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ANALYSIS OF FRACTIONAL DIFFERENTIAL SYSTEMS INVOLVING RIEMANN LIOUVILLE FRACTIONAL DERIVATIVE

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Abstract. This paper is devoted to studying the multiple positive solutions for a system of nonlinear fractional boundary value problems. Our analysis is based upon the Avery Peterson fixed point theorem. In addition, we include an example for the demonstration of our main result.

1. Introduction

Researchers have focused a great deal of attention on the fractional boundary value problems due to the rapid progress in the theory and applications of fractional calculus. Aside from various fields of mathematics, boundary value problems for fractional differential equations have many applications in the area of chemistry, physics, biology, aerodynamics, control theory, economics, viscoelasticity, electrical circuits, and so forth. Driven by the numerous applications, there are many works related to the existence of positive solutions for the nonlinear fractional boundary value problems. For an overview of these type of study, we mention Podlubny [12], Jiqiang Jiang, Hongchuan Wang [21], Kilbas, Srivastava, and Trujillo [9], Bai and Sun [1], Goodrich [3], Cabrera, Harjani and Sadarangani [15], He, Zhang, Liu, Yonghong Wu and Cui, [16], Wang, Liang and Wang [17],Kamal Shah,Salman Zeb,Rahmat Ali Khan [25]. Goodrich [4] studied the following fractional boundary value problem subject to the given boundary conditions

$$D^{\alpha}u(t)+f(t,u)=0, \quad 0<t<1, \quad n-1<\alpha\leq n,$$
$$u^{(i)}(0)=0,\ 0\leq i\leq n-2, \quad D^{\delta}u(1)=0, \quad 1\leq\delta\leq n-2,$$

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where D^{α} is the standard Riemann-Liouville fractional derivative of order α and $f \in \mathcal{C}([0,1] \times [0,\infty))$, $n > 3$. The existence of positive solutions was analyzed by means of the Krasnoselskii's fixed point theorem on cones.

In [20], C.F.Li et al. considered the following boundary value problem of fractional derivative equations

$$\begin{aligned} D^{\alpha}u(t) + f(t,u) &= 0, \quad 0 < t < 1, \\ u(0) &= 0, \\ D^{\beta}u(1) &= aD^{\beta}u(\eta), \end{aligned}$$

where D^{α} is the standard Riemann-Liouville fractional derivative of order α, $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $0 \leq a \leq 1$, $\eta \in (0,1)$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. Here, the argument relies on some fixed theorems on cones.

At the same time, boundary value problems for integer order differential systems are widely studied, despite fractional differential systems have emerged as a significant field of investigation quite recently. Thus intensive study of the existence theory of fractional systems has been carried out by means of methods of nonlinear analysis such as fixed point theory, lower and upper solutions, monotone iterative methods, see [11, 13, 14, 6, 7, 8, 5, 10, 22, 23, 24] and the references therein.

In this paper, we discuss the multiple positive solutions for the following systems of nonlinear fractional differential equations :

$$D^{q_1}u(t) + f_1(t,u(t),v(t)) = 0, \quad t \in (0,1), \tag{1}$$

$$D^{q_2}v(t) + f_2(t,u(t),v(t)) = 0, \quad t \in (0,1), \tag{2}$$

$$u(0) = u'(0) = 0, D^{p_1}u(1) = \mu D^{p_1}u(\eta) + g_1(\int_0^1 u(s)dA_1(s), \int_0^1 v(s)dA_1(s)), \tag{3}$$

$$v(0) = v'(0) = 0, D^{p_2}v(1) = \mu D^{p_2}v(\eta) + g_2(\int_0^1 u(s)dA_2(s), \int_0^1 v(s)dA_2(s)), \tag{4}$$

in which D is the Riemann-Liouville fractional derivative, $2 < q_i \leq 3$ and $0 < p_i \leq 1$, $0 < q_i - p_i - 1$ for $i = 1,2$, $0 < \eta < 1$, $\mu \in (0,\infty)$, $\mu\eta^{q_i - p_i - 1} < 1$, $\int_0^1 u(s)dA_i(s)$ and $\int_0^1 v(s)dA_i(s)$ are the Riemann- Stieltjes integrals with positive measures, A_1 and A_2 are functions of bounded variation, $f_i \in \mathcal{C}([0,1] \times [0,\infty) \times [0,\infty), [0,\infty))$, $g_i \in \mathcal{C}([0,\infty] \times [0,\infty), [0,\infty))$ for $i = 1,2$.

