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SS-SUPPLEMENTED MODULES

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Abstract. A module M is called *ss-supplemented* if every submodule U of M has a supplement V in M such that $U \cap V$ is semisimple. It is shown that a finitely generated module M is ss-supplemented iff it is supplemented and $Rad(M) \subseteq Soc(M)$. A module M is called *strongly local* if it is local and $Rad(M)$ is semisimple. Any direct sum of strongly local modules is ss-supplemented and coatomic. A ring R is semiperfect and $Rad(R) \subseteq Soc(_RR)$ iff every left R-module is (amply) ss-supplemented iff $_RR$ is a finite sum of strongly local submodules.

1. Introduction

Throughout this study, all rings are associative with identity and all modules are unitary left modules. Let R be a ring and M be an R-module. $U \subseteq M$ will mean that U is a submodule of M. $Rad(M)$ and $Soc(M)$ will indicate radical and socle of M. A submodule N of M is called *small* in M, denoted $N << M$, if $M \neq N + K$ for every proper submodule K of M. Let U and V be submodules of M. V is called a *supplement* of U in M if it is minimal with respect to $M = U + V$, equivalently $M = U + V$ and $U \cap V << V$. The module M is called *supplemented* if every submodule of M has a supplement in M. A submodule U of M has *ample supplements* in M if every submodule L of M such that $M = U + L$ contains a supplement of U in M. The module M is called *amply supplemented* if every submodule of M has ample supplements in M. For characterizations of supplemented and amply supplemented modules we refer to [7]

A non-zero module M is called *hollow* if every proper submodule of M is small in M and is called *local* if the sum of all proper submodules of M is also a proper submodule of M. Note that local modules are hollow and hollow modules are clearly amply supplemented. A ring R is called *local ring* if $_RR$ is a local module.

In [8], Zhou and Zhang generalized the concept of socle of a module M to that of $Soc_s(M)$ by considering the class of all simple submodules of M that are small in M

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in place of the class of all simple submodules of M, that is, $Soc_s(M) = \sum\{N << M \,|\, N \text{ is simple}\}$. It is clear that $Soc_s(M) \subseteq Rad(M)$ and $Soc_s(M) \subseteq Soc(M)$.

We call a module M *strongly local* if it is local and $Rad(M)$ is semisimple. We call a ring R *left strongly local ring* if ${}_RR$ is a strongly local module. Then we have that the following implications on modules:

$$\text{simple} \Longrightarrow \text{strongly local} \Longrightarrow \text{local}$$

Next we mention two examples which show that the above implications are proper. For the local left $\mathbb{Z}$-module $M = \mathbb{Z}_4$, we have $Rad(M) = Soc(M)$. Hence, M is strongly local but not simple. On the other hand, for the local left $\mathbb{Z}$-module $M = \mathbb{Z}_8$, $Soc(M)$ is a proper submodule of $Rad(M)$. Thus M is not a strongly local module.

In section 2 we study on strongly local modules and rings. We show that every left strongly local ring is left perfect and right perfect. A strongly local commutative domain is field.

Let U and V be submodules of a module M. V is called a *Rad-supplement* of U in M if $M = U + V$ and $U \cap V \subseteq Rad(V)$. Since $Soc_s(V) \subseteq Rad(V)$, it is of interest to investigate the analogue of this notion by replacing "$Rad(V)$"with "$Soc_s(V)$". Now, we give the following result playing a key role in our work as a proper generalization of direct summands. Firstly, we need the following well known facts that we include here for completeness.

Lemma 1. *Let M be a module and N be a semisimple submodule of M which is contained in $Rad(M)$. Then $N << M$.*

Proof. Let $N + K = M$ for some submodule K of M. Since N is semisimple, there exists a submodule N' of N such that $N = (N \cap K) \oplus N'$. Hence $M = N + K = [(N \cap K) \oplus N'] + K = N' + K$. Since $N' \cap K = (N' \cap N) \cap K = N' \cap (N \cap K) = 0$, we have $M = N' \oplus K$. It follows from [7, 21.6 (5)] that $Rad(M) = Rad(N') \oplus Rad(K) = Rad(K)$ since $Rad(N') \subseteq Rad(N) = 0$. Then $M = N + K \subseteq Rad(M) + K \subseteq K$. It means that $N << M$. □

Lemma 2. *Let M be a module. Then $Soc_s(M) = Rad(M) \cap Soc(M)$.*

Proof. Let $a \in Rad(M) \cap Soc(M)$. Then Ra is semisimple and so there exist $n \in \mathbb{Z}^+$ and simple submodules S_i of M $(1 \leq i \leq n)$ such that $Ra = S_1 \oplus S_2 \oplus ... \oplus S_n$ by [6, Proposition 3.3]. Since Ra is small in M, it follows from [7, 19.3 (2)] that each S_i is small in M. Thus $a \in Ra \subseteq Soc_s(M)$. □

Lemma 3. *Let M be a module and U, V be submodules of M. Then the following statements are equivalent:*

(1) *$M = U + V$ and $U \cap V \subseteq Soc_s(V)$,*
(2) *$M = U + V$, $U \cap V \subseteq Rad(V)$ and $U \cap V$ is semisimple,*
(3) *$M = U + V$, $U \cap V << V$ and $U \cap V$ is semisimple.*

Proof. (1) $\Longrightarrow$ (2) It follows that $U \cap V \subseteq Soc_s(V) \subseteq Rad(V) \cap Soc(V)$. Hence, we deduce that $U \cap V \subseteq Rad(V)$ and $U \cap V$ is semisimple.

