{b})$ is an orthonormal system, we may write

$$\alpha(s) = \lambda_1(s)\vec{t}(s) + \lambda_2(s)\vec{n}(s) + \lambda_3(s)\vec{b}(s).$$

Theorem 3.1 ([10]). *The position vector of the spherical curve in Euclidean 3-space is the equation where*

$$\lambda_1(s) = 0, \qquad \lambda_2(s) = -\frac{1}{\kappa(s)}, \qquad \lambda_3(s) = \left(-\frac{1}{\kappa(s)}\right)' \frac{1}{\tau(s)}.$$

Theorem 3.2 ([12]). *Let $\alpha : I \subset R \to E^3$ be a spherical curve. If the curvatures of α and radius of the sphere are κ, τ and r, respectively, then*

$$\left(\frac{1}{\kappa(s)}\right)^2 + \left(\frac{1}{\tau(s)}\left(\frac{1}{\kappa(s)}\right)'\right)^2 = r^2, \qquad (\kappa \neq 0,\ \ \tau \neq 0,\ \ r \neq 0).$$

Theorem 3.3. *The system of differential equations characterizing a unit speed spherical curve in Euclidean 3-space is*

$$\begin{aligned} \lambda_1'(s) &= \kappa(s)\lambda_2(s) + 1, \\ \lambda_2'(s) &= -\kappa(s)\lambda_1(s) + \tau(s)\lambda_3(s), \\ \lambda_3'(s) &= -\tau(s)\lambda_2(s). \end{aligned} \tag{3.1}$$

Proof. Let $\alpha(s)$ be a unit speed spherical curve in Euclidean 3-space. The position vector of spherical curve can be written as

$$\alpha(s) = \lambda_1(s)\vec{t}(s) + \lambda_2(s)\vec{n}(s) + \lambda_3(s)\vec{b}(s).$$

If we derivate this equation both sides with respect to s, then we get

$$\alpha'(s) = \lambda_1'(s)\vec{t}(s) + \lambda_1(s)\vec{t}'(s) + \lambda_2'(s)\vec{n}(s) + \lambda_2(s)\vec{n}'(s) + \lambda_3'(s)\vec{b}(s) + \lambda_3(s)\vec{b}'(s). \tag{3.2}$$

By substituting Eq. (2.1) into Eq. (3.2), we obtain

$$\begin{aligned}\vec{t}(s) = {} & \lambda_1'(s)\vec{t}(s) + \lambda_1(s)\kappa(s)\vec{n}(s) + \lambda_2'(s)\vec{n}(s) + \lambda_2(s)\left(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)\right) + \lambda_3'(s)\vec{b}(s) \\ & +\lambda_3(s)\left(-\tau(s)\vec{n}(s)\right).\end{aligned} \tag{3.3}$$

Thus, from Eq. (3.3), we can write the following system of differential equations

$$\begin{aligned}\lambda_1'(s) &= \kappa(s)\lambda_2(s) + 1,\\ \lambda_2'(s) &= -\kappa(s)\lambda_1(s) + \tau(s)\lambda_3(s),\\ \lambda_3'(s) &= -\tau(s)\lambda_2(s).\end{aligned}$$

□

4. Taylor Matrix Collocation Method

In this study, we consider the system of differential equations (3.1)

$$\left.\begin{aligned}\lambda_1'(s) &= \kappa(s)\lambda_2(s) + 1\\ \lambda_2'(s) &= -\kappa(s)\lambda_1(s) + \tau(s)\lambda_3(s)\\ \lambda_3'(s) &= -\tau(s)\lambda_2(s)\end{aligned}\right., \qquad a \le s \le b$$

under the initial conditions

$$\lambda_1(a) = \mu_1, \qquad \lambda_2(a) = \mu_2, \qquad \lambda_3(a) = \mu_3 \tag{4.1}$$

where μ_1, μ_2 and μ_3 are real constants. We use Taylor matrix collocation method for obtaining the approximate solutions of the system of differential equations. We assume the approximate solution of problem (3.1) and (4.1) in the truncated Taylor series form [6, 17, 18]

$$\lambda_i(s) \cong \lambda_{i,N}(s) = \sum_{k=0}^{N} a_{i,k}s^k, \qquad (i = 1, 2, 3); \tag{4.2}$$

where $\lambda_{i,N}(s)$, $(i = 1, 2, 3)$ are the approximate solutions of Eq. (3.1); N is chosen as any positive integer such that $N \geq 2$. On the other hand, we can write Eq. (4.2) in the matrix form

$$\lambda_{i,N}(s) = \mathbf{X}(s)\mathbf{A}_i, \qquad (i = 1, 2, 3); \tag{4.3}$$

$$\mathbf{X}(s) = \begin{bmatrix} 1 & s & \cdots & s^N \end{bmatrix}, \qquad \mathbf{A}_i = \begin{bmatrix} a_{i,0} & a_{i,1} & \cdots & a_{i,N} \end{bmatrix}^T.$$

