Conference Proceedings of Science and Technology, 2(3), 2019, 212–214

Conference Proceeding of 8th International Eurasian Conference on Mathematical Sciences and Applications (IECMSA-2019).

Fractional Solutions of a k-hypergeometric Differential Equation

ISSN: 2651-544X

http://dergipark.gov.tr/cpost

Resat Yilmazer [1,*] *Karmina K. Ali*[1,2]

[1] *Faculty of Science, Department of Mathematics, Firat University, Elazig, Turkey, ORCID:0000 0002 50593882*
[2] *Faculty of Science, Department of Mathematics, University of Zakho, Iraq, ORCID:0000-0002-3815-4457*
** Corresponding Author E-mail: rstyilmazer@gmail.com*

Abstract: In the present work, we study the second order homogeneous k-hypergeometric differential equation by utilizing the discrete fractional Nabla calculus operator. As a result, we obtained a novel exact fractional solution to the given equation.

Keywords: Discrete fractional, the k-hypergeometric differential equation, Nabla operator.

1 Introduction

Fractional calculus deal with derivatives and integrals of arbitrary orders, their applications seem in different areas of science such as physics, applied mathematics, chemistry, engineering [1–4]. Mathematical models have significant applications in physical and technical processing phenomena [5–9]. The solutions of the differential equations relevant to many interesting special functions in mathematics, physics, and engineering, such as the hypergeometric series [10], the zeta function [11], the continued fraction [12], the power series [13], the Fourier analysis [14]. The discrete fractional Nabla calculus operator have been applied to various singular ordinary equations such as the second-order linear ordinary differential equation of hypergeometric type [15], the modified Bessel differential equation [16], the radial equation of the fractional Schrödinger equation [17, 18], the Gauss equation [19], the non-Fuchsian differential equation [20], the Chebyshev's equation [21]. The aim of this study is to apply the Nabla calculus operator to a well-known ordinary differential equation k-hypergeometric equation [22], which is expressed by

$$kr\left(1-kr\right)\frac{d^2w}{dr^2}+\left[\alpha-\left(k+\rho+\sigma\right)kr\right]\frac{dw}{dr}-\rho\sigma w=v\left(r\right), \tag{1}$$

where $k\in\mathbb{R}^+,\alpha,\ \rho,\ \sigma\in\mathbb{R}^+$and $v\left(r\right)$ is holomorphic in an interval $D\subseteq\mathbb{C}$. If $k=1$ and the function $v\left(r\right)$ be vanishes identically, then Eq. (1) reduce to a linear homogenous hypergeometric ordinary differential equation (ODE) as follows

$$r\left(1-r\right)\frac{d^2w}{dr^2}+\left[\alpha-\left(1+\rho+\sigma\right)r\right]\frac{dw}{dr}-\rho\sigma w=0. \tag{2}$$

Many researchers have been studied the hypergeometric differential equation by different schemes, such as Kummer, presented the concurrent of hypergeometric equation in physical models [23]. Campos, finalize that this kind of equation contains complex calculations, and also the singularities of the differential equation are orderly. [24].

2 Preliminaries

Here, we have some imperative knowledge about the discrete fractional calculus theory and also some necessary notes, $\mathbb{N}$ is the set of natural numbers including zero, and $\mathbb{Z}$ is the set of integers. The $\mathbb{N}_b=\{b,b+1,b+2,...\}$ for $b\in\mathbb{Z}$. Let $f(t)$ and $g(t)$ are the real valued functions defined on $\mathbb{N}_0^+$. For more details see [15–21].
Definition 1. The rising factorial power is defined by

$$z^{\bar{n}}=t\left(z+1\right)\left(z+2\right)...\left(z+n-1\right),\ n\in\mathbb{N},\ z^{\bar{0}}=1.$$

Given α be a real number, then $z^{\bar{\alpha}}$ is expressed by

$$t^{\bar{\alpha}}=\frac{\Gamma\left(t+\alpha\right)}{\Gamma\left(t\right)}, \tag{3}$$

where $z\in\mathbb{R}\backslash\{...,-2,-1,0\}$, and $0^{\bar{\alpha}}=0$.
Let us symbolize that

$$\nabla\left(z^{\overline{\alpha}}\right)=\alpha z^{\overline{\alpha-1}}, \tag{4}$$

here $\nabla u(z) = u(z) - u(z-1)$. For $n = 2, 3, \ldots$ describe ∇^n by $\nabla^n = \nabla\nabla^{n-1}$.
Definition 2. The α^{th} order fractional sum of f is defined by

$$\nabla_b^{-\alpha} f(z) = \sum_{s=b}^{z} \frac{[s - \delta(z)]^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(s), \tag{5}$$

