Turk. J. Math. Comput. Sci.
11(2)(2019) 97–100
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The Principal Eigenvalue and The Principal Eigenfunction of A Boundary-Value-Transmission Problem

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Received: 24-06-2019 • Accepted: 21-11-2019

Abstract. It is well-know that the Sturm-Liouville theory justifies the "separation of variables"n method for voluminous partial differential equation problems. For Sturm-Liouville problems the Rayleigh quotient is the basis of an important approximation method that is used in physics, as well as in engineering. Although any eigenvalue can be related to its eigenfunction by the Rayleigh quotient, this quotient cannot be used to determine the exact value of the eigenvalue since eigenfunction is unknown. However, interesting and significant results can be obtained from the Rayleigh quotient without solving the differential equation(i.e. even in the case when the eigenfunction is not known). For example, Rayleigh quotient can be quite useful in estimating the eigenvalue.
It is the purpose of this paper to extend and generalize such important spectral properties as eigenfunction expansion and Parseval equality for Sturm-Liouville problems with interior singularities. We shall investigate certain spectral problems arising in the theory of the convergence of the eigenfunction expansion. Particularly, by modifying the Green's function method we shall extend and generalize such important spectral properties as Parseval's equality, Rayleigh quotient and Rayleigh-Ritz formula for the considered problem.

2010 AMS Classification: 34B24, 34L10.

Keywords: Boundary-value problem, Rayleigh quotient.

1. Introduction

The time-dependent Shrödinger equation for a single quantum-mechanical particle of mass m moving in one space dimension in a potential $V(x)$, has the form

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + V(x)\psi.$$

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Suppose that the solution has the form $\psi(x,t) = \varphi(x)h(t)$. Then we find that $h(t) = e^{-iEt/\hbar}$ and $\varphi(x)$ satisfies the two-order differential equation

$$-\frac{\hbar^2}{2m}\varphi'' + V(x)\varphi = E\varphi,$$

that is a classical Sturm-Liouville equation of the form $-u'' + qu = \lambda u$ The coefficient q is proportional to the potential V ($q(x) = \frac{2m}{\hbar^2}V(x)$) and the eigenvalue parameter λ is proportional to the energy E ($\lambda = \frac{2m}{\hbar^2}E$). The Rayleigh quotient is the basis of an important approximation method that is used in estimating of an eigenvalues. Let L is a differential operator in a Hilbert space H. Recall that the Rayleigh quotient $R(u)$ is defined as

$$R(u) = \frac{\langle Lu, u\rangle_H}{\langle u, u\rangle_H}$$

for the eigenvalue problem $Lu = \lambda u$ from which it follows that $\lambda_i = R(u_i)$ for any eigenvalue λ_i with the corresponding eigenfunction $u_i(x)$ of the differential operator L. Although any eigenvalue can be related to its eigenfunction by the Rayleigh quotient, this quotient cannot be used directly to determine the exact value of the eigenvalue since eigenfunction is unknown. However, some important results can be obtained from the Rayleigh quotient without solving the differential equation(i.e. even in the case when the eigenfunction is not known). For example, it can be useful in estimating the eigenvalue. It is the main purpose of this paper to extend and generalize such spectral properties as Rayleigh quotient, eigenfunction expansion and Rayleigh-Ritz formula for Sturm-Liouville problems with interior singularities. Not that in different areas of natural sciences many problems arise in the form of boundary value problems involving interior singularities(see, for example [1–7, 10]).

Consider the Sturm-Liouville equation

$$\Xi(u) := -u'' + q(x)u = \lambda u, \quad x \in [-1, 0) \cup (0, 1] \tag{1.1}$$

together with boundary conditions at the endpoints $x = -1$ and $x = 1$ given by

$$\alpha u(-1) + \alpha' u'(-1) = 0, \tag{1.2}$$

$$\beta u(1) + \beta' u'(1) = 0 \tag{1.3}$$

and the transmission conditions at the interior singular point $x = 0$ given by

$$u(-0) = u(+0), \tag{1.4}$$

$$u'(-0) = hu'(+0) \tag{1.5}$$

where the potential $q(x)$ is real-valued functions, which continuous in $[-1, 0)$ and $(0, 1]$ with finite limits $q(\mp 0)$. λ is a complex spectral parameter, $\alpha, \alpha', \beta, \beta'$ and h are real numbers. Throughout in this study we assume that $|\alpha| + |\alpha'| \neq 0$, $|\beta| + |\beta'| \neq 0$ and $h > 0$. Since the values of the solutions at the interior point $x = 0$ is not defined, an important question is how to introduce a new Hilbert space such a way that the considered problem can be generated self-adjoint operator in this space. Boundary value problems together with transmission conditions appear in various fields of physics and technics. For example, in electrostatics and magnetostatics the model problem which describes the heat transfer through an infinitely conductive layer is a transmission problem (see, [8] and the references listed therein). Some problems with transmission conditions arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness, see, [9]).