Therefore, matrices $\lambda_{i,N}(s)$, $(i = 1, 2, 3)$ can be expressed as

$$\mathbf{\Lambda}(s) = \overline{\mathbf{X}}(s)\mathbf{A};$$

$$\mathbf{\Lambda}(s) = \begin{bmatrix} \lambda_{1,N}(s) & \lambda_{2,N}(s) & \lambda_{3,N}(s) \end{bmatrix}, \ \overline{\mathbf{X}}(s) = \begin{bmatrix} \mathbf{X}(s) & 0 & \cdots & 0 \\ 0 & \mathbf{X}(s) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{X}(s) \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_N \end{bmatrix}^T.$$

It is obviously seen that the relation between the matrix $\mathbf{X}(s)$ and its derivative $\mathbf{X}'(s)$ is

$$\mathbf{X}'(s) = \mathbf{X}(s)\mathbf{B} \tag{4.4}$$

where

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By substituting the relations (4.3) and (4.4), we obtain the following matrix relation

$$\lambda'_{i,N}(s) = \mathbf{X}(s)\mathbf{B}\mathbf{A}_i. \tag{4.5}$$

Therefore, the matrices $\lambda'_{i,N}(s)$, $(i = 1, 2, 3)$ can be expressed as

$$\mathbf{\Lambda}'(s) = \overline{\mathbf{X}}(s)\overline{\mathbf{B}}\mathbf{A}; \tag{4.6}$$

$$\mathbf{\Lambda}'(s) = \begin{bmatrix} \lambda'_{1,N}(s) & \lambda'_{2,N}(s) & \lambda'_{3,N}(s) \end{bmatrix}, \qquad \overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & 0 & \cdots & 0 \\ 0 & \mathbf{B} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{B} \end{bmatrix}.$$

To find the approximate solution (4.2), we can use the collocation points defined by

$$s_\gamma = a + \frac{b-a}{N}\gamma, \quad \gamma = 0, 1, 2, ..., N. \tag{4.7}$$

On the other hand, Eq. (3.1) can be written as follows;

$$\sum_{j=1}^{3}\sum_{k=0}^{1} P^k_{ij}(s)\lambda_j^{(k)}(s) = g_i(s), \qquad (i = 1, 2, 3). \tag{4.8}$$

By substituting the relations (4.4)-(4.7) into Eq. (4.8), we have the fundamental matrix

$$\sum_{k=0}^{1} \mathbf{P}_k\overline{\mathbf{B}}^k\mathbf{X}\mathbf{A} = \mathbf{G} \Rightarrow \left\{\mathbf{P}_0\mathbf{X} + \mathbf{P}_1\overline{\mathbf{B}}\mathbf{X}\right\}\mathbf{A} = \mathbf{G}. \tag{4.9}$$

Here,

$$\mathbf{P}_k = \begin{bmatrix} \mathbf{P}_k(s_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_k(s_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_k(s_N) \end{bmatrix}; \qquad \mathbf{P}_k(s) = \begin{bmatrix} \mathbf{P}^k_{11}(s) & \mathbf{P}^k_{12}(s) & \mathbf{P}^k_{13}(s) \\ \mathbf{P}^k_{21}(s) & \mathbf{P}^k_{22}(s) & \mathbf{P}^k_{23}(s) \\ \mathbf{P}^k_{31}(s) & \mathbf{P}^k_{32}(s) & \mathbf{P}^k_{33}(s) \end{bmatrix},$$

$$\mathbf{X} = \begin{bmatrix} \overline{\mathbf{X}}(s_0) & \overline{\mathbf{X}}(s_1) & \cdots & \overline{\mathbf{X}}(s_N) \end{bmatrix}^T,$$

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}(s_0) & \mathbf{G}(s_1) & \cdots & \mathbf{G}(s_N) \end{bmatrix}^T; \qquad \mathbf{G} = \begin{bmatrix} g_1(s) & g_2(s) & g_3(s) \end{bmatrix}^T.$$

The matrix equation (4.9) can be written in the form

$$\mathbf{W}\mathbf{A} = \mathbf{G} \quad \text{or} \quad \begin{bmatrix} \mathbf{W}; \mathbf{G} \end{bmatrix}; \quad \mathbf{W} = \mathbf{P}_0\mathbf{X} + \mathbf{P}_1\overline{\mathbf{B}}\mathbf{X}. \tag{4.10}$$

By using the conditions (4.1) and the relation (4.6), the matrix form for the conditions is obtained as

$$\overline{\mathbf{X}}(a)\mathbf{A} = \boldsymbol{\mu}; \;\; \boldsymbol{\mu} = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3 \end{bmatrix}^T. \tag{4.11}$$

Hence, the matrix equation (4.11) can be written in the form

$$\mathbf{M}\mathbf{A} = \boldsymbol{\mu} \quad \text{or} \quad \begin{bmatrix} \mathbf{M}; \boldsymbol{\mu} \end{bmatrix}; \quad \mathbf{M} = \overline{\mathbf{X}}(a). \tag{4.12}$$