Motivated by the above papers, our goal is to obtain the existence of multiple positive solutions for the fractional differential system (1)-(4). Here, we employ Riemann-Stieltjes integral boundary conditions. As they include multi-point and integral conditions as special cases, the system (1)-(4) is more general than the problems mentioned in some literature. Applying the Avery Peterson fixed point theorem, multiple positive solutions are established. An example is also presented to illustrate our main result.

In order to present our main result, we will make use of the following concepts and the Avery Peterson fixed point theorem.

Let φ and θ be nonnegative continuous convex functionals on the cone P, ϕ be a nonnegative continuous concave functional on P, and ψ be a nonnegative continuous functional on P. Then, for positive numbers a,b,c,d we define the following sets:

$$\begin{aligned} P(\varphi, d) &= \{x \in P : \varphi(x) < d\}, \\ P(\varphi, \phi, b, d) &= \{x \in P : b \le \phi(x), \varphi(x) \le d\}, \\ P(\varphi, \theta, \phi, b, c, d) &= \{x \in P : b \le \phi(x), \theta(x) \le c, \varphi(x) \le d\}, \\ R(\varphi, \psi, a, d) &= \{x \in P : a \le \psi(x), \varphi(x) \le d\}. \end{aligned}$$

Theorem 1. [18] *Let P be a cone in a real Banach space E. and φ, θ, ϕ, ψ be defined as above, furthermore ψ holds $\psi(kx) \le k\psi(x)$ for $0 \le k \le 1$ such that, for some positive numbers $\overline{M}$ and d,*

$$\phi(x) \le \psi(x) \text{ and } \|x\| \le \overline{M}\varphi(x)$$

for all $x \in \overline{P(\varphi, d)}$. Assume $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$ is completely continuous and there exist positive numbers a,b,c with $a < b$, such that

$(S_1) : \{x \in P(\varphi, \theta, \phi, b, c, d) : \phi(x) > b\} \ne \emptyset$ and $\phi(Tx) > b$ for $x \in P(\varphi, \theta, \phi, b, c, d)$,

$(S_2) : \phi(Tx) > b$ for $x \in P(\varphi, \phi, b, d)$ with $\theta(Tx) > c$,

$(S_3) : 0 \notin R(\varphi, \psi, a, d)$ and $\psi(Tx) < a$ for $x \in R(\varphi, \psi, a, d)$ with $\psi(x) = a$.

Then, T has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\varphi, d)}$, such that

$\varphi(x_i) \le d$, for $i = 1, 2, 3$; $b < \phi(x_1)$, $a < \psi(x_2)$, with $\phi(x_2) < b$ and $\psi(x_3) < a$.

2. Existence Results

During the last decade, many definitions on the fractional calculus have been carried out. In our paper, our work is based upon the Riemann Liouville fractional operator defined by

$$D^{\nu} g(t) = \frac{1}{\Gamma(n-\nu)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\nu-1} g(s) ds,$$

where $g : (0, \infty) \to \mathcal{R}$ is a function, n is the smallest integer greater than or equal to ν whenever the right hand side is defined. In particular, for $\nu = n$, $D^{\nu} g(t) = D^n g(t)$.

In order to derive the main result of the system (1)-(4), we present the following lemma:

Lemma 2. *If $h, y \in \mathcal{C}[0, 1]$, then the fractional differential equation*

$$\begin{aligned} D^{q_1} u(t) + h(t) &= 0, \quad t \in (0, 1), &(5)\\ D^{q_2} v(t) + y(t) &= 0, \quad t \in (0, 1), &(6) \end{aligned}$$

with the boundary conditions (3) and (4) has the solution

$$u(t) = \int_0^1 H_1(t, s) h(s) ds + \frac{t^{q_1-1} \Gamma(q_1 - p_1)}{\Gamma(q_1) \Delta_1} g_1(\int_0^1 u(s) dA_1(s), \int_0^1 v(s) dA_1(s)),$$

$$v(t) = \int_0^1 H_2(t,s)y(s)ds + \frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2} g_2(\int_0^1 u(s)dA_2(s), \int_0^1 v(s)dA_2(s)),$$

where

$$H_i(t,s) = G_i(t,s) + \frac{t^{q_i-1}\mu}{\Gamma(q_i)\Delta_i}\overline{G_i}(\eta,s), \tag{7}$$

$$G_i(t,s) = \frac{1}{\Gamma(q_i)}\begin{cases} t^{q_i-1}(1-s)^{q_i-p_i-1} - (t-s)^{q_i-1}, & 0 \le s \le t \le 1, \\ t^{q_i-1}(1-s)^{q_i-p_i-1}, & 0 \le t \le s \le 1, \end{cases} \tag{8}$$

$$\overline{G}_i(\eta,s) = \begin{cases} \eta^{q_i-p_i-1}(1-s)^{q_i-p_i-1} - (\eta-s)^{q_i-p_i-1}, & 0 \le s \le \eta \le 1, \\ \eta^{q_i-p_i-1}(1-s)^{q_i-p_i-1}, & 0 \le \eta \le s \le 1, \end{cases} \tag{9}$$

and $\Delta_i = 1 - \mu\eta^{q_i-p_i-1}$, $(i \in \{1,2\})$.

Proof. The equations (5) and (6) can be translated into the following equations:

$$u(t) = -\frac{1}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}h(s)ds + c_1t^{q_1-1} + c_2t^{q_1-2} + c_3t^{q_1-3},$$
$$v(t) = -\frac{1}{\Gamma(q_2)}\int_0^t (t-s)^{q_2-1}y(s)ds + d_1t^{q_2-1} + d_2t^{q_2-2} + d_3t^{q_2-3}.$$

Taking into account of (3)-(4) and $D^\sigma[t^{q-1}] = \frac{\Gamma(q)}{\Gamma(q-\sigma)}t^{q-\sigma-1}$ $(\sigma, q > 0)$, we obtain $c_2 = c_3 = 0$, $d_2 = d_3 = 0$ and

$$\begin{aligned} c_1 = & \frac{1}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})}\int_0^1 (1-s)^{q_1-p_1-1}h(s)ds \\ & -\frac{\mu}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})}\int_0^\eta (\eta-s)^{q_1-p_1-1}h(s)ds \\ & +\frac{\Gamma(q_1-p_1)}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})}g_1(\int_0^1 u(s)dA_1(s)), \int_0^1 v(s)dA_1(s)), \end{aligned}$$

$$\begin{aligned} d_1 = & \frac{1}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})}\int_0^1 (1-s)^{q_2-p_2-1}y(s)ds \\ & -\frac{\mu}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})}\int_0^\eta (\eta-s)^{q_2-p_2-1}y(s)ds \\ & +\frac{\Gamma(q_2-p_2)}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})}g_2(\int_0^1 u(s)dA_2(s)), \int_0^1 v(s)dA_2(s)). \end{aligned}$$

So, the solution is

$$\begin{aligned} u(t) = & -\frac{1}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}h(s)ds \\ & +\frac{t^{q_1-1}}{\Gamma(q_1)(1-\mu\eta^{q_1-p_1-1})}\int_0^1 (1-s)^{q_1-p_1-1}h(s)ds \end{aligned}$$

$$
\begin{aligned}
&-\frac{t^{q_1-1}\mu}{\Gamma(q_1)\Delta_1}\int_0^\eta(\eta-s)^{q_1-p_1-1}h(s)ds\\
&+\frac{t^{q_1-1}\Gamma(q_1-p_1)}{\Gamma(q_1)\Delta_1}g_1(\int_0^1 u(s)dA_1(s)),\int_0^1 v(s)dA_1(s))\\
=\;&\int_0^1 H_1(t,s)h(s)ds+\frac{t^{q_1-1}\Gamma(q_1-p_1)}{\Gamma(q_1)\Delta_1}g_1(\int_0^1 u(s)dA_1(s)),\int_0^1 v(s)dA_1(s)),
\end{aligned}
$$

