Conference Proceedings of Science and Technology, 2(3), 2019, 164–168

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

On a*-*I*-open Sets and a Decomposition of Continuity

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Aynur Keskin Kaymakci [1*]

[1] *Department of Mathematics, Faculty of Science, Selcuk University, Campus, Konya, Turkey, ORCID:0000-0001-5909-8477*
** Corresponding Author E-mail: akeskin@selcuk.edu.tr*

Abstract: In this paper, we introduce a new set namely a*-*I*-open set in ideal topological spaces. Besides, we give some properties and characterizations of it. We obtain that it is stronger than pre*-I-open set with b-open set and weaker than $\delta\beta_I$-open set. Finally, we give a decomposition of continuity by using a*-*I*-open set as stated the following:" $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ is continuous if and only if it is a*-I-continuous and strongly A$_I$-continuous."

Keywords: a*-*I*-open set, Decomposition of continuity, Ideal.

1 Introduction and preliminaries

Topic of ideals in topological spaces has been studied since beginning of 20th century. It has won reputain and importance in citevai.Throughout this paper, we will denote topological spaces by (X, τ) and (Y, φ). For a subset A of a space (X, τ), the closure of A and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively. It is well known that a subset A of a space (X, τ) is said to be regular open citevel if $A = Int(Cl(A))$. A subset A of a space (X, τ) is said to be δ-$open$ citevel if for each $x \in A$ there exists a regular open set U such that $x \in U \subseteq A$. A is δ-$closed$ citevel if $(X$-$A)$ is δ-$open$. The set $\{x \in X \mid x \in U \subseteq A$ for some regular open set U of $X\}$ is called the δ-interior of A and is denoted by $Int_\delta(A)$ citevel. A point $x \in X$ is called a δ-cluster point of A if $A \cap Int(Cl(V)) \neq \varnothing$ for each open set V containing x. The set of all δ-cluster points of A is called the δ-closure of A and is denoted by $\delta Cl(A)$ citevel. Of course, δ-$open$ sets form a topology τ^δ and then $\tau^\delta \subset \tau$ holds citevel.

An ideal I on X is defined as a nonempty collection of subsets of X satisfying the following two conditions:
(1) If $A \in I$ and $B \subset A$, then $B \in I$;
(2) If $A \in I$ and $B \in I$, then $A \cup B \in I$.

Let (X, τ) be a topological space and I an ideal on X . An ideal topological space is a topological space (X, τ) with an ideal I on X and is denoted by (X, τ, I). For a subset $A \subset X$, $A^*(I, \tau) = \{x \in X \mid U \cap A \notin I$ for each neighbourhood U of $x\}$ is called the local function of A with respect to I and τ (citekur). Throught this paper, we use A^* instead of $A^*(I, \tau)$. Besides, in citejan, authors introduced a new Kuratowski closure operator $Cl^*(.)$ defined by $Cl^*(A) = A \cup A^*(I, \tau)$ and obtained a new topology on X which is called an $*$-$topo\log y$. This topology is denoted by $\tau^*\ (I)$ which is finer than τ.

A point x in an ideal topological space is called δ_I-$cluster\ point$ of A if $Int(Cl^*(U) \cap A \neq \varnothing$ for each neighborhood U of x. The set of all δ_I-$cluster\ points$ of A is called the δ_I-$closure\ of\ A$ and will be denoted by $\delta Cl_I(A)$ citey/"uk. A is said to be δ_I-$closed$ citey/"uk if $A = \delta Cl_I(A)$. Of course, the complement of δ_I-open set is said δ_I-$closed$ citey/"uk. The family of all δ_I-$open$ sets in any ideal topological space (X, τ, I) form a topology $\tau^{\delta I}$ and then $\tau^{\delta I} \subset \tau$ holds citey/"uk.

Definition 1. *some label A subset A of an ideal topological space (X, τ, I) is said to be α-open citenja (resp. semi-open citelev, pre-open citemas1, b-open citeand (or γ open citeel-a), β-open citeabd) if $A \subset Int(Cl(Int(A)))$ (resp. $A \subset Cl(Int(A))$, $A \subset Int(Cl(A))$, $A \subset Int(Cl(A)) \cup Cl(Int(A))$, $A \subset Cl(Int(Cl(A)))$).*

Definition 2. *some label A subset A of an ideal topological space (X, τ, I) is said to be pre-I-$open$ citedon (resp. $semi$-I-$open$ citehat1, α-I-$open$ citehat1, b-I-$open$citeg/"ul, β-I-$open$ citehat 1) if $A \subset Int(Cl^*(A))$ (resp. $A \subset Cl^*(Int(A))$, $A \subset Int(Cl^*(Int(A)))$, $A \subset Cl^*(Int(A)) \cup Int(Cl^*(A))$, $A \subset Cl(Int(Cl^*(A)))$).*