In order to obtain the solution of Eq. (3.1) under the initial conditions (4.1), following the augmented matrix is constructed by replacing any rows of the matrix (4.10) with rows of matrix (4.12); so we have the new augmented matrix

$$\widetilde{\mathbf{W}}\mathbf{A} = \widetilde{\mathbf{G}} \quad \text{or} \quad \begin{bmatrix} \widetilde{\mathbf{W}}; \widetilde{\mathbf{G}} \end{bmatrix}.$$

If $rank(\widetilde{\mathbf{W}}) = rank(\widetilde{\mathbf{W}}; \widetilde{\mathbf{G}}) = 3(N+1)$, the solution of the augmented matrix $\begin{bmatrix} \widetilde{\mathbf{W}}; \widetilde{\mathbf{G}} \end{bmatrix}$ is $\mathbf{A} = (\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}$ and $\mathbf{A}$ is uniquely determined. In this way, the unknown coefficients are obtained and the approximate solutions $\lambda_{i,N}(s)$, $(i = 1, 2, 3)$ are found in the form (4.2).

5. Residual Error Estimation

Since the truncated Taylor series are the approximate solutions of Eq. (3.1), when the approximate solution $\lambda_{i,N}(s)$, $(i = 1, 2, 3)$ and their derivatives are substituted in Eq. (3.1), the resulting equation must be satisfied approximately; that is, for, $a \le s_\gamma \le b$:

$$R_{i,N}(s_\gamma) = \left| \sum_{j=1}^{3} \sum_{k=0}^{1} P_{ij}^{k}(s_\gamma)\lambda_{j,N}^{(k)}(s_\gamma) - g_i(s_\gamma) \right| \cong 0, \quad (i = 1, 2, 3)$$

and

$$R_{i,N}(s_\gamma) \le 10^{-k_\gamma} \qquad (k_\gamma \in \mathbb{Z}^+).$$

If $\max 10^{-k_\gamma} = 10^{-k}$ is prescribed, then the truncation limit N is increased until the difference $R_{i,N}(s_\gamma)$ at each of the points becomes smaller than the prescribed 10^{-k}. On the other hand, by means of residual function $R_{i,N}(s)$ and the mean value of function $\left|R_{i,N}(s)\right|$, the accuracy of the solution can be controlled and the error can be estimated. For this aim, by using linear operator L and mean value theorem, the upper bound of the mean error $\overline{R_N}$ can be estimated as follows [2, 5, 6, 9, 11, 14, 18]:

$$R_{i,N}(s) = L\left[\ \lambda_{i,N}(s)\ \right] - g_i(s); \qquad L\left[\ \lambda_i(s)\ \right] = g_i(s), \ \ (i = 1, 2, 3)$$

$$\left| \int_a^b R_{i,N}(s)ds \right| \le \int_a^b \left|R_{i,N}(s)\right| ds, \ \ s \in \left[\ a, b\ \right]$$

$$R_{i,N}(s_0) = \frac{\int_a^b R_{i,N}(s)ds}{(b-a)}; \ \ s_0 \in \left[\ a, b\ \right]$$

from this relations

$$\left| \int_a^b R_{i,N}(s)ds \right| = \left|R_{i,N}(s_0)\right| |(b-a)|$$

$$\left|R_{i,N}(s_0)\right| |(b-a)| \le \int_a^b \left|R_{i,N}(s)\right| ds$$

$$\left|R_{i,N}(s_0)\right| \le \frac{\int_a^b \left|R_{i,N}(s)\right| ds}{(b-a)} = \overline{R_{i,N}}.$$

Also, absolute error is used for measuring errors. If $\lambda_{i,N}(s)$, $(i = 1, 2, 3)$ is approximation to $\lambda_i(s)$, $(i = 1, 2, 3)$, then the absolute error is

$$e_{i,N}(s) = \left|\lambda_i(s) - \lambda_{i,N}(s)\right|, \ \ (i = 1, 2, 3).$$

6. Numerical Example

In this section, we give an example to illustrate the efficiency of the approximation method based on Taylor polynomials used to find approximate solutions of the system of differential equations characterizing spherical curve.

Let us consider the curve $\alpha : \left[0, \frac{3\pi}{10}\right] \to E^3$ given by

$$\alpha(s) = \left(\sqrt{1-s^2}\cos\left(\sqrt{2}\arcsin(s)\right) + \frac{s\sin\left(\sqrt{2}\arcsin(s)\right)}{\sqrt{2}}, -\sqrt{1-s^2}\sin\left(\sqrt{2}\arcsin(s)\right) + \frac{s\cos\left(\sqrt{2}\arcsin(s)\right)}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right).$$

This curve is unit speed curve which lie on unit sphere. Curvature and torsion of the this curve are as follows

$$\kappa(s) = \frac{1}{\sqrt{1-s^2}}, \qquad \tau(s) = -\frac{1}{\sqrt{1-s^2}}.$$