$$
\begin{aligned}
v(t)\;=\;&-\frac{1}{\Gamma(q_2)}\int_0^t(t-s)^{q_2-1}y(s)ds\\
&+\frac{t^{q_2-1}}{\Gamma(q_2)(1-\mu\eta^{q_2-p_2-1})}\int_0^1(1-s)^{q_2-p_2-1}y(s)ds\\
&-\frac{t^{q_2-1}\mu}{\Gamma(q_2)\Delta_2}\int_0^\eta(\eta-s)^{q_2-p_2-1}y(s)ds\\
&+\frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2}g_2(\int_0^1 u(s)dA_2(s)),\int_0^1 v(s)dA_2(s))\\
=\;&\int_0^1 H_2(t,s)y(s)ds+\frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2}g_2(\int_0^1 u(s)dA_2(s)),\int_0^1 v(s)dA_2(s)).
\end{aligned}
$$

□

Lemma 3. *(See [2]) The function $G_i(t,s)$, $i\in\{1,2\}$ holds the following properties :*

(i) $G_i(t,s)\geq 0$ for any $t,s\in[0,1]$,

(ii) $p_i t^{q_i-1}L_i(s)\leq G_i(t,s)\leq L_i(s)$ for any $t,s\in[0,1]$,
where

$$L_i(s)=\frac{s(1-s)^{q_i-p_i-1}}{\Gamma(q_i)}. \tag{10}$$

One can easily obtain the following lemma.

Lemma 4. *The function $H_i(t,s)$, $i\in\{1,2\}$ holds the following properties :*

(i) $H_i(t,s)\geq 0$ for any $t,s\in[0,1]$,

(ii) $p_i t^{q_i-1}K_i(s)\leq H_i(t,s)\leq K_i(s)$ for any $t,s\in[0,1]$,

where $K_i(s)=\dfrac{s(1-s)^{q_i-p_i-1}}{\Gamma(q_i)}+\dfrac{\mu\overline{G}_i(\eta,s)}{\Gamma(q_i)\Delta_i}$.

Let us introduce the Banach space $\mathcal{B} = \mathcal{C}[0,1] \times \mathcal{C}[0,1]$ with the norm $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in \mathcal{B}$ and $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define a cone

$$P = \left\{(u,v) \in \ \mathcal{B} : u(t) \geq 0, v(t) \geq 0, t \in [0,1], \min_{t\in[\eta,1]}(u(t)+v(t)) \geq p\|(u,v)\|\right\}$$

where $p = min\left\{p_1\eta^{q_1-1}, p_2\eta^{q_2-1}\right\}$and operators $T_i : P \to \mathcal{B}$, $i \in \{1,2\}$given by

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 H_1(t,s) f_1(s,u(s),v(s))ds \\ &\quad + \frac{t^{q_1-1}\Gamma(q_1-p_1)}{\Gamma(q_1)\Delta_1} g_1\left(\int_0^1 u(s)dA_1(s), \int_0^1 v(s)dA_1(s)\right), \\ T_2(u,v)(t) &= \int_0^1 H_2(t,s) f_2(s,u(s),v(s))ds \\ &\quad + \frac{t^{q_2-1}\Gamma(q_2-p_2)}{\Gamma(q_2)\Delta_2} g_2\left(\int_0^1 u(s)dA_2(s), \int_0^1 v(s)dA_2(s)\right). \end{aligned}$$

Let us set

$$\begin{aligned} N_i &= 4\int_0^1 K_i(s)ds, \\ m_i &= 2p\int_\eta^1 K_i(s)ds, \\ \overline{L_i} &= \frac{4\Gamma(q_i-p_i)\int_0^1 dA_i(s)}{\Gamma(q_i)\Delta_i}. \end{aligned}$$

To prove that the system $(1)-(4)$ has three positive solutions, the following three functionals are defined by

$$\phi(u,v) = \min_{t\in[\eta,1]}(u(t)+v(t)), \quad \psi(u,v) = \theta(u,v) = \varphi(u,v) = \|u\| + \|v\|.$$