where $z \in \mathbb{N}_b,\ \delta(z) = z - 1$ is backward jump operator.
Theorem 1. Let $f(z)$ and $g(z) :\ \mathbb{N}_0^+ \to \mathbb{R},\ \alpha,\ \beta > 0$, and $h,\ v$ are constants, then

$$\nabla^{-\alpha}\nabla^{-\beta} f(z) = \nabla^{-(\alpha+\beta)} f(z) = \nabla^{-\beta}\nabla^{-\alpha} f(z) \tag{6}$$

$$\nabla^{\alpha}[hf(z) + vg(z)] = h\nabla^{\alpha} f(z) + v\nabla^{\alpha} g(z) \tag{7}$$

$$\nabla\nabla^{-\alpha} f(z) = \nabla^{-(\alpha-1)} f(z) \tag{8}$$

$$\nabla^{-\alpha}\nabla f(z) = \nabla^{(1-\alpha)} f(z) - \begin{pmatrix} z + \alpha - 2 \\ z - 1 \end{pmatrix} f(0) \tag{9}$$

Lemma 1. For all $\alpha > 0,\ \alpha^{th}$ order fractional difference of the product fg is expressed by

$$\nabla_0^{\alpha}(fg)(z) = \sum_{n=0}^{z} \begin{pmatrix} \alpha \\ n \end{pmatrix} \left[\nabla_0^{\alpha-n} f(z-n)\right] \left[\nabla^n g(z)\right]. \tag{10}$$

Lemma 2. If the function $f(t)$ is single valued and analytic, then

$$[f_{\alpha}(z)]_{\beta} = f_{\alpha+\beta}(z) = \left[f_{\beta}(z)\right]_{\alpha},\ \left[f_{\alpha}(z) \neq 0,\ f_{\beta}(z) \neq 0,\ \alpha,\ \beta \in \mathbb{R},\ z \in \mathbb{N}\right]. \tag{11}$$

3 Main results

Theorem 2. Let $w \in \{w : 0 \neq |w_{\vartheta}| < \infty,\ \vartheta \in \mathbb{R}\}$, and then the homogeneous k-hypergeometric equation is given by

$$w_2 kr(1 - kr) + w_1[\alpha - (k + \rho + \sigma) kr] - w\rho\sigma = 0, \tag{12}$$

has a particular solution of the form

$$w = h\left\{(r)^{-\left(\frac{1}{k}(\vartheta\theta k+\alpha)\right)}(1 - kr)^{-\left(\frac{1}{k}(\vartheta\theta k+\rho+\sigma-\alpha+k)\right)}\right\}_{-(\vartheta+1)},\ r \neq \left\{0, \frac{1}{k}\right\}. \tag{13}$$

where $w_m(r) = \frac{d^m w}{dr^m},\ (m = 0, 1, 2),\ w_0 = w(r)$, and $\alpha,\ \rho,\ \sigma$ are given constants as well as h is a constant of integration.
Proof. When we applied the discrete fractional calculus operator to both sides of Eq. (12), we have

$$\nabla^{\vartheta} w_2 kr(1 - kr) + \nabla^{\vartheta} w_1[\alpha - (k + \rho + \sigma) kr] - \nabla^{\vartheta}(w\rho\sigma) = 0, \tag{14}$$

using Eq. (8), and Eq. (9) together with Eq. (14), one may obtain

$$\begin{aligned} &w_{\vartheta+2} kr(1 - kr) + w_{\vartheta+1}[\vartheta\theta k(1 - 2kr) + \alpha - (k + \rho + \sigma) kr] \\ &+ w_{\vartheta}\left[-\vartheta(\vartheta - 1)\theta^2 k^2 + \vartheta\theta(-(k + \rho + \sigma) k) - \rho\sigma\right] = 0, \end{aligned} \tag{15}$$

where θ is a shift operator.
We choose ϑ such that

$$\vartheta(\vartheta - 1)\theta^2 k^2 + \vartheta\theta\left(k^2 + k\rho + k\sigma\right) + \rho\sigma = 0,$$

$$\vartheta = \frac{\left[\theta k - (k+\rho+\sigma) \pm \sqrt{((k+\rho+\sigma)-\theta k)^2 - 4\rho\sigma}\right]}{2\theta k}, \tag{16}$$

and let $(k+\rho+\sigma-\theta k)^2 \geq 4\rho\sigma$, then we have

$$w_{\vartheta+2} kr\,(1-kr) + w_{\vartheta+1}\left[\vartheta\theta k\,(1-2kr) + \alpha - (k+\rho+\sigma)\,kr\right] = 0, \tag{17}$$

and set

$$w_{\vartheta+1} = W = W\,(r)\,,\ \left(w = W_{-(\vartheta+1)}\right). \tag{18}$$

