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The Space bv_k^θ and Matrix Transformations

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Abstract: In this study, we introduce the space bv_k^θ, give its some algebraic and topological properties, and also characterize some matrix operators defined on that space. Also we extend some well known results.

Keywords: BK spaces, Matrix transformations, Sequence spaces.

1 Introduction

Let ω be the set of all complex sequences, ℓ_k and c be the sets of k-absolutely convergent series and convergent sequences, respectively. By bv we denote the space of all sequences of bounded variation, i.e.,

$$bv = \{x \in w : \Delta x \in \ell_k\}.$$

Let U and V be subspaces of w and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. By $A(x) = (A_n(x))$, we denote the A-transform of the sequence $x = (x_v)$, i.e.,

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

provided that the series is convergent for $n \geq 0$. Then, we say that A defines a matrix transformation from U into V, and denote it by $A \in (U, V)$ if the sequence $A(x) = (A_n(x)) \in V$ for every sequence $x \in U$, also the sets $U^\beta = \{\varepsilon = (\varepsilon_v) : \Sigma \varepsilon_v x_v \text{ converges for all } x \in U\}$ and

$$U_A = \{x \in \omega : A(x) \in U\} \tag{1}$$

are called the β dual of U and the domain of a matrix A in U. Further, $U \subset w$ is said to be a BK-space if it is a Banach space with continuous coordinates $p_n : U \to \mathbb{C}$ defined by $p_n(x) = x_n$ for $n \geq 0$. The sequence (e_v) is called a Schauder base (or briefly base) for a normed sequence space U if for each $x \in U$ there exist unique scalar coefficients (x_v) such that

$$\lim_{m\to\infty} \left\| x - \sum_{v=0}^{m} x_v e_v \right\| = 0,$$

and we write

$$x = \sum_{v=0}^{\infty} x_v e_v.$$

An infinite matrix $A = (a_{nv})$ is called a triangle if $a_{nn} \neq 0$ and $a_{nv} = 0$ for all $v > n$ for all n, v [1].
We define the notations Γ_c, Γ_∞ and Γ_s for $v = 1, 2, ...$, as follows:

$$\Gamma_c = \left\{ \varepsilon = (\varepsilon_v) : \lim_m \sum_{v=r}^{m} \varepsilon_v \text{ exists for } r = 1, 2, ... \right\},$$

$$\Gamma_\infty = \left\{ \varepsilon = (\varepsilon_v) : \sup_{m,r} \left| \sum_{v=r}^{m} \varepsilon_v \right| < \infty,\ r = 1, 2, ... \right\},$$

and

$$\Gamma_s = \left\{ \varepsilon = (\varepsilon_v) : \sup_m \sum_{r=1}^{m} \left| \theta_r^{-1/k^*} \sum_{v=r}^{m} \varepsilon_v \right|^{k^*} < \infty \right\},$$

where $k^*$is the conjugate of k, that is, $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

More recently some new sequence spaces by means of the matrix domain of a particular limitation method or absolute summability methods have been defined and studied by several authors in many research papers (see, for instance [2–8]). In this study, we introduce the space bv_k^{θ}, give its some algebraic and topological properties and characterize some matrix operators defined on that space. Also we extend some well known results.

The following lemmas are needed in proving our theorems.

Lemma 1. Let $1 \leq k < \infty$. Then, $A \in (\ell, \ell_k)$ if and only if

$$\sup_v \sum_{n=0}^{\infty} |a_{nv}|^k < \infty,$$

[9].

Lemma 2.
a-)

$$A \in (\ell, c) \Leftrightarrow (i) \quad \lim_n a_{nv} \text{ exists for each } v, \text{ and } \quad (ii) \quad \sup_{n,v} |a_{nv}| < \infty.$$

b-) Let $1 < k < \infty$.Then $A \in (\ell_k, c) \Leftrightarrow (i)$ holds and

$$\sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty$$

[10].

2 The space bv_k^{θ} and matrix operators

In this section we introduce the space bv_k^{θ} as

$$bv_k^{\theta} = \left\{ x = (x_k) \in w : \left(\theta_n^{1/k^*} \triangle x_n \right) \in \ell_k \right\},$$

where (θ_n) is a sequence of nonnegative terms, $1 \leq k < \infty$ and $\triangle x_n = x_n - x_{n-1}$ for all n. Note that it includes some known spaces. For example, it is reduced to bv^k for $\theta_n = 1$ for all n and $bv_1^{\theta} = bv$, which have been studied by Malkowsky et al [11] and Jarrah and Malkowsky [6]. Moreover, recently, Başar et al [3] have defined the sequence space $bv\,(u,p)$ and proved that this space is linearly isomorphic to the space $\ell(p)$ of Maddox [12] as generalized to paranormed space.

