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Geometric Interpretation of Curvature Circles in Minkowski Plane

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Abstract: In this study, we investigate the geometric interpretation of the curvature circles of motion at the initial position in Minkowski plane. We consider the equations of the circling-point and centering-point curves of one-parameter motion in Minkowski plane and then determine the positions of these curves relative to each other.

Keywords: Centering-point curve, Circling-point curve, Minkowski plane.

1 Introduction

The concept of instantaneous invariants was first given by Bottema to determine the geometric properties of a moving rigid body at a given moment. Therefore, the geometric and kinematic properties of planar motions in Euclidean space are investigated according to these invariants [1] and this method has also guided many studies in the field of kinematics [2–6]. Later, the instantaneous invariants were called B-invariants (Bottema-invariants) by Veldkamp [7]. Besides, Veldkamp found special geometrical ground curves such as the inflection curve, the circling-point curve and the centering-point curve with the help of B-invariants, as well as the intersection points of these curves, Ball and Burmester points [8, 9]. The special geometrical ground curves in Minkowski (Lorentz) plane and their intersection points were analyzed by recent studies [10, 11], however, the positions of these curves relative to each other have not been studied yet. Therefore, it is aimed to present the geometric interpretation of curvature circles relative to each other throughout one-parameter planar motion in Minkowski plane based on the above-mentioned studies.

2 Preliminaries

The Minkowski plane L is the plane R^2 endowed with the Lorentzian scalar product given by $\langle x, y\rangle = x_1y_1 - x_2y_2$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. The norm of a vector is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$. An arbitrary vector $x \in L$ is called timelike if $\langle x, x\rangle < 0$, spacelike if $\langle x, x\rangle > 0$ or $x = 0$, lightlike if $\langle x, x\rangle = 0$ whereby $x \neq 0$. Two vectors x and y are said to be orthogonal if $\langle x, y\rangle = 0$. Let L_m be a Minkowski plane in continuous motion relative to a fixed Minkowski plane L_f. Then one-parameter planar motion L_m with respect to L_f is represented by

$$\begin{aligned} X &= x\cosh\theta + y\sinh\theta + a \\ Y &= x\sinh\theta + y\cosh\theta + b \end{aligned} \tag{1}$$

with respect to Cartesian frames of reference xoy and XOY in L_m and L_f, respectively. Here $a,\ b$ and θ are functions depending on time t. The position corresponding to $\varphi = 0$ of L_m is called initial position. The values for the initial position of the nth $(n = 0, 1, 2, \ldots)$ derivative of a function f of φ with respect to φ is denoted by f_n.

The Minkowski plane L_m is chosen to rotate with a constant angular velocity relative to the fixed Minkowski plane L_f, that is, $\theta = t$. The canonical relative system of motion is constructed by

$$a_0 = b_0 = a_1 = b_1 = a_2 = 0 \tag{2}$$

and the instantaneous invariants a_n and b_n characterize completely the infinitesimal properties of motion of Minkowski planes up the n-th order as

$$\begin{aligned} &X = x, \quad X' = y, \quad X'' = x, \qquad X''' = y + a_3, \\ &Y = y, \quad Y' = x, \quad Y'' = y + b_2, \quad Y''' = x + b_3 \end{aligned} \tag{3}$$

for $t = 0$ [10, 11].

3 The curvature circles in Minkowski plane

In this section, let's first recall the definitions of curvature circles in Minkowski plane.

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Definition 1. *The locus of the points of moving Minkowski plane L_m, whose curvature of the trajectory is constant at initial position, is called circling-point curve in Minkowski plane and denoted by cp.*

The equation of the circling-point curve cp in Minkowski plane is

$$\left(x^2 - y^2\right)(a_3x - b_3y) + 3x\left(x^2 - y^2 + y\right) = 0, \quad (x, y) \neq (0, 0) \tag{4}$$

where $(x, y) \neq (0, 0)$ or $x \neq \mp y$, [10, 11].

