Turk. J. Math. Comput. Sci.
11(2)(2019) 101–106

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On Co-Ideals of Implicative Semigroups with Apartness

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Received: 28-08-2019 • Accepted: 27-12-2019

Abstract. The setting of this research is the Bishop's constructive mathematics - a mathematics based on the Intuitionistic Logic and the principled-philosophical constructive orientation. Implicative semigroups with apartness were introduced and analyzed in 2016-17 in two published articles (*An introduction to implicative semigroups with apartness*, Sarajevo J. Math., 12(25)(2)(2016), 155-165 and *Strongly extensional homomorphism of implicative semigroups with apartness*. Sarajevo J. Math., 13(2)(2017), 155-162). In this paper, as a continuation of the mentioned articles, the concept of co-ideals was introduced in the implicative semigroups with apartness. Some of the important properties of these substructures in such semigroups have been proven. In addition, it has been shown that the family of all co-ideals in these semigroups forms a complete lattice.

2010 AMS Classification: 03F65, 20M12, 06F05, 06A06, 06A12.

Keywords: Constructive mathematics, semigroup with apartness, co-ordered semigroups, implicative semigroups with apartness, co-ideals.

1. Introduction

The notions of implicative semigroup were introduced by Chan and Shum [9]. For the first generalization of implicative semilattice see Nemitz [20] and Blyth [6]. Moreover, there exists a close relationship between implicative semigroups and other domains. For example, there is a lot of implications in mathematical logic and set theory (see Birkhoff [5]). For the general development of implicative semilattice theory, the ordered ideals and filters play an important role. It has been shown by Nemitz [20]. Motivated by this, Chan and Shum [9] established some elementary properties and constructed quotient structure of implicative semigroups via ordered filters. Jun [13, 14], Jun, Meng and Xin [15], Jun and Kim [16] and Lee, Shum and Wu [17, 18] discussed ordered ideals and filters of implicative semigroups. Bang and So [1] analyzed some special substructures in implicative semigroups.

In paper [26], in setting of Bishop's constructive mathematics, following the ideas of Chan and Shum and other authors mentioned above, the author introduced the notion of implicative semigroups with tight apartness and gave some fundamental characterization of these semigroups. In [26, 27] and in this article, using sets with apartness and co-order relations introduced by the author, instead of partial order. See for example [22–25, 28–30]. In this case, it is an excise relation, researched by Greenleaf [12], Negri [21] and von Plato [32]. So, in this research, the author studied side effects induced by existence of apartness and co-orders. Additionally, in [26] the author introduced the notion of co-filter in an implicative semigroup and described its connections with filter. Further, in [27] he analyzed a connection between co-filters and strongly extensional homomorphisms of implicative semigroup with apartness.

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In this article, as a continuation of his mentioned articles [26, 27], the author discuss about concept of co-ideals (Definition 3.1) in such semigroups. Author contributions in this article in addition to the concept of co-ideals, are properties of co-ideals (Theorem 3.1), and also the properties of the family of such substructures in an implicative semigroup with apartness (Theorem 3.2). The properties of the mentioned entities are the result of the influence of the characteristics of strongly extensionality of the internal binary operations and of co-order relation in implicit semigroups with apartness.

2. Preliminaries

2.1. **The Bishop's constructive framework.** This report is in Bishop's constructive algebra in a sense of papers [8, 10–12, 22–24, 28, 30] and books [2–4, 7], [31] (Chapter 8: Algebra). Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation "$\neq$"). The *diversity relation* "$\neq$" ([3]) is a binary relation on S, which satisfies the following properties:

$$\neg(x \neq x),\ x \neq y \Longrightarrow y \neq x,\ x \neq y \wedge y = z \Longrightarrow x \neq z\,.$$

If it satisfies the following condition

$$(\forall x, z \in S)(x \neq z \Longrightarrow (\forall y \in S)(x \neq y \ \vee\ y \neq z)),$$

then, it is called *apartness* (A. Heyting). In this article, it is assume that the basic apartness is *tight*, i.e. that it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \Longrightarrow x = y).$$

Subset X of S is a *strongly extensional subset* of S if and only if

$$(\forall x \in X)(\forall y \in S)(x \neq y \ \vee\ y \in S)$$

holds. A subsets X is *set-set apart* from the subset Y, and it is denoted by $X \bowtie Y$, if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$ is a valid formula. It's labeled like this $x \lhd Y$, instead of $\{x\} \bowtie Y$, and, of course, $x \neq y$ instead of $\{x\} \bowtie \{y\}$. With $X^{\lhd} = \{x \in S : x \lhd X\}$ is denoted the *strong complement* of X in S. A subset G of set $(S, =, \neq)$ is a *detachable subset* of S if $(\forall x \in S)(x \in G \ \vee\ x \lhd G)$ holds.