It is redefined as $bv_k^{\theta} = (\ell_k)_A$ with the notation (1), where the matrix A is defined by

$$a_{nv} = \begin{cases} -\theta_n^{1/k^*}, & v = n-1, \\ \theta_n^{1/k^*}, & v = n, \\ 0, & v \neq n, n-1. \end{cases}$$

Further, $\left|N_p^{\theta}\right|_k = \left(bv_k^{\theta}\right)_A$ and $|C_\alpha|_k = \left(bv_k^{\theta}\right)_B$ where A and B are Cesàro and Nörlund means of series Σx_n (see [8],[5, 13]).

Now we begin with topological properties of bv_k^{θ}, which also can be deduced from [3] .

Lemma 3. Let $1 \leq k < \infty$ and (θ_n) be a sequence of nonnegative numbers. Then,
a-) The space bv_k^{θ} is a BK-space and norm isomorphic to the space ℓ_k, $i.e.$, $bv_k^{\theta} \approx \ell_k$.
b-) $\left(bv_k^{\theta}\right)^{\beta} = \Gamma_c \cap \Gamma_s$ for $1 < k < \infty$ and $(bv)^{\beta} = \Gamma_c \cap \Gamma_\infty$ for $k = 1$.
c-) Define the sequence $b^{(j)} = \left(b_n^{(j)}\right)$ such that, for $j, n \geq 0$,

$$b_n^{(j)} = \begin{cases} \theta_j^{-1/k^*}, & n \geq j, \\ 0, & n < j. \end{cases}$$

Then, the sequence $b^{(j)} = \left(b_n^{(j)}\right)$ is the base of bv_k^{θ}.

Proof: a-) Since ℓ_k is a BK-space with respect to its usual norm and A is a triangle matrix, Theorem 4.3.2 of Wilansky [1, p. 61] gives the fact that bv_k^{θ} is a BK-space for $1 \leq k < \infty$. Now, consider $T : bv_k^{\theta} \to \ell_k$ defined by $y = T(x) = \left(\theta_n^{1/k^*} \Delta x_n\right)$ for all $x \in bv_k^{\theta}$. Then, it is clear that T is a linear operator, and surjective since, if $y = (y_n) \in \ell_k$, then $x = (x_n) = \left(\Sigma_{j=0}^n \theta_j^{-1/k^*} y_j\right) \in bv_k^{\theta}$, and also one to one. Further, it preserves the norm, since

$$\|T(x)\|_{\ell_k} = \left(\sum_{n=0}^{\infty} \theta_n^{k-1} |\triangle x_n|^k \right)^{1/k} = \|x\|_{bv_k^{\theta}},$$

which completes the proof.

b-) This part can be proved together with Lemma 2.

c-) Since the sequence $e^{(j)}$ is a base of ℓ_k, where $e^{(j)}=\left(e_n^{(j)}\right)_{n=0}^{\infty}$ is the sequence whose only non-zero term is 1 in the nth place for each $n\in\mathbb{N}$, it is clear that the sequence $b^{(j)}$ is the base of bv_k^{θ}. In fact, we first note that $T^{-1}(e^{(j)})=b^{(j)}$. Now, if $x\in bv_k^{\theta}$, then there exists $y\in\ell_k$ such that $y=T(x)$, and so it follows from (a) that

$$\left\|x-\sum_{j=0}^{m}x_jb^{(j)}\right\|_{bv_k^{\theta}}=\left\|y-\sum_{j=0}^{m}y_je^{(j)}\right\|_{\ell_k}\to 0 \text{ as } m\to\infty,$$

and it is easy to see that the representation $x=\sum_{j=0}^{\infty}x_jb^{(j)}$ is unique. □

Theorem 1. *Let $A=(a_{nv})$ be an infinite matrix of complex numbers for all $n,v\geq 0$, (θ_n) be a sequence of nonnegative numbers and $1\leq k<\infty$. Then, $A\in\left(bv,bv_k^{\theta}\right)$ if and only if*

$$\lim_{n\to\infty}\sum_{j=\nu}^{\infty}a_{nj} \text{ exists for each } v, \tag{2}$$

$$\sup_{n,v}\left|\sum_{j=v}^{\infty}a_{nj}\right|<\infty \tag{3}$$

and

$$\sup_{\nu}\sum_{n=0}^{\infty}\left|\theta_n^{1/k^*}\sum_{j=\nu}^{\infty}\left(a_{nj}-a_{n-1,j}\right)\right|^k<\infty. \tag{4}$$