Therefore

$$W_1 + W\left[\frac{\vartheta\theta k\,(1-2kr) + \alpha - (k+\rho+\sigma)\,kr}{kr\,(1-kr)}\right] = 0, \tag{19}$$

by using Eq. (17), and Eq. (18), then the solution of the ODE Eq. (19) has the form

$$W = h\,(r)^{-\left(\frac{1}{k}(\vartheta\theta k+\alpha)\right)}(1-kr)^{-\left(\frac{1}{k}(\vartheta\theta k+\rho+\sigma-\alpha+k)\right)}. \tag{20}$$

4 Conclusion

In the present study, we applied the discrete fractional Nabla calculus operator to the homogeneous k-hypergeometric differential equation. As a result, we obtained a new exact discrete fractional solution.

5 References

[1] K. S. Miller, and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, 1993.
[2] K. Oldham, and J. Spanier, *The fractional calculus theory and applications of differentiation and integration to arbitrary order*, Elsevier, 1974.
[3] I. Podlubny, *Matrix approach to discrete fractional calculus. Fractional calculus and applied analysis*, **3**(4) (2000), 359-386.
[4] H. T. Michael, *The Laplace transform in discrete fractional calculus*, Computers and Mathematics with Applications **62**(3) (2011) 1591-1601.
[5] M. N. Özişik, H. R. B. Orlande, M. J. Colac, R. M. Cotta, *Finite difference methods in heat transfer*, CRC press, 2017.
[6] P. T. Kuchment, *Floquet theory for partial differential equations*, Birkhäuser, 2012.
[7] A. H. Khater, M. H. M. Moussa, and S. F. Abdul-Aziz, *Invariant variational principles and conservation laws for some nonlinear partial differential equations with variable coefficients part II*, Chaos, Solitons and Fractals **15**(1) (2013), 1-13.
[8] P. Verdonck, *The role of computational fluid dynamics for artificial organ design*, Artificial organs **26**(7) (2002), 569-570.
[9] A. Mandelis, *Diffusion-wave fields: mathematical methods and Green functions*, Springer Science and Business Media, 2013.
[10] G. M. Viswanathan, *The hypergeometric series for the partition function of the 2D Ising model*, Journal of Statistical Mechanics: Theory and Experiment **2015**(7) (2015), 07004.
[11] C. M. Bender, C. B. Dorje, and P.M. Markus, *Hamiltonian for the zeros of the Riemann zeta function*, Physical Review Letters **118**(13) (2017), 130201.
[12] P. Flajolet, *Combinatorial aspects of continued fractions*, Discrete mathematics **306**(10-11) (2006), 992-1021.
[13] G. Plonka, D. Potts, G. Steidi, M. Tasche, *Fourier series, Numerical Fourier Analysis*, Birkhäuser, Cham, 1-59, 2018.
[14] J. W. Cooley, J. W. Tukey, *An algorithm for the machine calculation of complex fourier series*, Mathematics of computation, **19**(90), 297-301, 1965.
[15] R. Yilmazer, M. Inc, F. Tchier, D. Baleanu, *Particular solutions of the confluent hypergeometric differential equation by using the nabla fractional calculus operator,* Entropy **18**(2) (2016), 49.
[16] R. Yilmazer, and O. Ozturk, *On Nabla discrete fractional calculus operator for a modified Bessel equation*, Therm. Sci. **22** (2018) 203-209.
[17] R. Yilmazer, *N-fractional calculus operator N^{μ} method to a modified hydrogen atom equation*, Mathematical Communications **15**(2) (2010), 489-501.
[18] R. Yilmazer, *Discrete fractional solutions of a Hermite equation*, Journal of Inequalities and Special Functions, **10**(1) (2019), 53-59.
[19] R. Yilmazer, *Discrete fractional solution of a non-Homogeneous non-Fuchsian differential equations*, Thermal Science, **23**(1) (2019), 121-127.
[20] R. Yilmazer, M. Inc, and M. Bayram, *On discrete fractional solutions of Non-Fuchsian differential equations*, Mathematics **6**(12) (2018), 308.
[21] M. Inc and R. Yilmazer, *On some particular solutions of the Chebyshev's equation by means of Ña discrete fractional calculus operator*, Prog. Fract. Differ. Appl. **2**(2) (2016), 123-129.
[22] L. Shengfeng, and Y. Dong, *k-Hypergeometric series solutions to one type of non-homogeneous k-Hypergeometric Equations*, Symmetry **11**(2) (2019), 262.
[23] E. E. Kummer, *De integralibus quibusdam definitis et seriebus infinitis*, Journal für die reine und angewandte Mathematik **17** (1837), 228-242.
[24] L. Campos, *On some solutions of the extended confluent hypergeometric differential equation*, Journal of computational and applied mathematics **137**(1) (2001), 177-200.