Definition 3. *A subset A of an ideal topological space (X, τ, I) is said to be δ-α-I-$open$ citehat 4, pre*-I-open citeeki (resp. semi*-I-open , $\delta\beta$-I-$open$ citehat 4) if $A \subset Int(Cl(\delta Int_I(A)))$ (resp. $A \subset Int(\delta Cl_I(A))$, $A \subset Cl(\delta Int_I(A))$, $A \subset Cl(Int(\delta Cl_I(A)))$).*

Related to above definitions, one can find the following diagram in citehat 4. None of these implications are reversible in generally as shown in the related papers.

open → α-I-open → semi-I-open
↑ ↓ ↘ ↓ ↘
↑ ↓ pre-I-open → ↓ b-I-open → β-I-open
↑ ↓ ↓ ↓ ↓ ↓
↑ α-open ↓ → semi-open ↓ ↓
↑ ↑ ↘ ↓ ↘ ↓ ↓
↑ ↑ pre-open → b-open → β-open
↑ ↑ ↘ ↘ ↓
↑ ↑ pre*-I-open → a*-I-open ↓
↑ ↑ ↗ ↘ ↓
δ-I-open → δ-α-I-open → semi*-I-open → $\delta\beta_I$-open

Diagram II

Lemma 1. *For a subset A of an ideal topological space (X, τ, I) , the following properties are hold:*
(1) If U is an open set, then $U \cap Cl^*(A) \subseteq Cl^*(U \cap A)$ citehat2,
(2) If U is an open set, then $\delta Cl_I(U)) = Cl(U)$ citehat3.

2 a*-*I*-open sets

In this section, to give a decomposition of open set we introduce a new set which name is a*-I-open set and obtain some properties and characterizations of it.

Definition 4. *A subset A of an ideal topological space (X, τ, I) is said to be an a*-I-open if $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$. The complement of an a*-I-open set said to be an a*-I-closed. It is obvious that A is an a*-I-closed if and only if $Cl(\delta Int_I(A)) \cap Int(Cl(A)) \subset A$.*

Corollary 1. *It is obtained from Definition 4, $\varnothing$ and X are both a*-I-open sets and a*-I-closed sets.*

Proposition 1. *Let (X, τ, I) be an ideal topological space. Then, the following properties are hold:*
(1) If A is pre-I-open, then it is a*-I-open,*
(2) If A is b-open, then it is a-I-open,*
(3) If A is a-I-open, then it is $\delta\beta_I$-open.*

Proof: The proof of (1) is clear from Definitions 1, 3 and 4. The others are obtained by using related set definitions.The following diagram is obtained by using Proposition 3 and several sets defined above. □

Remark 1. *The converses of each statements in Proposition 3 are not true in generally as shown in the next examples.*

Example 1. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ and $I = \{\varnothing\}$.(1) Set $A = \{a, d\}$. Then, A is an a*-I-open but it is not pre*-I-open (2) Set $A = \{a, b\}$. Then, A is an a*-I-open but it is not b-open.*

Example 2. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\varnothing\}$. For $A = \{b, d\}$ is $\delta\beta_I$-open, but it isn't a*-I-open.*

We have the following diagram.