The main theorem of this paper is stated as follows :

Theorem 5. *Assume that there exist constants* $0 < a < b < \frac{b}{p} < c < d$ *such that* $b \leq \frac{m_i d}{N_i}$ *and* f_i, g_i *hold the following conditions:*

(C_1) $f_i(t,u,v) \leq \dfrac{d}{N_i}$ *for* $t \in [0,1]$, $(u+v) \in [0,d]$,

(C_2) $f_i(t,u,v) > \dfrac{b}{m_i}$ *for* $t \in [\eta,1]$, $(u+v) \in [b,c]$,

(C_3) $f_i(t,u,v) \leq \dfrac{a}{N_i}$ *for* $t \in [0,1]$, $(u+v) \in [0,a]$,

(C_4) $g_i(u,v) \leq \dfrac{u+v}{\overline{L}_i}$ *for* $(u+v) \in [0, d\int_0^1 dA_i(s)]$.

Then the system $(1)-(4)$ *has at least three positive solutions* (u_i, v_i) $(i = 1, 2, 3)$ *such that* $\|(u_i, v_i)\| \leq d$, $i = 1, 2, 3$; $b \leq \phi(u_1, v_1)$, $a < \|\psi(u_2, v_2)\|$ *with* $\phi(u_2, v_2) < b$ *and* $\|(u_3, v_3)\| < a$.

Proof. Define the completely continuous operator $T : P \to \mathcal{B}$ by

$$T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t)).$$

As easily seen, the fixed point of the operator T is the solution of the system $(1)-(4)$. First, we check that $T : P \to P$. Lemma 4 and the nonnegativity of f_i and g_i imply that $T_1(u, v)(t) \geq 0$, $T_2(u, v)(t) \geq 0$ for $t \in [0, 1]$. Besides, for $(u, v) \in P$

$$\begin{aligned}
\|T_1(u,v)\| &\leq \int_0^1 K_1(s) f_1(s, u(s), v(s)) ds \\
&\quad + \frac{\Gamma(q_1 - p_1)}{\Gamma(q_1)\Delta_1} g_1\left(\int_0^1 u(s) dA_1(s), \int_0^1 v(s) dA_1(s)\right), \\
\|T_2(u,v)\| &\leq \int_0^1 K_2(s) f_2(s, u(s), v(s)) ds \\
&\quad + \frac{\Gamma(q_2 - p_2)}{\Gamma(q_2)\Delta_2} g_2\left(\int_0^1 u(s) dA_2(s), \int_0^1 v(s) dA_2(s)\right)
\end{aligned}$$

and

$$\begin{aligned}
\min_{t \in [\eta, 1]} T_1(u, v)(t) &\geq p_1 \eta^{q_1 - 1} \int_0^1 K_1(s) f_1(s, u(s), v(s)) ds \\
&\quad + \frac{\eta^{q_1 - 1}\Gamma(q_1 - p_1)}{\Gamma(q_1)\Delta_1} g_1\left(\int_0^1 u(s) dA_1(s), \int_0^1 v(s) dA_1(s)\right) \\
&\geq p_1 \eta^{q_1 - 1} \|T_1(u, v)\|.
\end{aligned}$$

In a similar manner, we obtain $\min\limits_{t \in [\eta, 1]} T_2(u, v)(t) \geq p_2 \eta^{q_2 - 1} \|T_2(u, v)\|$. Thus,

$$\begin{aligned}
\min_{t \in [\eta, 1]} \{T_1(u, v)(t) + T_2(u, v)(t)\} &\geq p_1 \eta^{q_1 - 1} \|T_1(u, v)\| + p_2 \eta^{q_2 - 1} \|T_2(u, v)\| \\
&\geq p[\|T_1(u, v)\| + \|T_2(u, v)\|] \\
&= p\|T(u, v)\|,
\end{aligned}$$

so $T : P \to P$. Furthermore by employing standard methods, T is a completely continuous operator.