A function $f : (S, =, \neq) \longrightarrow (T, =, \neq)$ is *strongly extensional* if and only if

$$(\forall a, b \in S)(f(a) \neq f(b) \Longrightarrow a \neq b)$$

holds.

A relation $\alpha \subseteq S \times S$ is an *co-order relation* on semigroup S, if it is consistent, co-transitive and linear

$$\alpha \subseteq \neq \text{ (consistency)},\ \ \alpha \subseteq \alpha * \alpha \text{ (co-transitivity)},\ \neq \subseteq \alpha \cup \alpha^{-1} \text{ (linearity)},$$

where α has to be *compatible* with the semigroup operation in the following way

$$(\forall x, y, z \in S)(((xz, yz) \in \alpha \ \vee\ (zx, zy) \in \alpha) \Longrightarrow (x, y) \in \alpha).$$

In addition to this term, the term 'anti-order relation' is used (See, for example: [24–27]). In this article both terms are used. The α is said to be a *co-quasiorder* if it is consistent and co-transitive relation.

Speaking by the language of the classical algebra, the relation α is left and right cancellative. Here, "$*$" is the *filed product* between relations defined by the following way: If α and β are relations on set S, then filed product $\beta * \alpha$ of relation α and β is the relation given by $\{(x, z) \in X \times X : (\forall y \in X)((x, y) \in \alpha \ \vee\ (y, z) \in \beta)\}$.

For undefined notions and notations a reader can referred to the following papers [8, 10, 11, 22–30].

2.2. **Implicative semigroup with apartness.** In this subsection, some definitions and the necessary results will be repeated. When it comes to a *negatively anti-ordered* semigroup (briefly, n.a-o. semigroup) ([26, 27]), then it is mean a set S with a co-order α and a binary internal operation "$\cdot$" (sometime we write as xy instead of $x \cdot y$) such that for all $x, y, z \in S$ the following holds:

(1) $(xy)z = x(yz)$,

(2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and

(3) $(xy, x) \lhd \alpha$ and $(xy, y) \lhd \alpha$.

In that case for anti-order α we will say that it is a *negative anti-order relation* on semigroup. The operation "$\cdot$" is a *extensional* and *strongly extensional* function from $S \times S$ into S, i.e. it has to be

$$(x,y) = (x',y') \Longrightarrow xy = x'y' \text{ and}$$
$$(xy \neq x'y \ \vee \ yx \neq yx') \Longrightarrow x \neq x'$$

for any elements x, x', y, y' of S.

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \longrightarrow S$ such that the following is true

(4) $(z, x \otimes y) \in \alpha \Longleftrightarrow (zx, y) \in \alpha$ for any elements x, y, z of S.

In addition, let us recall that the internal binary operation "$\otimes$" must satisfy the following implications:

$$(a,b) = (u,v) \Longrightarrow a \otimes b = u \otimes v \text{ and}$$
$$a \otimes b \neq u \otimes v \Longrightarrow (a,b) \neq (u,v).$$

The operation "$\otimes$" is called *implication*. From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup*.

An implicative semigroup is to be *commutative* if it satisfies the following condition

$$(\forall x \in S)(\forall y \in S)(x \cdot y = y \cdot x).$$

Let α be a relation on S. For an element a of S we put $a\alpha = \{x \in S : (a,x) \in \alpha\}$ and $\alpha a = \{x \in S : (x,a) \in \alpha\}$. In the following proposition we give some properties of negative anti-order relation on semigroup.

Theorem 2.1 ([26], Theorem 3.1)**.** *If $\alpha \subseteq S \times S$ is an anti-order relation on a semigroup S, then the following statements are equivalent:*

(i) *α is a negative co-order relation;*
(ii) *αb for any b in S has the following properties:*
$xy \in \alpha b \Longrightarrow x \in \alpha b \ \wedge \ y \in \alpha b$,
$x \in \alpha b \Longrightarrow (x,y) \in \alpha \ \vee \ y \in \alpha b$;
(iii) $(\forall a, b \in S)(\alpha a \cup \alpha b \subseteq \alpha(ab))$;
(iv) *$a\alpha$ is an ideal of S for any a in S;*
(v) $(\forall a, b \in S)((ab)\alpha \subseteq a\alpha \cap b\alpha)$.