Spherical curve can be expressed as

$$\alpha(s) = \lambda_1(s)\vec{t}(s) + \lambda_2(s)\vec{n}(s) + \lambda_3(s)\vec{b}(s),$$

where $\lambda_1(s)$, $\lambda_2(s)$ and $\lambda_3(s)$ are the unknown functions. The system of differential equations characterizing this spherical curve is

$$\begin{aligned} \lambda_1'(s) &= \kappa(s)\lambda_2(s) + 1, \\ \lambda_2'(s) &= -\kappa(s)\lambda_1(s) + \tau(s)\lambda_3(s), \\ \lambda_3'(s) &= -\tau(s)\lambda_2(s). \end{aligned}$$

Approximate solutions of this system of differential equations under the initial conditions

$$\lambda_1(0) = 0, \qquad \lambda_2(0) = -1, \qquad \lambda_3(0) = 0$$

is calculated by using Taylor matris collocation method for $N = 5, 10, 15, 20, 25, 30$. For solving this problem, we suppose $\lambda_{i,N}(s)$ is approximated by the truncated Taylor series form

$$\lambda_{i,N}(s) = \sum_{k=0}^{N} a_{i,k} s^k, \qquad (i = 1, 2, 3).$$

The main matrix equation of $\alpha(s)$ with respect to Taylor matrix collocation method:

$$\left\{\mathbf{P}_0\mathbf{X} + \mathbf{P}_1\overline{\mathbf{B}}\mathbf{X}\right\}\mathbf{A} = \mathbf{G}.$$

Here

$$\mathbf{P}_0 = \begin{bmatrix} \mathbf{P}_0(s_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_0(s_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_0(s_N) \end{bmatrix}, \quad \mathbf{P}_1 = \begin{bmatrix} \mathbf{P}_1(s_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_1(s_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_1(s_N) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}(s_0) \\ \mathbf{G}(s_1) \\ \vdots \\ \mathbf{G}(s_N) \end{bmatrix};$$

$$\mathbf{P}_0(s) = \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}, \quad \mathbf{P}_1(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{G}(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

By using the procedure in Section 4, the fundamental matrix relations for the equation and conditions are computed and then the Taylor coefficients are found. Numerical results can be seen in Figure 1-4 and Table 1-6.

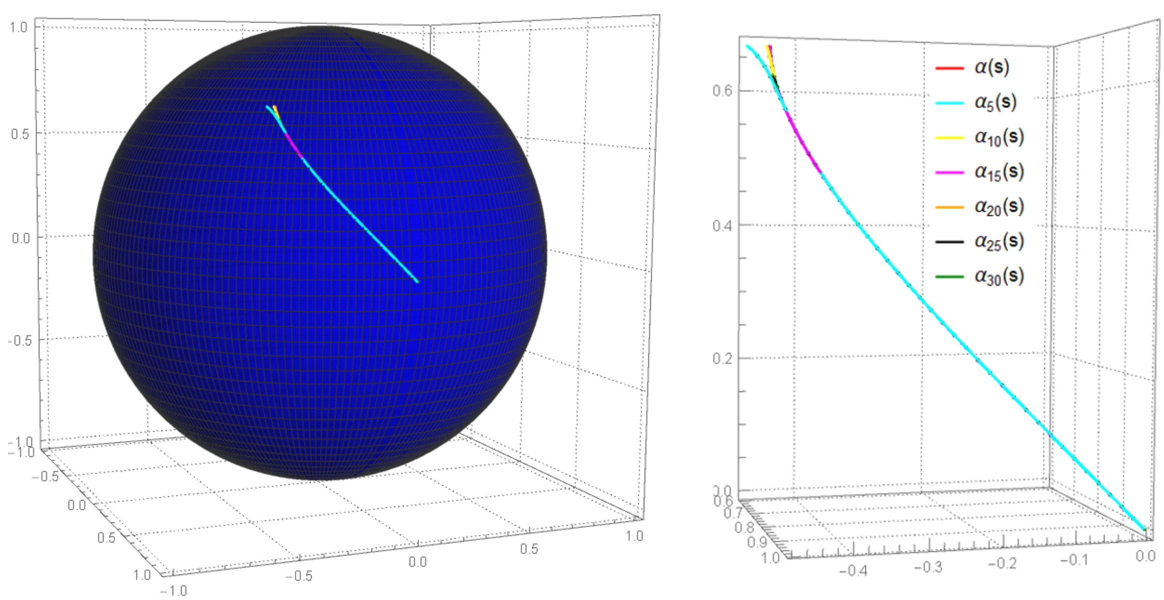

Figure 1. Comparison of analytical solution and approximate solutions

The approximate solutions $\lambda_{i,5}(s)$ $(i = 1, 2, 3)$ is obtained as

$$\lambda_{1,5}(s) = -2.613410227258 \times 10^{-16} - 1.463344849233 \times 10^{-16} s - 0.0036205434120 s^2 + 0.016265409561 s^3 - 0.0277239399594 s^4 + 0.0161267934050626 s^5,$$

$$\lambda_{2,5}(s) = -1.00000000000000 + 4.738301313483 \times 10^{-17} s + 0.46182718521305 s^2 + 0.260289657037436 s^3 - 0.48975648592404 s^4 + 0.52448584105127 s^5,$$

$$\lambda_{3,5}(s) = -s - 0.00362054341205 s^2 + 0.0162654095610 s^3 - 0.02772393995942 s^4 + 0.01612679340506 s^5.$$