(2) $\Longrightarrow$ (3) It is clear by Lemma 1.

(3) $\Longrightarrow$ (1) It is clear by Lemma 2 □

We say that V an *ss-supplement* of U in M if the equal conditions in the above lemma are satisfied. It is clear that the following implications on submodules of a module hold:

Direct summand $\Longrightarrow$ ss-supplement $\Longrightarrow$ supplement $\Longrightarrow$ Rad-supplement

We call a module M *ss-supplemented* if every submodule of M has an ss-supplement in M. A submodule U of a module M has *ample ss-supplements* in M if every submodule V of M such that $M = U + V$ contains an ss-supplement of U in M. We call a module M *amply ss-supplemented* if every submodule of M has ample ss-supplements in M. It is clear that every ss-supplemented module is supplemented. Of course there exists the same relationship between amply ss-supplemented modules and amply supplemented modules. Later we shall give examples of (amply) supplemented modules which are not (amply) ss-supplemented (see Example 17 and Example 18).

In section 3 we characterize ss-supplemented and amply ss-supplemented modules. For modules with small radical, we give some conditions which are equivalent to being an ss-supplemented module in Theorem 20. It follows that a finitely generated module M is ss-supplemented if and only if it is supplemented and $Rad(M) \subseteq Soc(M)$. Any direct sum of strongly local modules is ss-supplemented and coatomic. A module M is amply ss-supplemented if and only if every submodule of the module M is ss-supplemented. We show that a ring R is semiperfect and $Rad(R) \subseteq Soc(_RR)$ if and only if every left R-module is (amply) ss-supplemented.

2. Strongly Local Modules and Rings

As we mentioned at introduction, we denote by $Soc_s(M)$ the sum of all simple submodules of a module M that are small in M. Then we have:

Let M be a non-zero module. M is called *indecomposable* if the only direct summands of M are 0 and M.

Lemma 4. *Let M be an indecomposable module. Then M is simple or $Soc(M) \subseteq Rad(M)$.*

Proof. Suppose that M is not simple. Let $M = Soc(M) + X$ for some submodule X of M. Since $Soc(M)$ is semisimple, there exists a submodule Y of $Soc(M)$ such that $Soc(M) = (Soc(M) \cap X) \oplus Y$. Therefore, $M = Soc(M) + X = [(Soc(M) \cap X) \oplus Y] + X = X \oplus Y$. Since M is indecomposable and not simple, it follows that $Y = 0$. It means that $X = M$. Hence $Soc(M) << M$, that is, $Soc(M) \subseteq Rad(M)$. □

Using Lemma 2 and Lemma 4, we have the following result.

Corollary 5. *Let M be a local module which is not simple. Then $Soc_s(M) = Soc(M)$.*

Recall that a module M is called *radical* if M has no maximal submodules, that is, $M = Rad(M)$. Let $P(M)$ be the sum of all radical submodules of M. It is easy to see that $P(M)$ is the largest radical submodule of M. If $P(M) = 0$, M is called *reduced*.

Proposition 6. *Let M be a strongly local module. Then M is reduced.*

Proof. Since M is strongly local, we get $P(M) \subseteq Rad(M) \subseteq Soc(M)$. This implies that $P(M)$ is semisimple and so $P(M) = Rad(P(M)) = 0$. This completes the proof. □

Note that the condition "strongly" in the above proposition is necessary. The following example shows that in general a local module need not be reduced.

Example 7. *Let K be a field. In the polynomial ring $K[x_1, x_2, \ldots]$ with countably many indeterminates x_n, $n \in \mathbb{Z}^+$, consider the ideal $I = (x_1^2, x_2^2 - x_1, x_3^2 - x_2, \cdots)$ generated by x_1^2 and $x_{n+1}^2 - x_n$ for each $n \in \mathbb{Z}^+$. Then as shown in* [**?**, Example 6.2]*, the quotient ring $R = \frac{K[x_1,x_2,\ldots]}{I}$ is a local ring with the unique maximal ideal $J = \frac{(x_1,x_2,\ldots)}{I} = J^2$. Let M be the left R-module ${}_RR$. Then M is a local module. On the other hand, M is not reduced because $P(M) = Rad(J) = J \neq 0$.*

Proposition 8. *Every factor module of a strongly local module is strongly local.*

Proof. Let M be a strongly local module and N be a submodule of M. Then the factor module $\frac{M}{N}$ is local. Since $Rad(M)$ is the unique maximal submodule of M, it follows from [7, 21.2 (1)] that $Rad(\frac{M}{N}) = \frac{Rad(M)}{N} \subseteq \frac{Soc(M)}{N} = \pi(Soc(M)) \subseteq Soc(\frac{M}{N})$, where $\pi : M \longrightarrow \frac{M}{N}$ is the canonical projection. Hence $\frac{M}{N}$ is strongly local. □