Proof: $A\in\left(bv,bv_k^{\theta}\right)$ iff $\left(a_{nj}\right)_{j=0}^{\infty}\in bv^{\beta}$ and $A\left(x\right)\in bv_k^{\theta}$ for every $x\in bv$, and also, by Lemma 3, $\left(a_{nj}\right)_{j=0}^{\infty}\in bv^{\beta}$ iff (2) and (3) hold. Now, to prove necessity and sufficiency of the condition (4), consider the operators $B:bv\to\ell$ and $B':bv_k^{\theta}\to\ell_k$ defined by

$$B_n\left(x\right)=\Delta x_n,\ B'_n(x)=\theta_n^{1/k^*}\Delta x_n,$$

respectively. As in Lemma 3, these operators are bijection and the matrices corresponding to these operators are triangles. Further, let $x\in bv$ be given. Then, $B\left(x\right)=y\in\ell$ iff $x=S\left(y\right)$, where S is the inverse of B and it is given by

$$s_{n\nu}=\begin{cases}1, & 0\leq\nu\leq n,\\ 0, & \nu>n.\end{cases}$$

On the other hand, if any matrix $R=(r_{nv})\in(\ell,c)$, then, the series $R_n(x)=\Sigma r_{nv}x_v$ is convergent uniformly in n, since, by Lemma 2, the remaining term tends to zero uniformly in n, that is,

$$\left|\sum_{v=m}^{\infty}r_{nv}x_v\right|\leq\left(\sup_{n,v}|r_{nv}|\right)\sum_{v=m}^{\infty}|x_v|\to 0 \text{ as } m\to\infty,$$

and so

$$\lim_n R_n(x)=\sum_{v=0}^{\infty}\lim_n r_{nv}x_v. \tag{5}$$

Now, it is easily seen from (2) and (3) that $H=\left(h_{mr}^{(n)}\right)\in(\ell,c)$, which gives us, by (5), that

$$A_n(x)=\lim_m\sum_{r=0}^{m}h_{mr}^{(n)}y_r=\sum_{r=0}^{\infty}\left(\sum_{v=r}^{\infty}a_{nv}\right)y_r,$$

converges for all $n\geq 0$,where, for $r,m=0,1,...,$

$$h_{mr}^{(n)}=\begin{cases}\sum_{v=r}^{m}a_{nv}s_{vr}, & 0\leq r\leq m,\\ 0, & r>m.\end{cases}$$

This shows that the mapping sequence $A(x)=(A_n(x))$ exists. On the other hand, since S is the infinite triangle matrix, it is clear that $A(x)=A\left(S(y)\right)\in bv_k^{\theta}$ for every $x\in bv$ iff $B'\left(A\left(S(y)\right)\right)\in\ell_k$, i.e., $\left(B'oAoS\right)(y)\in\ell_k$, which implies that $D=B'oAoS:\ell\to\ell_k$.

Therefore, it can be written that $A : bv \to bv_k^\theta$ iff $D : \ell \to \ell_k$, and also $D = B'o\widehat{A}$, where $\widehat{A} = AoS$. Now, a few calculations reveal that

$$\widehat{a}_{nv} = \sum_{j=v}^{\infty} a_{nj} s_{jv} = \sum_{j=v}^{\infty} a_{nj}$$

and so

$$d_{nv} = \sum_{j=0}^{n} b'_{nj} \widehat{a}_{jv} = \theta_n^{1/k^*} \sum_{j=\nu}^{\infty} \left(a_{nj} - a_{n-1,j}\right)$$

Now, let us apply Lemma 1 with the matrix D. Then, it can be easily obtained from the definition of the matrix D that $D : \ell \to \ell_k$ iff condition (4) holds. This completes the proof.

□

If A is an infinite triangle matrix in Theorem 1, then (2) and (3) hold, and so it reduces to the following result.

Corollary 1. If A is an infinite triangle matrix of complex numbers for all $n, v \geq 0$ and $1 \leq k < \infty$, then, $A \in \left(bv, bv_k^\theta\right)$ if and only if

$$\sup_{\nu} \sum_{n=0}^{\infty} \left| \theta_n^{1/k^*} \sum_{j=\nu}^{n} \left(a_{nj} - a_{n-1,j}\right) \right|^k < \infty.$$

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