Definition 2. *The locus of the curvature centers of the points of moving Minkowski plane L_m is called centering-point curve in Minkowski plane and denoted by $\tilde{cp}$.*

The equation of the centering-point curve $\tilde{cp}$ in Minkowski plane is

$$\left(x^2 - y^2\right)(a_3x - b_3y) + 3xy = 0 \tag{5}$$

where $(x, y) \neq (0, 0)$ or $x \neq \mp y$, [10, 11].

Now, let us examine the positions of circling-point and centering-point curves relative to each other in Minkowski plane. The curve cp given by equation (4) and the curve $\tilde{cp}$ given by equation (5) can be arranged as

$$\left(x^2 - y^2\right)\left(\frac{(a_3+3)}{3}x - \frac{b_3}{3}y\right) + xy = 0$$

and

$$\left(x^2 - y^2\right)\left(\frac{a_3}{3}x - \frac{b_3}{3}y\right) + xy = 0,$$

respectively.

On the other hand, a third-order cubic curve γ in Minkowski plane can be given by

$$(\alpha x + \beta y)\left(x^2 - y^2\right) + xy = 0. \tag{6}$$

Let γ be an irreducible curve, this means that $\alpha\beta \neq 0$.

If $\alpha = \frac{a_3+3}{3}$ and $\beta = -\frac{b_3}{3}$ are satisfied, then the curve given by the equation (6) corresponds to the circling-point curve cp according to the canonical system in Minkowski plane.

Moreover, if there are the relations $\alpha = \frac{a_3}{3}$ and $\beta = -\frac{b_3}{3}$, then the curve given by the equation (6) corresponds to the centering-point curve $\tilde{cp}$ according to the canonical system in Minkowski plane.

Theorem 1. *The parametric equation of the curve γ is given by*

$$x = \frac{u}{(u^2 - 1)(\alpha + \beta u)}, \quad y = \frac{u^2}{(u^2 - 1)(\alpha + \beta u)} \tag{7}$$

where $u \neq \pm 1$.

Proof: If we substitute $y = ux$, such that $u \neq \pm 1$, in the equation (6), then we get $x^3(\alpha + \beta u)\left(1 - u^2\right) + ux^2 = 0$. Afterwards, some direct calculations completes the proof. □

Specifically, the parametric value $\frac{-\alpha}{\beta}$ corresponds to the infinity point of the curve γ. We can examine the reducible states of this curve in the following corollaries:

Corollary 1. *In Minkowski plane, the parametric equation of the curvature circle Γ_0, which is tangent to the curve γ along the axis y, is represented by*

$$x = \frac{1}{\beta(u^2 - 1)}, \quad y = \frac{u}{\beta(u^2 - 1)}. \tag{8}$$

Proof: If $\alpha = 0$ is taken in the equation (7) then the proof is obvious. □

Corollary 2. *In Minkowski plane, the parametric equation of the curvature circle Γ_1, which is tangent to the curve γ along the axis x, is given by*

$$x = \frac{u}{\alpha(u^2 - 1)}, \quad y = \frac{u^2}{\alpha(u^2 - 1)}. \tag{9}$$

Proof: Taking $\beta = 0$ in the equation (7) completes the proof. □

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From the equation (8), the Cartesian equation of the curvature circle Γ_0 in Minkowski plane is represented as

$$\beta\left(x^2-y^2\right)+x=0. \tag{10}$$

Similarly, by taking the equation (9) the Cartesian equation of the curvature circle Γ_1 in Minkowski plane is given by

$$\alpha\left(x^2-y^2\right)+y=0. \tag{11}$$

Let the points A_i $(i=1,2,3)$ be on the curve γ. In that case, these points are given as

$$A_i=\left(\frac{u_i}{({u_i}^2-1)(\alpha+\beta u_i)},\quad \frac{{u_i}^2}{({u_i}^2-1)(\alpha+\beta u_i)}\right),\quad (i=1,2,3).$$