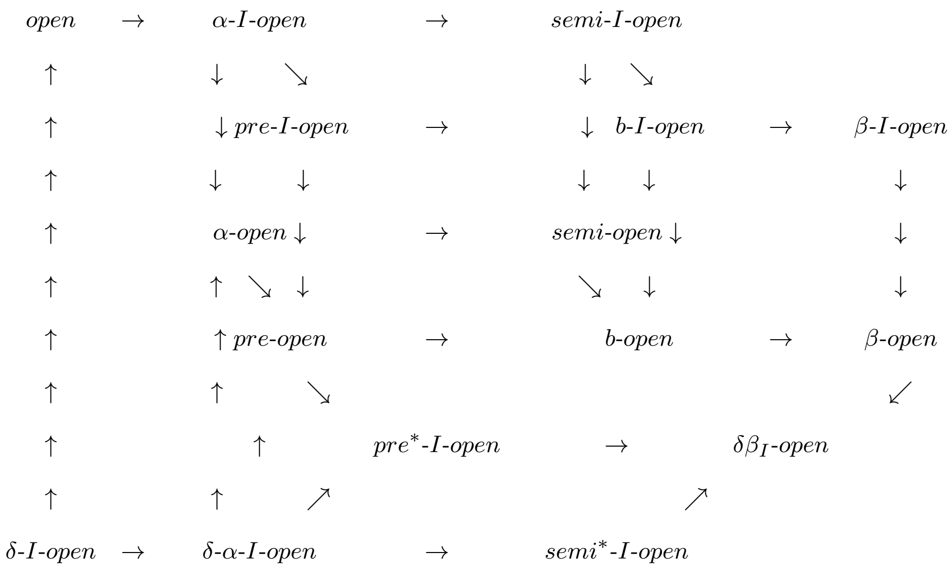

Diagram I

Proposition 2. *For an ideal topological space (X, τ, I) and a subset A of X, the following property is hold: "If $I = \wp(X)$, then A is an a^*-I-open if and only if A is an b-open."*

Proof: Since sufficiency is stated in Proposition 3(2), we prove only necessity. Let $I = \wp(X)$. Then, $A^* = \varnothing$ and $Cl^*(A) = A \cup A^* = A$ for every subset A of X. So, we have $\delta Cl_I(A) = Cl(A)$. If A is an a^*-I-$open$ set, then we obtain that $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A)) \subset Int(Cl(A)) \cup Cl(Int(A))$ and hence every a^*-I-$open$ set is a b-$open$. □

Remark 2. *The notions of a^*-I-open set and β-open set are independent each other. Indeed in Example 2, set $A = \{b, d\}$ is β-open, but it isn't a^*-I-open. Besides in Example 1(2), set $A = \{a, b\}$ is an a^*-I-open but it is not β-open.*

Proposition 3. *Let (X, τ, I) be an ideal topological space with an arbitrary index set Δ. If $\{A_\alpha : \alpha \in \Delta\} \subset a^*IO(X, \tau)$, then $\cup\{A_\alpha : \alpha \in \Delta\} \in a^*IO(X, \tau)$.*

Proof: Since $\{A_\alpha : \alpha \in \Delta\} \subset a^*IO(X, \tau)$, $A_\alpha \subset Int(\delta Cl_I(A_\alpha)) \cup Cl(Int(A_\alpha))$ for every $\alpha \in \Delta$. Since δCl_I is a Kuratowski closure operator, we have

$$\begin{aligned}(\bigcup_{\alpha\in\triangle} A_\alpha) &\subset \left(\bigcup_{\alpha\in\triangle} Int(\delta Cl_I(A_\alpha)) \cup Cl(Int(A_\alpha))\right) \\ &= \left(\bigcup_{\alpha\in\triangle} Int(\delta Cl_I(A_\alpha))\right) \cup \left(\bigcup_{\alpha\in\triangle} Cl(Int(A_\alpha))\right) \\ &\subset Int\left(\bigcup_{\alpha\in\triangle} \delta Cl_I(A_\alpha)\right) \cup Cl\left(\bigcup_{\alpha\in\triangle} Int(A_\alpha)\right) \\ &\subset Int(\delta Cl_I\left(\bigcup_{\alpha\in\triangle} A_\alpha\right) \cup Cl(Int\left(\bigcup_{\alpha\in\triangle} A_\alpha\right).\end{aligned}$$
□

Proposition 4. *Let (X, τ, I) be an ideal topological space and A, U are subsets of X. If A is an a^*-I-open set and U is δ-I-open set. Then $(A \cap U)$ is an a^*-I-open set.*

Proof: Since A is an a^*-I-$open$ set and U is δ-I-$open$ set, we have $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$ and $U \subset \delta Int\,(U)$. By using some properties of closure, interior and δ-I-closure operations, we have

$$\begin{aligned}(A \cap U) &\subset ((Int(\delta Cl_I(A))) \cup Cl(Int(A))) \cap \delta Int_I(U) \\ &= (Int(\delta Cl_I(A)) \cap \delta Int_I(U)) \cup (Cl(Int(A)) \cap \delta Int_I(U)) \\ &\subseteq (Int(\delta Cl_I(A)) \cap Int(U)) \cup (Cl(Int(A)) \cap Int(U)) \\ &\subseteq Int\,[\delta Cl_I(A) \cap Int(U)] \cup Cl\,[Int(A) \cap Int(U)] \\ &\subseteq Int\,(\delta Cl_I\,(A \cap Int(U))) \cup Cl(Int(A \cap U)) \\ &\subseteq Int\,(\delta Cl_I\,(A \cap U)) \cup Cl(Int(A \cap U)).\end{aligned}$$

This shows that $(A \cap U)$ is an a^*-I-$open$ set. □

Definition 5. *A subset A of an ideal topological space (X, τ, I) is called*
(1) strongly t-I-set citeeki if $Int(\delta Cl_I\,(A) = Int\,(A)$,
(2) strongly A_I-set if $A = U \cap V$, where $U \in \tau$ and V is strongly t-I-set and $Int(\delta Cl_I\,(V) = Cl\,(Int\,(V))$.