The approximate solutions $\lambda_{i,10}(s)$ $(i = 1, 2, 3)$ is obtained as

$$\lambda_{1,10}(s) = 9.5766597921 \times 10^{-15} - 2.33812646898 \times 10^{-15} s + 0.000574274057321 s^2 - 0.00856375956916 s^3 + 0.063504398391 s^4 - 0.2748077702 s^5 + 0.73748787433 s^6 - 1.24431590117 s^7 + 1.2842693682 s^8 - 0.74052407508445 s^9 + 0.182651283383836 s^{10},$$

$$\lambda_{2,10}(s) = -1.0000000000000 - 1.59510000875085 \times 10^{-14} s + 0.50941750155470 s^2 - 0.182679974295995 s^3 + 1.708338410452 s^4 - 7.76012318598 s^5 + 23.3599011621 s^6 - 43.7819511775 s^7 + 50.393529766 s^8 - 32.510754053442 s^9 + 9.1045094038588 s^{10},$$

$$\lambda_{3,10}(s) = -1.00000000000000 s + 0.00057427405827708 s^2 - 0.0085637595843814 s^3 + 0.06350439849463 s^4 - 0.274807770610470 s^5 + 0.737487875207460 s^6 - 1.2443159023993200 s^7 + 1.28426936933350 s^8 - 0.74052407558578 s^9 + 0.1826512834866 s^{10}.$$

The approximate solutions $\lambda_{i,15}(s)$ $(i = 1, 2, 3)$ is obtained as

$$\lambda_{1,15}(s) = -1.39607006931 \times 10^{-12} - 4.575308656 \times 10^{-13} s - 0.000119203356871 s^2 + 0.00324950360028 s^3 - 0.046441646884 s^4 + 0.415129021783 s^5 - 2.51655777469 s^6 + 10.81258303 s^7 - 33.737079157 s^8 + 77.286305019 s^9 - 129.959379170 s^{10} + 158.48012215 s^{11} - 136.31402897 s^{12} + 78.373211641 s^{13} - 27.0181446563461 s^{14} + 4.2212591738136 s^{15},$$

$$\lambda_{2,15}(s) = -1.0000000000004 + 3.462406597727 \times 10^{-15} s + 0.497308708590039 s^2 + 0.091157420414001 s^3 - 1.34244147356 s^4 + 14.3420090101 s^5 - 93.742548804 s^6 + 431.78096429 s^7 - 1437.437962542 s^8 + 3505.6266918 s^9 - 6268.38776712 s^{10} + 8126.59001975 s^{11} - 7434.855439 s^{12} + 4552.2107536 s^{13} - 1674.50052793038 s^{14} + 280.001022366432 s^{15},$$

$$\lambda_{3,15}(s) = -1.00000000000 s - 0.000119201070710364 s^2 + 0.0032494286064063 s^3 - 0.04644048709727 s^4 + 0.415118222008017 s^5 - 2.516490995659360 s^6 + 10.812294512042 s^7 - 33.73618345241140 s^8 + 77.28428016 s^9 - 129.95604117 s^{10} + 158.47615246 s^{11} - 136.310713148 s^{12} + 78.371366541 s^{13} - 27.01753069243 s^{14} + 4.2211667790255 s^{15}.$$

The approximate solutions $\lambda_{i,20}(s)$ $(i = 1, 2, 3)$ is obtained as

$$\begin{aligned}\lambda_{1,20}(s) = {} & 4.02401118456\times10^{-10} - 5.7950108780\times10^{-11}s + 0.0000298579019933s^2 - 0.00122171833002s^3\\ & +0.02669897025s^4 - 0.37460769291s^5 + 3.67791381947s^6 - 26.5574310329s^7 + 145.5617424906s^8\\ & -618.29208988s^9 + 2062.7224138s^{10} - 5447.21229599s^{11} + 11421.0183424s^{12} - 18982.0250242s^{13}\\ & +24843.5928134s^{14} - 25278.7059214s^{15} + 19573.0404136s^{16} - 11136.564398s^{17} + 4387.856794s^{18}\\ & -1069.05593892533s^{19} + 121.291818876649s^{20},\end{aligned}$$