In any implicative semigroup S there exists a special element 1 of S, the biggest element in $(S, \alpha^{\triangleleft})$, which is the left neutral element in $(S, \cdot)$.

Some elementary properties of semigroup with apartness are given in the following proposition ([26], Theorem 3.3, Theorem 3.4, Corollary 3.2 and Corollary 3.3).

Theorem 2.2.
(a) $(\forall x \in S)(x \otimes x = 1)$;
(b) $(\forall x \in S)(\forall y \in S)((x,y) \in \alpha \Longleftrightarrow 1 \neq x \otimes y)$;
(c) $(\forall x \in S)(1 = x \otimes 1)$ *and* $(\forall x \in S)(x = 1 \otimes x)$.

3. The Concept of Co-Ideals

In this section it is introduced and analyzed the concept of co-ideals of an implicative semigroup with apartness:

Definition 3.1. A subset K of S is called *co-ideal* if the following holds:

(K1) $(\forall x, y \in S)(xy \in K \Longrightarrow y \in K)$ and

(K2) $(\forall x, y, z \in S)(x \otimes z \in K \Longrightarrow ((xy, z) \in \alpha \ \vee \ y \in K))$.

We say for co-ideal K of S that it is *proper* co-ideal if $K \subset S$ is valid.

The condition (K2) is equivalent to

(K2') $(\forall x, y, z \in S)(x \otimes z \in K \Longrightarrow ((x, y \otimes z) \in \alpha \ \vee \ y \in K))$

according to (4).

It is easy to check that the following holds:

Proposition 3.2. *If K is a co-ideal in an implicative semigroup S, then*

(K3) $(\forall y, z \in S)(z \in K \Longrightarrow ((y,z) \in \alpha \ \vee \ y \in K))$.

Proof. If we put x = 1 in (K2), we get

$$1 \otimes z \in K \Longrightarrow ((1 \cdot y, z) \in \alpha \ \vee \ y \in K)).$$

Given that $1 \otimes z = z$ according to (c) of Theorem 2.2 and $1 \cdot y = y$ according to Corollary 3.1 in [26], we obtain (K3). □

Corollary 3.3. *A co-ideal K of an implicative semigroup S with apartness is a strongly extensional subset of S.*

Proof. The claim of this lemma follows from (K3) with the assumption that α is a consistent relation. □

Corollary 3.4. *For a proper co-ideal K in an implicative semigroup S, the following holds*

(K4) $1 \lhd K$.

Proof. let y be an arbitrary element in S. Suppose $1 \in K$. Then $(y, 1) \in \alpha \ \vee \ y \in K$. Since the first option $(y, 1) \in \alpha$ is impossible because $(y, 1) \in \alpha \iff 1 \neq y \otimes 1 = 1$ according to claims (b) and (c) in Theorem 2.2, we have to $y \in K$. So, $S \subseteq K$ which is impossible because K is a proper co-ideal in S. Therefore, have to be $\neg(1 \in K)$. Assertion $1 \lhd K$ follows from the strongly extensionality of K in S. Indeed, for any $y \in K$ we have $(1, y) \in \alpha \ \subseteq \neq$ or $1 \in K$. As the second option is impossible, we have $1 \neq y \in K$. □

More information about the properties of the set $K^{\lhd}$ is given by the following theorem

Theorem 3.5. *The strong compliment $K^{\lhd}$ of a co-ideal K in a semigroup S satisfies the following two conditions:*

(i) *$K^{\lhd}$ is a right ideal in $(S, \cdot)$; and*

(ii) $(\forall x, y \in S)((y \in K^{\lhd} \ \wedge \ (y, x) \in \alpha^{\lhd}) \implies x \in K^{\lhd})$.

Proof. Let $x, y, u \in S$ be arbitrary elements such that $y \in K^{\lhd}$ and $u \in K$. Then from $u \in K$ follows $(xy, u) \in \alpha$ or $xy \in K$. Since the option $xy \in K$ gives $y \in K$, according to (K1), we have a contradiction with the hypothesis. So, it must be $(xy, u) \in \alpha \ \subseteq \neq$. From here it follows $xy \neq u \in K$. i.e. $xy \in K^{\lhd}$. Thus, $K^{\lhd}$ is a right ideal in $(S, \cdot)$.