Theorem 1. *The following properties hold for a subset A of an ideal topological space (X, τ, I):*
(1) If A is strongly t-I-set and $Int(\delta Cl_I\,(A) = Cl\,(Int\,(A))$, then it is strongly A_I-set,
(2) If A is open set, then it is strongly A_I-set.

Proof:
(1) : Since A is strongly t-I-set with $Int(\delta Cl_I\,(A) = Cl\,(Int\,(A))$ and $X \in \tau$, the proof of 1) is obvious.
(2) : Since X is strongly t-I-set with $Int(\delta Cl_I\,(X) = Cl\,(Int\,(X))$ and $A \in \tau$, the proof of 2) is obtained. □

Theorem 2. *For a subset A of (X, τ, I), the following properties are equivalent:*
(1) A is open,
(2) A is an a^*-I-open and strongly A_I-set.

Proof: $(1) \Longrightarrow (2)$: By Diagram II, every open set is a^*-I-$open$. Besides, we have every open set is strongly A$_I$-set according to Theorem 7(2).
$(2) \Longrightarrow (1)$: Let A is an a^*-I-$open$ and strongly A$_I$-set. Then, we have $A \subset Int(\delta Cl_I(A)) \cup Cl(Int(A))$ and strongly A$_I$-set if $A = U \cap V$, where $U \in \tau$ and V is strongly t-I-set and $Int(\delta Cl_I\,(V) = Cl\,(Int\,(V))$, respectively. Therefore, we have $A \subset Int(\delta Cl_I(U \cap V)) \cup Cl(Int(U \cap V)) \subseteq [Int(\delta Cl_I(U)) \cap Int(\delta Cl_I(V))] \cup [Cl(Int(U) \cap Cl(Int(V)] = [Int(\delta Cl_I(U)) \cap Cl(Int(V)] \cup [Cl(Int(U) \cap Cl(Int(V)]$. According to Lemma 1(2), since $U \in \tau$, it is obvious that $\delta Cl_I(U) = Cl(U)$ and $Int(\delta Cl_I(U)) = Int(Cl(U))$. So, we have
$A \subset [Int(Cl(U)) \cap Cl(Int(V)] \cup [Cl(Int(U) \cap Cl(Int(V)] = [Int(Cl(U)) \cup Cl(Int(U)] \cap Cl(Int(V)$. Consequently, since $A \subset U$, we obtain $A \subset U \cap \{[Int(Cl(U)) \cup Cl(Int(U)] \cap Cl(Int(V))\} = \{U \cap [Int(Cl(U)) \cup Cl(Int(U)]\} \cap Cl(Int(V)) = [(U \cap Int(Cl(U))) \cup (U \cap Cl(Int(U)))] \cap Cl(Int(V)) = U \cap Int(V) = Int(U \cap V) = Int(A)$. Hence A is an open. □

The notions of a^*-I-$open$ set and strongly A$_I$-set are independent each other as shown in the following examples.

Example 3. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = \{\varnothing, \{d\}\}$. For $A = \{a\}$, then it is a^*-I-open but it isn't strongly A_I-set.*

Example 4. *Let $X = \{a, b, c, d\}$, $\tau = \{X, \varnothing, \{a\}, \{c\}, \{a, c\}\}$ and $I = \{\varnothing, \{a\}\}$. For $A = \{b, d\}$, then it is strongly A_I-set but it isn't a^*-I-open.*

3 Decomposition of continuity

In this section, we introduce the notions of a*-I-continuity, strongly A$_I$-continuity and obtain a decomposition of continuity.

Definition 6. *A function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is said to be b-continuous citeel-a if $f^{-1}\,(V)$ is a b-open set in (X, τ) for every open set V in (Y, φ).*

Definition 7. *A function $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ is said to be pre*-I-continuous citeeki (resp. δ beta$_I$-continuous citehat4, a*-I-continuous strongly A_I-continuous) if $f^{-1}\,(V)$ is a pre*-I-open (resp. $\delta\beta_I$-open, a*-I-open set, strongly A_I-set).*
(resp. $\delta\beta_I$-open, a*-I-open set, strongly A_I