Now, all the conditions of Theorem 1 will be shown to be verified. First, we indicate that $T : \overline{P(\varphi, d)} \to \overline{P(\varphi, d)}$. If $(u, v) \in \overline{P(\varphi, d)}$, then $\varphi(u, v) \leq d$, $\|u\| + \|v\| \leq d$. In view of C_4, we can get

$$g_i\left(\int_0^1 u(s) dA_i(s), \int_0^1 v(s) dA_i(s)\right) \leq \frac{\int_0^1 (u(s) + v(s)) dA_i(s)}{\overline{L}_i}$$

$$\leq \frac{d\int_0^1 dA_i(s)}{\overline{L}_i}.$$

Hence, (C_1) yields that

$$\begin{aligned}
\max_{t\in[0,1]} T_1(u,v)(t) &= \max_{t\in[0,1]}\Big|\int_0^1 H_1(t,s)f_1(s,u(s),v(s))ds \\
&\quad + \frac{t^{q_1-1}\Gamma(q_1-p_1)g_1(\int_0^1 u(s)dA_1(s),\int_0^1 v(s)dA_1(s))}{\Gamma(q_1)\Delta_1}\Big| \\
&\leq \frac{d}{N_1}\int_0^1 K_1(s)ds + \frac{\Gamma(q_1-p_1)d}{\Gamma(q_1)\Delta_1\overline{L}_1}\int_0^1 dA_1(s) \\
&\leq \frac{d}{2}.
\end{aligned}$$

In the same way, one has $\max_{t\in[0,1]} T_2(u,v)(t) \leq \frac{d}{2}$. So, we have $T : \overline{P}(\varphi,d) \to \overline{P}(\varphi,d)$. Next, we indicate that (S_1) of Theorem 1 is fulfilled. Take $(\frac{b}{2p},\frac{b}{2p})$. Then, one may verify that $(\frac{b}{2p},\frac{b}{2p}) \in P(\varphi,\theta,\phi,b,c,d)$ and $\phi(u,v) > b$. Hence, $\{(u,v)\in P(\varphi,\theta,\phi,b,c,d) : \phi(u,v) > b\} \neq \emptyset$. Choose $(u,v)\in P(\varphi,\theta,\phi,b,c,d)$, then this means $(u(t)+v(t)) \in [b,c]$ for any $t\in[\eta,1]$. By C_2 we get

$$\begin{aligned}
\phi(T(u,v)) &= \min_{t\in[\eta,1]}(T_1(u,v)(t)+T_2(u,v)(t)) \\
&\geq p\int_\eta^1 K_1(s)f_1(s,u(s),v(s))ds + p\int_\eta^1 K_2(s)f_2(s,u(s),v(s))ds \\
&> p\frac{b}{m_1}\int_\eta^1 K_1(s)ds + p\frac{b}{m_2}\int_\eta^1 K_2(s)ds \\
&> b.
\end{aligned}$$

Thus (S_1) of Theorem 1 holds.

Finally, we need to show that the last condition of Theorem 1 is fulfilled. In fact, if $(u,v)\in P(\varphi,\phi,b,d)$ with $\theta(T(u,v)) > c$, then

$$\begin{aligned}
\min_{t\in[\eta,1]}(T_1(u,v)(t)+T_2(u,v)(t)) &\geq p\|T(u,v)\| \\
&> pc > b,
\end{aligned}$$

so, (S_2) holds.

Since $a > 0$, 0 is not member of $R(\varphi,\psi,a,d)$ with $\psi(u,v) = a$. Let $(u,v)\in R(\varphi,\psi,a,d)$ and $\psi(u,v) = a$, then using (C3), we get

$$\begin{aligned}
\psi(T(u,v)) &= \|T(u,v)\| \\
&\leq \int_0^1 K_1(s)f_1(s,u(s),v(s))ds + \int_0^1 K_2(s)f_2(s,u(s),v(s))ds
\end{aligned}$$

$$
\begin{aligned}
& +\frac{\Gamma(q_1-p_1)g_1(\int_0^1 u(s)dA_1(s),\int_0^1 v(s)dA_1(s))}{\Gamma(q_1)\Delta_1} \\
& +\frac{\Gamma(q_2-p_2)g_2(\int_0^1 u(s)dA_2(s),\int_0^1 v(s)dA_2(s))}{\Gamma(q_2)\Delta_2} \\
\leq \quad & \frac{a}{N_1}\int_0^1 K_1(s)ds+\frac{a}{N_2}\int_0^1 K_2(s)ds \\
& +\frac{\Gamma(q_1-p_1)a}{\Gamma(q_1)\Delta_1\overline{L}_1}\int_0^1 dA_1(s)+\frac{\Gamma(q_2-p_2)a}{\Gamma(q_2)\Delta_2\overline{L}_2}\int_0^1 dA_2(s) \\
= \quad & a.
\end{aligned}
$$