Let $x, y, u \in S$ be elements such that $y \in K^{\lhd}$, $(y, x) \in \alpha^{\lhd}$ and $u \in K$. From $u \in K$ follows $(x, u) \in \alpha$ or $x \in K$ by (K3). From $x \in K$ it follows $(y, x) \in \alpha$ or $y \in K$ which is a contradiction with the starting hypothesis. So, it must be $(x, u) \in \alpha \ \subseteq \neq$. This means $x \neq u \in K$, i.e. $x \in K^{\lhd}$. □

Proposition 3.6. *If S is a commutative semigroup, then (K3) implies (K2).*

Proof. Let $x, y, z \in S$ be arbitrary elements such that $x \otimes z \in K$. Then

$$\begin{aligned} x \otimes z \in K \implies & ((y, x \otimes z) \in \alpha \ \vee \ y \in K) && \text{by (K3)} \\ & ((yx, z) \in \alpha \ \vee \ y \in K) && \text{by (4)} \\ & ((xy, z) \in \alpha \ \vee \ y \in K) && \text{by commutativity of } S. \end{aligned}$$ □

Theorem 3.7. *The family $\Re(S)$ of all co-ideals in an implicative semigroup S with apartness forms a complete lattice.*

Proof. Let $\{K_i\}_{i \in I}$ be a family of co-ideals in an implicative semigroup S with apartness. Then:

(1) Let $x, y, z \in S$ be elements such that $xy \in \bigcup_{i \in I} K_i$ and $x \otimes z \in \bigcup_{i \in I} K_i$. Then there exists an index $j \in I$ such that $xy \in K_j$ and $x \otimes z \in K_j$. Thus $y \in K_j \subseteq \bigcup_{\in I} K_i$ and $(xy, z) \in \alpha \ \vee \ y \in K_j \subseteq \bigcup_{i \in I} K_i$. Therefore, the conditions (K1) and (K2) are proven for the union $\bigcup_{i \in I} K_i$.

(2) Let $\mathcal{B}$ be the family of all co-ideals in S included in $\bigcap_{i \in I} K_i$. Then $\cup \mathcal{B}$ is the maximal co-ideal included in $\bigcap_{i \in I} K_i$.

(3) If we put $\sqcup_{i \in I} K_i = \bigcup_{\in I} K_i$ and $\sqcap_{i \in I} K_i = \cup \mathcal{B}$, then $(\Re(S), \sqcup, \sqcap)$ is a complete lattice. □

Corollary 3.8. *Let B be a subset of an implicative semigroup S with apartness. Then there exists the maximal co-ideal included in B.*

Proof. The proof of this Corollary follows directly from the first part of the proof of the previous theorem. □

Corollary 3.9. *Let a be an arbitrary element in an implicative semigroup S with apartness. Then there exists the maximal co-ideal M_a in S such that $a \lhd M_a$.*

Proof. The proof of this Corollary follows directly from the previous Corollary if we put $B = S \setminus \{a\}$. □

4. Final Observation

Bishop's constructive mathematics ([2–4, 7]) includes the following two basements:

(1) The Intuitionistic logic ([31]) and

(2) The principled-philosophical orientations of constructivism.

Intuitionistic logic does not accept the TND principle ('tertium non datur' i.e. the logical princess 'exclusion of the third') as an axiom. In addition, Intuitionistic logic does not accept the validity of the 'double negation' principle. This makes it possible to have a difference relation in sets which is not a negation of equality relation. Therefore, we accept that in Bishop's constructive mathematics we consider set A as a relational system $(A, =, \neq)$.

In Bishop's constructive algebra ([19, 30]) we always encounter with at least the following three problems:

(a) How to choose a predicate (or more predicates) between several classically equivalent ones by which an algebraic concept is determined?

(b) Since every predicate has at least one of its duales, how to construct a dual of given an algebraic structure?

(c) What are the specifics of this approach to looking at a given algebraic structure and what are the particularities of the case that cannot be found in classical algebra?

The concept of implicit semigroup with apartness was introduced and analyzed in articles [26, 27]. An interested reader can find much information about semigroups with apartness in articles [8, 10, 11, 22, 25]. The author has also studied some specific semigroups such as 'semilattice-ordered semigroups' (See, [24,28]). In all these and other studies of algebraic structures with apartness, which are not mentioned here, it is always pointed out that every algebraic term has at least one dual with characteristic features that cannot be found in classical algebra. A reader can find more about these features in the article [30].

It remains to verify the connection between the concept of co-ideals introduced by Definition 3.1 and the concept of ideals introduced in article [16]. This will be done in our next article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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