Proposition 5. *For a function f:$(X, \tau, I) \longrightarrow (Y, \varphi)$, the following properties are hold: (1) If f is pre*-I-continuous, then f is a*-I(2) If f is b-continuous, then f is a*-I-continuous, (3) If f is a*-I-continuous, then f is $\delta\beta_I$*

Proof: The proofs are omitted from Proposition 3 as consequences by using Definitions 6 and 7. □

Remark 3. *The converses of each statements in Proposition 9 are not true in generally as shown in the next examples.*

Example 5. *Let (X, τ, I) be an ideal topological space as same as in Example 1 and $Y = \{a, b\}$, $\varphi = \{Y, \varnothing, \{a\}\}$. (1) Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function defined as $f(a) = f(d) = a$, $f(b) = f(c) = b$. Then f is a*-I-continuous, but it isn't pre*-I-continuous.*
(2) Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function defined as $f(b) = f(d) = a$, $f(a) = f(c) = b$. Then f is a*-I-continuous, but it isn't b-continuous.

Example 6. *Let (X, τ, I) be an ideal topological space as same as in Example 2 and $Y = \{a, b\}$, $\varphi = \{Y, \varnothing, \{a\}\}$. Let $f : (X, \tau, I) \longrightarrow (Y, \varphi)$ be a function defined as $f(a) = f(d) = a$, $f(b) = f(c) = b$. Then f is $\delta\beta_I$-continuous, but it isn't a*-I-continuous.*

It is known that a function $f : (X, \tau) \longrightarrow (Y, \varphi)$ is continuous if $f^{-1}\,(V)$ is an open set in (X, τ) for every open set V in (Y, φ).

Theorem 3. *For a function $f : (X, \tau, I) \longrightarrow (Y, \varphi)$, the following statements are equivalent: (1) f is continuous, (2) f is a*-I-continuous and strongly A_I-continuous.*

Proof: This follows from Theorem 8. □

Acknowledgement

This work is supported by Scientific Research Projects Coordination Office(BAP) of Selcuk University with 19701234 number project.

4 References

[1] M. E. Abd El-Monsef, S. N. El-Deeb, R. A. Mahmoud, *β-open sets and β-continuous mapping*, Bull. Fac. Sci. Assiut Univ. **A12**(1) (1983), 77-90.
[2] D. Andrijević, *On b-open sets*, Mat. Vesnik, **48** (1996), 59-64.

[3] A. Caksu Güler, G. Aslim, *b-I-open sets and decomposition of continuity via idealization*, Proceedings of Institute of Mathematics and Mechanics, Natural Academy of Sciences of Azerbaijan, **22** (2005), 27-32.
[4] J. Dontchev, *On pre-I-open sets and decompositon of I-continuity*, Banyan Math. J., **2** (1996).
[5] E. Ekici, T. Noiri, *On subsets and decompositions of continuity in ideal topological spaces*, Arab. J. Sci. Eng. Sect. A Sci.,**34** (2009), 165-167.
[6] A.A. El-Atik, A study on some types of mappings on topological spaces, MSc. Thesis, Tanta University, Egypt, 1997.
[7] E. Hatir, T. Noiri, *On decompositions of continuity via idealization*, Acta Math. Hungar., **96**(4) (2002), 341-349.
[8] E. Hatir, A. Keskin, T. Noiri, *A note on strong β-I-open sets and strongly β-I-continuous functions*, Acta Math. Hungar., **108**(1-2) (2005), 87-94.
[9] E. Hatir, *A note on $\delta\alpha$-I-open sets and $semi^*$-I-open sets*, Math. Commun., **16** (2011), 433-445.
[10] E. Hatir, *On decompositions of continuity and complete continuity in ideal topological spaces*, Europan Journal of Pure and Applied Math., **6**(3) (2013), 352-362.
[11] D. Jankovič, T.R. Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, **97** (1990), 295-310.
[12] K. Kuratowski, *Topology*, Vol. I, Academic Press, New York, 1966.
[13] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36-41.
[14] A. S. Mashhour, M. E. Abd El-Monsef, S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proceedings of the Mathematical and Physical Society of Egypt, **53** (1982), 47-53.
[15] O. Njastad, *On Some Classes of Nearly Open Sets*, Pasific J. Math., **15** (1965), 961-970.
[16] R. Vaidyanathaswamy, *The localisation theory in set topology*, Proc. Indian Acad. Sci. Math. Sci., **20** (1945), 51-61.
[17] N. V. Velićko, *H-closed topological spaces*, Amer. Math. Soc. Transl., **78** (1968), 103-118.
[18] S. Yüksel, A. Acikgöz, T. Noiri, *On δ-I-continuous functions*, Turkish Journal of Mathematics, **29** (2005), 39-51.