Because all the condition of Theorem 1 fulfilled, the assertion of Theorem 5 is satisfied. The proof is complete. □

Example 6. *Consider*

$$
\begin{cases}
D^{5/2}u(t)+f_1(t,u(t),v(t))=0, \quad t\in(0,1), \\
D^{5/2}v(t)+f_2(t,u(t),v(t))=0, \quad t\in(0,1), \\
u(0)=u'(0)=v(0)=v'(0)=0, \\
D^{1/2}u(1)=1/2D^{1/2}u(1/2)+g_1(\int_0^1 u(s)dA_1(s),\int_0^1 v(s)dA_1(s)), \\
D^{1/2}v(1)=1/2D^{1/2}v(1/2)+g_2(\int_0^1 u(s)dA_2(s),\int_0^1 v(s)dA_2(s)),
\end{cases} \tag{11}
$$

in which $q_1=q_2=\frac{5}{2}$, $p_1=p_2=\frac{1}{2}$, $\mu=\frac{1}{2}$, $A_1(s)=A_2(s)=s^2$, $\eta=\frac{1}{2}$,

$$
f_1(t,u,v)=\begin{cases}
\frac{t}{7}+\frac{4(u+v)}{5}, & (u+v)\in[0,10], \\
\frac{t}{7}+\frac{642(u+v)-6340}{10}, & (u+v)\in[10,20], \\
\frac{t}{7}+\frac{5(u+v)+37600}{58}, & (u+v)\in[20,600], \\
\frac{t}{7}+700, & (u+v)\in[600,\infty),
\end{cases}
$$

$$
f_2(t,u,v)=\begin{cases}
\frac{t}{10}+\frac{4(u+v)}{5}, & (u+v)\in[0,10], \\
\frac{t}{10}+\frac{642(u+v)-6340}{10}, & (u+v)\in[10,20], \\
\frac{t}{10}+\frac{5(u+v)+37600}{58}, & (u+v)\in[20,600], \\
\frac{t}{10}+700, & (u+v)\in[600,\infty).
\end{cases}
$$

And

$$
g_i(u,v)=\begin{cases}
\frac{9\sqrt{\pi}}{64}ln(u+v+1), & (u+v)\in[0,600], \\
\frac{9\sqrt{\pi}}{64}ln(601), & (u+v)\in[600,\infty).
\end{cases}
$$

It is easily seen that $\Delta_1=\Delta_2=\frac{3}{4}$. *We obtain,* $N_1=N_2=\frac{4}{3\sqrt{\pi}}$, *then* $p=(\frac{1}{2})^{\frac{5}{2}}$, $m_1=m_2=\frac{1}{2^{\frac{5}{2}}3\sqrt{\pi}}$. *And* $\overline{L_1}=\overline{L_2}=\frac{64}{9\sqrt{\pi}}$. *Choosing,*

$f_1(t,u,v)\leq\frac{d}{N_1}\approx 1413,7$, *for* $t\in[0,1]$, $(u+v)\in[0,600]$,

$$f_1(t,u,v) \geq \frac{b}{m_1} \approx 601,59, \textit{ for } t \in [\tfrac{1}{2},1],\ (u+v) \in [20,200],$$

$$f_1(t,u,v) \leq \tfrac{a}{N_1} \approx 13,29 \textit{ for } t \in [0,1],\ (u+v) \in [0,10],$$

$$g_i(u,v) \leq \tfrac{u+v}{L_i} \textit{ for } (u+v) \in [0,600].$$

We conclude that all the assumptions of Theorem 5 are verified, thus the problem (11) has at least three positive solutions.

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