Theorem 2. *The points A_i $(i=1,2,3)$ with parametric value u_i $(i=1,2,3)$ are on the same line does not pass through the origin if and only if*

$$u_3u_2u_1=\frac{\alpha}{\beta}. \tag{12}$$

Proof: The points A_i are on the same line that does not pass through the origin if and only if the slopes of the lines A_1A_2 and A_2A_3 are equal the each other. Thus, there is the relationship

$$\frac{\frac{-{u_3}^2}{(1-{u_3}^2)(\alpha+\beta u_3)}+\frac{{u_2}^2}{(1-{u_2}^2)(\alpha+\beta u_2)}}{\frac{-u_3}{(1-{u_3}^2)(\alpha+\beta u_3)}+\frac{u_2}{(1-{u_2}^2)(\alpha+\beta u_2)}}=\frac{\frac{-{u_2}^2}{(1-{u_2}^2)(\alpha+\beta u_2)}+\frac{{u_1}^2}{(1-{u_1}^2)(\alpha+\beta u_1)}}{\frac{-u_2}{(1-{u_2}^2)(\alpha+\beta u_2)}+\frac{u_1}{(1-{u_1}^2)(\alpha+\beta u_1)}}.$$

In this manner, we get

$$\beta^2u_1{u_2}^2u_3+\beta\alpha\left(u_2\left(u_1u_3-1\right)\right)-\alpha^2.$$

If this equation is factored, we find

$$(\beta u_1u_2u_3-\alpha)=0 \text{ or } (\beta u_2+\alpha)=0.$$

So, we can write

$$u_1u_2u_3=\tfrac{\alpha}{\beta} \text{ or } u_2=\tfrac{-\alpha}{\beta}.$$

Here $u_2\neq\frac{-\alpha}{\beta}$ must be satisfied since the parametric value $\frac{-\alpha}{\beta}$ corresponds to the infinity point of the curve γ. □

If one of these three points is at the infinity, i.e., $u_3^*=\frac{-\alpha}{\beta}$, this means that this line is parallel to the asymptotes of the curve γ and cuts the curve at two points with the parameters u_1^* and u_2^*. Then the correlation between the parameters u_1^* and u_2^* is given by

$$u_1^*u_2^*=-1. \tag{13}$$

If the points A_1 and A_2 of the curve γ are represented with respect to the parameters u_1 and u_2, then the equation of the line A_1A_2 is found as

$$\left(\alpha\left(u_2+u_1\right)+\beta u_1u_2\left(u_1u_2+1\right)\right)x-\left(\alpha\left(u_1u_2+1\right)+\beta u_1u_2\left(u_2+u_1\right)\right)y+u_1u_2=0. \tag{14}$$

After the formation this equation we have

$$\alpha\left(\left(u_1+u_2\right)x-\left(u_1u_2+1\right)y\right)-\beta u_1u_2\left(-\left(u_1u_2+1\right)x+\left(u_2+u_1\right)y-\frac{1}{\beta}\right)=0. \tag{15}$$

If we denote the slopes of the lines d_1 and d_2 given by the equations

$$\left(u_1+u_2\right)x-\left(u_1u_2+1\right)y=0 \tag{16}$$

and

$$-\beta\left(u_1u_2+1\right)x+\beta\left(u_2+u_1\right)y-1=0 \tag{17}$$

by m_{d_1} and m_{d_2}, respectively, we see that these lines are perpendicular in Minkowski plane since there is the relationship $m_{d_1}m_{d_2}=1$. Hence, we can interpret that the line given by the equation (14) passes through the intersection of the lines d_1 and d_2 which are perpendicular to each other in the Minkowski plane.