2. Construction of The Green's Function in Terms of The Left-Hand and Right-Hand Eigensolutions

Let $\psi_1(x,\lambda)$ and $\chi_2(x,\lambda)$ be the solutions of the equation (1.1) on $[-1, 0)$ and $(0, 1]$ satisfying the initial conditions

$$\psi_1(-1) = \alpha', \ \psi_1'(-1) = -\alpha,$$

and

$$\chi_2(1) = \beta', \ \chi_2'(1) = -\beta,$$

respectively and let the solutions $\psi_2(x,\lambda)$ and $\chi_1(x,\lambda)$ of the same equation on $(0, 1]$ and $[-1, 0)$ satisfying by the initial conditions:

$$\psi_2(+0) = \psi_1(-0), \ \ \psi_2'(+0) = \frac{1}{h}\psi_1'(+0)$$

and

$$\chi_1(-0) = \chi_2(+0), \ \chi_1'(-0) = -\frac{1}{h}\chi_2'(a).$$

respectively. The existence and uniqueness of these solutions follows from the well-known theorems of differential equations theory. Moreover, we can show that each of the solutions $\psi_i(x,\lambda)$ and $\chi_i(x,\lambda)$(i=1,2) is entire function of spectral parameter λ for each fixed x. It is well-known that each of the Wronskians $W[\psi_i(x,\lambda),\chi_i(x,\lambda)]$(i=1,2) is independent of variable x. Denoting $\Delta_i(\lambda) := W[\psi_i(.,\lambda),\chi_i(.,\lambda)]$ we have

$$\begin{aligned}\Delta_2(\lambda) &= W(\psi_2,\chi_2;+0)\\ &= -hW(\psi_1,\chi_1;-0)\\ &= -h\Delta_1(\lambda).\end{aligned}$$

It is convenient to define the characteristic function $\Delta(\lambda)$ as

$$\Delta(\lambda) := \Delta_2(\lambda) = -h\Delta_1(\lambda).$$

Theorem 2.1. *The set of eigenvalues of the problem (1.1)-(1.5) and the set of the zeros of the characteristic function $\Delta(\lambda)$ are coincide.*

Theorem 2.2. *All eigenvalues are real and simple of the problem (1.1)-(1.5).*

Remark 2.3. Taking in view the fact that all eigenvalues of the problem (1.1)-(1.5) are real and all coefficients of this problem are real valued the corresponding eigenfunctions can be chosen to be real-valued. Therefore from now on we will assume that all eigenfunctions are real-valued.

Theorem 2.4. *If λ is not an eigenvalue, then the nonhomogeneous equation $\Xi(u) = \lambda u + f$ together with the boundary-transmission conditions (1.2)-(1.5) has a unique solution $u = u(x,\lambda)$ given by*

$$u(x,\lambda) = \int_{-1}^{-0} G(x,\phi;\lambda)f(\phi)d\phi + \frac{1}{h}\int_{+0}^{1} G(x,\phi;\lambda)f(\phi)d\phi$$

where the kernel $G(x,\phi;\lambda)$ is defined by the formula

$$G(x,\phi;\lambda) = \begin{cases} \frac{1}{\Delta_1(\lambda)}\psi_1(x,\lambda)\chi_1(\phi,\lambda), & for\ -1 \le x \le \phi < 0\\ \frac{1}{\Delta_1(\lambda)}\psi_1(\phi,\lambda)\chi_1(x,\lambda), & for\ -1 \le \phi \le x < 0\\ \frac{1}{\Delta_1(\lambda)}\psi_1(x,\lambda)\chi_2(\phi,\lambda), & for\ -1 \le x < 0 < \phi \le 1\\ \frac{1}{h\Delta_2(\lambda)}\psi_1(\phi,\lambda)\chi_2(x,\lambda), & for\ -1 \le \phi < 0 < x \le 1\\ \frac{1}{h\Delta_2(\lambda)}\psi_2(x,\lambda)\chi_2(\phi,\lambda), & for\ 0 < x \le \phi \le 1\\ \frac{1}{h\Delta_2(\lambda)}\psi_2(\phi,\lambda)\chi_2(x,\lambda), & for\ 0 < \phi \le x \le 1 \end{cases}$$

which is called the Green's function for the problem (1.1)-(1.5).

3. Completeness of The Eigenfunction System in Suitable Hilbert Space. The Generalized Parseval Equality

We can prove the next results.

Theorem 3.1. *There are infinitely many real eigenvalues $\lambda_1 < \lambda_2 < \ldots$ of the problem (1.1)-(1.5) with $\lambda_n \to \infty$ as $n \to \infty$.*

Theorem 3.2. *The set of eigenfunctions $(\Phi_i(x))$, is complete in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$.*

Theorem 3.3. *For any $f \in L_2(-1,0) \oplus L_2(0,1)$*

$$f = \sum_{i=1}^{\infty}\{\int_{-1}^{\gamma 0} f\Phi_i ds + \frac{1}{h}\int_{+0}^{1} f\Phi_i ds\}\psi_i$$

where, the series converges in the Hilbert space $L_2(-1,0) \oplus L_2(0,1)$

Theorem 3.4. *(Generalized Parseval's equality) For any $f \in L_2(-1,0) \oplus L_2(0,1)$, we have the following generalized Parseval's equality*

$$\int_{-1}^{\gamma 0} \|f(x)\|^2 dx + \frac{1}{h}\int_{+0}^{1} \|f(x)\|^2 dx = \sum_{n=1}^{\infty} |\int_{-1}^{-0} f\Phi_n ds + \frac{1}{h}\int_{+0}^{1} f\Phi_n ds|^2.$$

4. The Principal Eigenvalue and The Principal Eigenfunction

Definition 4.1. the first eigenvalue of Sturm-Liouville problem (1.1)-(1.5) is called the principal eigenvalue and the corresponding eigenfunction is called the principal eigenfunction.

Theorem 4.2. *The principal eigenvalue for the boundary value -transmission problem (1.1)-(1.5) is the minimum value of the Rayleigh quotient*

$$R(u) = \frac{\int_{-1}^{-0}(u'^2 + qu^2)dx + \frac{1}{h}\int_{+0}^{1}(u'^2 + qu^2)dx}{\int_{-1}^{-0} u^2 dx + \frac{1}{h}\int_{+0}^{1} u^2 dx}$$

over the set of all functions u satisfying (1.2)-(1.5). Moreover, the principal eigenvalue is

$$\lambda_1 = \min R(u) = R(\Phi_1)$$

where Φ_1 is the principal eigenfunction.

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