$$\begin{aligned}\lambda_{2,20}(s) = {} & -1.00000000005 - 2.85810163\times10^{-14}s + 0.50083215903s^2 - 0.041201196627s^3 + 1.12294081901s^4\\ & -15.12733826201s^5 + 158.7606193073s^6 - 1217.47757311s^7 + 7066.55740376s^8 - 31722.0613774s^9\\ & +111698.678553420s^{10} - 311047.81780954s^{11} + 687252.499694300s^{12} - 1.20307963815469\times10^6s^{13}\\ & +1.657823334002\times10^6s^{14} - 1.7755190807\times10^6s^{15} + 1.44675204469\times10^6s^{16} - 866206.7483114s^{17}\\ & +359160.24913335s^{18} - 92110.715165560s^{19} + 11005.851810934s^{20},\end{aligned}$$

$$\begin{aligned}\lambda_{3,20}(s) = {} & -0.99999999999998s + 0.0000369063463053581s^2 - 0.00156771640430988s^3 + 0.03499413661455s^4\\ & -0.498825874852s^5 + 4.96267630935s^6 - 36.2542441362s^7 + 200.8109202117s^8 - 861.1816368645s^9\\ & +2898.2058292s^{10} - 7714.05890928s^{11} + 16287.7298526s^{12} - 27237.15334589s^{13} + 35834.6429358s^{14}\\ & -36619.44351853s^{15} + 28449.804339s^{16} - 16226.8591928s^{17} + 6403.29615156s^{18} - 1561.12452927s^{19}\\ & +177.088098661405s^{20}\end{aligned}.$$

The approximate solutions $\lambda_{i,25}(s)$ $(i = 1, 2, 3)$ is obtained as

$$\begin{aligned}\lambda_{1,25}(s) = {} & 2.89072292625\times10^{-9} + 3.91396547092\times10^{-11}s + 6.933547437881\times10^{-6}s^2 - 0.00044850875423s^3\\ & +0.0138320046214s^4 - 0.2642766566702s^5 + 3.46645679337s^6 - 32.98190676s^7 + 235.3267201151s^8\\ & -1285.38555863s^9 + 5438.76749796s^{10} - 17906.1579196s^{11} + 45730.626523s^{12} - 89507.11585964s^{13}\\ & +131121.097173s^{14} - 139239.82350s^{15} + 110773.913640s^{16} - 108390.944873s^{17} + 211832.412329s^{18}\\ & -399901.550671s^{19} + 528130.35896s^{20} - 479231.14933s^{21} + 297640.5263436s^{22} - 121965.195933s^{23}\\ & +29882.9141276940s^{24} - 3328.8534109014s^{25},\end{aligned}$$

$$\begin{aligned}\lambda_{2,25}(s) = {} & -0.99999999997883 - 1.3618760842433\times10^{-9}s + 0.49994078266185s^2 + 0.00358303915682761s^3\\ & +0.019036123244s^4 + 1.96056132294s^5 - 25.0918350811s^6 + 237.083423214s^7 - 1702.21855902s^8\\ & +9534.48303221s^9 - 42291.371888s^{10} + 149778.649726s^{11} - 424398.300546s^{12} + 957832.49438s^{13}\\ & -1.70015200352\times10^6s^{14} + 2.3134842114\times10^6s^{15} - 2.2940072256\times10^6s^{16} + 1.47691429\times10^6s^{17}\\ & -421026.09379s^{18} - 29646.5164313s^{19} - 310303.5132852s^{20} + 790592.7853s^{21} - 809195.846816s^{22}\\ & +457462.436669927s^{23} - 142330.83684636s^{24} + 19240.9994220730s^{25},\end{aligned}$$

$$\begin{aligned}\lambda_{3,25}(s) = {} & -1.0000000000s + 0.0000221677245332501s^2 - 0.00134278550417024s^3 + 0.039106616139171s^4\\ & -0.70315121883s^5 + 8.6250499285s^6 - 76.0827826896s^7 + 497.167448028s^8 - 2440.689718968s^9\\ & +8990.461525s^{10} - 24243.14617609s^{11} + 43992.6428806s^{12} - 35814.5808002s^{13} - 64799.99707s^{14}\\ & +286665.023136400s^{15} - 493059.7423752900s^{16} + 369848.2792201800s^{17} + 288624.949488450s^{18}\\ & -1.2196390847836800\times10^6s^{19} + 1.7980850869144800\times10^6s^{20} - 1.64699408493708000\times10^6s^{21}\\ & +1.01009556877158\times10^6s^{22} - 407214.819440462s^{23} + 98290.708823136s^{24} - 10815.62014425s^{25}.\end{aligned}$$

TABLE 1. Comparison of exact solution $\lambda_1(s)$ and approximate solutions $\lambda_{1,N}(s)$ for $N = 5, 10, 15, 20, 25$

s	$\lambda_1(s)$	$\lambda_{1,5}(s)$	$\lambda_{1,10}(s)$	$\lambda_{1,15}(s)$	$\lambda_{1,20}(s)$	$\lambda_{1,25}(s)$
0	0	0.000000	0.000000	0.000000	0.0000004	0.000000
$\frac{3\pi}{40}$	0	-0.000061	0.000002	0.000000	0.000000	0.000000
$\frac{3\pi}{20}$	0	-0.000094	0.000004	0.000000	0.000000	0.000000
$\frac{9\pi}{40}$	0	-0.000139	0.000006	0.000000	0.000000	0.000000
$\frac{3\pi}{10}$	0	0.000518	0.000069	0.000008	0.000001	0.000000