Also, considering the equation of distance from a point to a line in the Minkowski plane we find the equation of the distance from origin to the line A_1A_2 as

$$d=\frac{|u_1u_2|}{\sqrt{\left|\left(-\alpha^2+\beta^2u_1^2u_2^2\right)\left(u_1^2-1\right)\left(u_2^2-1\right)\right|}} \tag{18}$$

where $u_i\neq\pm1$, $i=1,2$.

Let A_3 be a point with the parameter $-u_1$ on the curve γ. From the equation (18), the lines A_2A_1 and A_2A_3 have equal distance from origin, that is, the lines A_2A_1 and A_2A_3 are symmetrical according to the point A_2.

Now let's give the formation of the circles Γ_0 and Γ_1. Since the geometric location of the curvature centers of the curve *cp* is the centering-point curve $\tilde{cp}$, the curvature center of a point with the parameter u of the curve *cp* coincides with the same parameter point of the curve $\tilde{cp}$, [11]. Let A_1 and A_2 be two points on the curve *cp*. Also, let α_1 and α_2 be the centers of curvature of these points. If the points A_1 and A_2 are given by the parameters u_1 and u_2 , respectively, the equation of line A_1A_2 is found by writing $\alpha = \frac{a_3+3}{3}$, $\beta = -\frac{b_3}{3}$ in the equation (14) and the equation of line $\alpha_1\alpha_2$ is found by writing $\alpha = \frac{a_3}{3}$, $\beta = -\frac{b_3}{3}$ in the equation (14).

Thus, we get the equations of A_1A_2 and $\alpha_1\alpha_2$ lines as

$$((3+a_3)(u_1+u_2) - b_3u_1u_2(1+u_1u_2))\,x - ((3+a_3)(1+u_1u_2) - b_3u_1u_2(u_1+u_2))\,y - 3u_1u_2 = 0$$

and

$$(a_3(u_1+u_2) - b_3u_1u_2(1+u_1u_2))\,x - (a_3(1+u_1u_2) - b_3u_1u_2(u_1+u_2))\,y - 3u_1u_2 = 0,$$

respectively. Here, the lines A_1A_2 and $\alpha_1\alpha_2$ pass through the intersection of the lines given by the equations (16) and (17), which are perpendicular to each other in the Minkowski plane. Here, the equation (16) indicates a line and this line passes through the pole point P and the intersection point Q of the lines $\alpha_1\alpha_2$ and A_1A_2. The equation (17) refers to the equation of the line perpendicular to the line PQ passing through the point Q.

In case of $\alpha = 0$, by substituting the parameter equation (18) into the equation (17), for Γ_0 we get

$$u^2 - (u_2+u_1)\,u + u_1u_2 = 0. \tag{19}$$

Corollary 3. *u_1 and u_2 (the roots of the equation (19)) give the parametric expression of the intersection points of circle Γ_0 with the line given by the equation (17).*

In addition, these points are on the PA_1 and PA_2 lines. Similarly, the above statements can be investigated for the curvature circle Γ_1 in Minkowski plane. For this, let's first examine the line passing through the pole point P perpendicular to the line PQ. This line is given by the following equation taking into consideration the equation (16) such that the product of the slopes of these lines is 1 and these lines pass from pole P:

$$(u_1+u_2)\,y - (u_1u_2+1)\,x = 0.$$

If the above equation and (14) are considered together, the intersection point (is denoted by R) of this line with line A_1A_2 is on the line below

$$\alpha\left((u_1+u_2)\,x - (u_1u_2+1)\,y\right) + u_1u_2 = 0. \tag{20}$$

So the line passing through the point R is parallel to the line PQ. By substituting the parameter equation of circle Γ_1 into the equation (20), we get

$$u^2 - (u_2+u_1)\,u + u_1u_2 = 0. \tag{21}$$

The equation (21) is the previously obtained equation (19).

Corollary 4. *u_1 and u_2 (the roots of the equation (21)) give the parametric expression of the intersection point of the circle Γ_1 and the line given by equation (20).*

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