TABLE 2. Comparison of exact solution $\lambda_2(s)$ and approximate solutions $\lambda_{2,N}(s)$ for $N = 5, 10, 15, 20, 25$

s	$\lambda_2(s)$	$\lambda_{2,5}(s)$	$\lambda_{2,10}(s)$	$\lambda_{2,15}(s)$	$\lambda_{2,20}(s)$	$\lambda_{2,25}(s)$
0	-1	-1.000000	-1.000000	-1.000000	-1.000000	-0.999999
$\frac{3\pi}{40}$	-0.971845	-0.972085	-0.971836	-0.971846	-0.971845	-0.971845
$\frac{3\pi}{20}$	-0.882006	-0.882169	-0.881998	-0.882006	-0.882005	-0.882006
$\frac{9\pi}{40}$	-0.707355	-0.707032	-0.707352	-0.707356	-0.707355	-0.707355
$\frac{3\pi}{10}$	-0.334269	-0.368271	-0.339344	-0.335284	-0.334497	-0.334343

TABLE 3. Comparison of exact solution $\lambda_3(s)$ and approximate solutions $\lambda_{3,N}(s)$ for $N = 5, 10, 15, 20, 25$

s	$\lambda_3(s)$	$\lambda_{3,5}(s)$	$\lambda_{3,10}(s)$	$\lambda_{3,15}(s)$	$\lambda_{3,20}(s)$	$\lambda_{3,25}(s)$
0	0	0	0	0	0	0
$\frac{3\pi}{40}$	-0.235619	-0.235681	-0.234617	-0.235620	-0.235619	-0.235619
$\frac{3\pi}{20}$	-0.471239	-0.471333	-0.471234	-0.471239	-0.471239	-0.471239
$\frac{9\pi}{40}$	-0.706858	-0.706998	-0.706852	-0.706859	-0.706858	-0.706858
$\frac{3\pi}{10}$	-0.942478	-0.941959	-0.942408	-0.942469	-0.942476	-0.942480

TABLE 4. Upper bound of the error and the absolute error of the $\lambda_{1,N}(s)$ for $N = 5, 10, 15, 20, 25$

s	$e_{1,5}(s)$	$e_{1,10}(s)$	$e_{1,15}(s)$	$e_{1,20}(s)$	$e_{1,25}(s)$
0	2.61E-16	9.57E-15	1.39E-12	4.02E-10	2.89E-09
$\frac{3\pi}{40}$	6.19E-05	2.68E-06	2.91E-07	4.60E-08	1.80E-09
$\frac{3\pi}{20}$	9.42E-05	4.87E-06	5.35E-07	8.46E-08	6.42E-10
$\frac{9\pi}{40}$	1.39E-04	6.67E-06	7.33E-07	1.15E-07	3.32E-09
$\frac{3\pi}{10}$	5.18E-04	6.98E-05	8.96E-06	1.85E-06	3.59E-07
$\overline{R_{1,N}}$	4.04E-03	3.30E-04	4.60E-05	7.85E-06	1.20E-06

TABLE 5. Upper bound of the error and the absolute error of the $\lambda_{2,N}(s)$ for $N = 5, 10, 15, 20, 25$

s	$e_{2,5}(s)$	$e_{2,10}(s)$	$e_{2,15}(s)$	$e_{2,20}(s)$	$e_{2,25}(s)$
0	2.22E-16	1.33E-15	4.86E-13	5.86E-11	2.11E-11
$\frac{3\pi}{40}$	2.39E-04	9.39E-06	1.05E-06	1.67E-07	9.46E-09
$\frac{3\pi}{20}$	1.63E-04	7.77E-06	8.48E-07	1.34E-07	9.60E-09
$\frac{9\pi}{40}$	13.22E-04	3.04E-06	4.78E-07	7.39E-08	8.03E-09
$\frac{3\pi}{10}$	3.40E-02	5.07E-03	1.01E-03	2.28E-04	7.45E-05
$\overline{R_{2,N}}$	3.79E-02	5.51E-03	1.09E-03	2.45E-04	8.01E-05

TABLE 6. Upper bound of the error and the absolute error of the $\lambda_{3,N}(s)$ for $N = 5, 10, 15, 20, 25$

s	$e_{3,5}(s)$	$e_{3,10}(s)$	$e_{3,15}(s)$	$e_{3,20}(s)$	$e_{3,25}(s)$
0	0	0	0	0	0
$\frac{3\pi}{40}$	6.19E-05	2.68E-06	2.91E-07	4.71E-08	1.12E-09
$\frac{3\pi}{20}$	9.42E-05	4.87E-06	5.35E-07	8.57E-08	1.36E-09
$\frac{9\pi}{40}$	1.39E-04	6.67E-06	7.33E-07	1.16E-07	4.03E-09
$\frac{3\pi}{10}$	5.18E-04	6.98E-05	8.96E-06	2.08E-06	2.53E-06
$\overline{R_{3,N}}$	4.04E-03	3.30E-04	4.60E-05	8.11E-06	3.15E-06

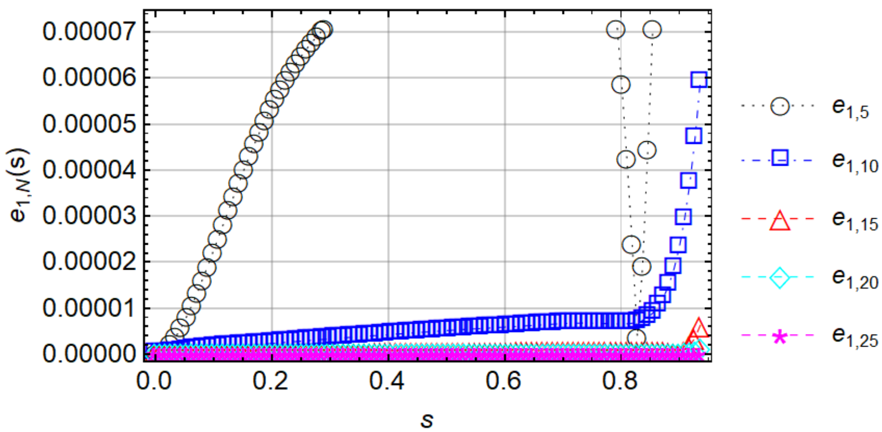

FIGURE 2. Comparison of absolute errors

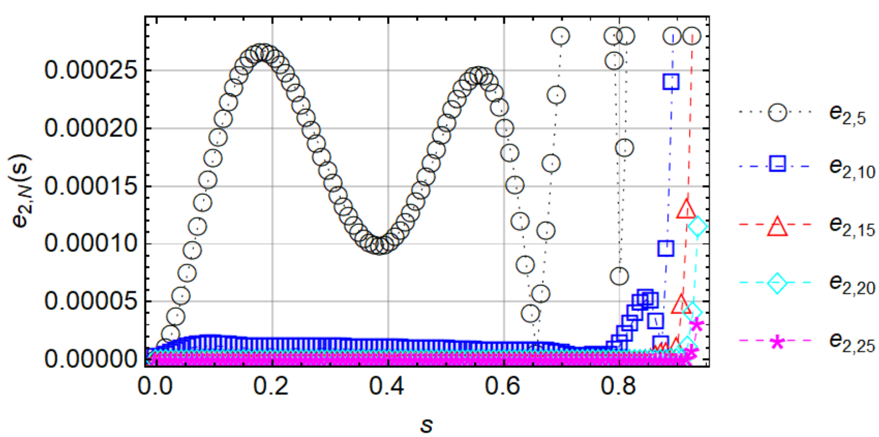

FIGURE 3. Comparison of absolute errors

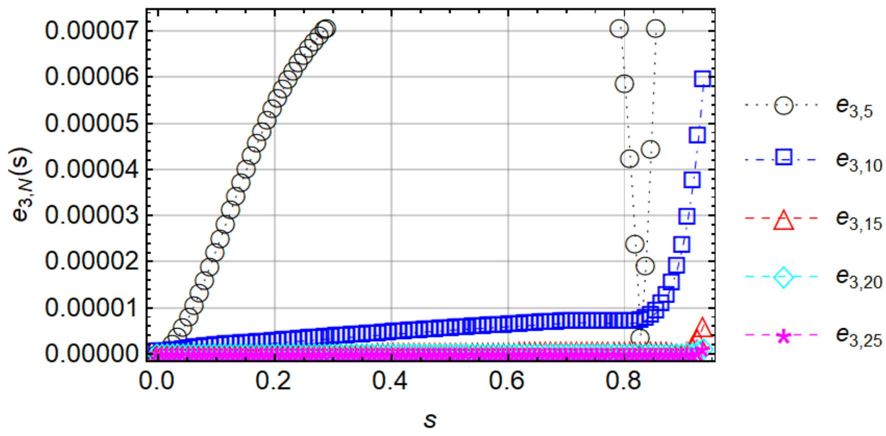

FIGURE 4. Comparison of absolute errors

7. CONCLUSION

In the present study, we deal with a system of differential equation with variable coefficients which is characterizing the curve that lies on a sphere. The approximate solutions of the this system of differential equation is obtained by using Taylor matrix collocation method. Also, an error analysis technique based on residual function is developed for our problem.

It is seen from Tables 1-6 and Figures 1-4 that the approximate solutions are close to the analytical solution when the values of N are selected big enough. In other words, the numerical results show that the accuracy improves when the values of N are increased.

This method can be extended to the system of differential equations of the spherical curves in Minkowski space but some modifications are required.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

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