Proposition 9. *Let R be a left strongly local ring. Then $(Rad(R))^2 = 0$. In particular, $Rad(R)$ is nilpotent.*

Proof. Since $Rad(R) \subseteq Soc({}_RR)$, it follows from [7, 21.12 (4)] that $(Rad(R))^2 = 0$. It means that $Rad(R)$ is nilpotent. □

Recall from [7] that an ideal I of a ring R is *right t-nilpotent* if for every sequence $a_1, a_2, ..., a_k$ of elements in I, there is a $k \in \mathbb{Z}^+$ with $a_1 a_2 ... a_k = 0$. Similarly *left t-nilpotent* is defined. Following [7, 43.9], R is called *left perfect* (respectively, *right perfect*) if R is semilocal and $Rad(R)$ is right t-nilpotent (respectively, left t-nilpotent). Here a ring R is *semilocal* if $\frac{R}{Rad(R)}$ is an artinian semisimple ring (see [4]). Note that nilpotent ideals are left and right t-nilpotent. Using this fact, we have the following:

Corollary 10. *Every left strongly local ring is left perfect and right perfect.*

Proof. Let R be a left strongly local ring. Since local rings are semilocal, it follows from Proposition 9 that R is left perfect and right perfect. □

It is well known that an artinian commutative domain is field. We have:

Proposition 11. *A strongly local commutative domain is field.*

Proof. Let R be a strongly local commutative domain and a be any element of R. If $a \in R \backslash Rad(R)$, we can write $Ra = R$ because R is local. Therefore, a is an invertible element of R. Suppose that $a \in Rad(R)$. It follows from Proposition 9 that $a^2 \in (Rad(R))^2 = 0$. By the hypothesis, we get $a = 0$. Hence, R is field. □

3. *SS*-Supplemented Modules

It is known that a ring R is semiperfect if and only if every finitely generated R-module is (amply) supplemented (see [7, 42.6]). In this section we obtain new characterizations of semiperfect rings via their *ss*-supplemented modules.

Recall that for a maximal submodule U of a module M, a submodule V of M is a supplement of U in M if and only if $M = U + V$ and V is local (see [7, 41.1 (3)]). Analogous to that we have:

Proposition 12. *Let M be a module and U be a maximal submodule of M. A submodule V of M is an ss-supplement of U in M if and only if $M = U + V$ and V is strongly local.*

Proof. Let V be an *ss*-supplement of U in M. By [7, 41.1.(3)], V is local and $U \cap V = Rad(V)$ is the unique maximal submodule of V. Since $U \cap V$ is semisimple, we have $Rad(V) \subseteq Soc(V)$. Thus V is strongly local.

Conversely, since V is local and $M = U + V$, we can write $U \cap V \subseteq Rad(V)$. It follows from assumption that $U \cap V$ is semisimple. Hence, V is an *ss*-supplement of U in M. □

Now, we give examples of (amply) supplemented modules which are not (amply) *ss*-supplemented. We first need the following facts.

Lemma 13. *Let M be an ss-supplemented module and N be a small submodule of M. Then $N \subseteq Soc_s(M)$.*

Proof. By the assumption, M is the unique *ss*-supplement of N in M and so $N \cap M = N$ is semisimple. Hence, $N \subseteq Soc_s(M)$ by Lemma 2. □

The following result is a direct consequence of Lemma 13.

Corollary 14. *Let M be an ss-supplemented module and $Rad(M) << M$. Then $Rad(M) \subseteq Soc(M)$.*

It is well known that every local module is amply supplemented. Now we give an analogous characterization of this fact for amply *ss*-supplemented modules.

Proposition 15. *Every strongly local module is amply ss-supplemented.*

Proof. Let M be a strongly local module. Then, M is local and so it is amply supplemented. Note that M has no supplement submodule except for 0 and M. Since $Rad(M) \subseteq Soc(M)$, M is amply ss-supplemented. □

Proposition 16. *Let R be a ring and M be a hollow R-module. M is (amply) ss-supplemented if and only if it is strongly local.*

Proof. Suppose that M is ss-supplemented. Let $m \in Rad(M)$. Then we get $Rm << M$. Since M is ss-supplemented, it follows from Lemma 13 that $Rm \subseteq Soc_s(M)$. It means that $m \in Soc(M)$ and so $Rad(M) \subseteq Soc(M)$. Suppose that $M = Rad(M)$. Since $M = Rad(M) = Soc(M)$ and the radical of a semisimple module is zero, we have that $M = 0$. This is a contradiction because M is hollow. It means that $M \neq Rad(M)$, that is, M is local by [7, 41.4]. Therefore M is strongly local. The converse follows from Proposition 15. □

Example 17. *For any prime integer p, consider the left $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$. Note that M is a hollow module which is not local. Since hollow modules are (amply) supplemented, M is (amply) supplemented. However, M is not (amply) ss-supplemented module by Proposition 16.*

Every artinian module is supplemented. The next example shows that in general artinian modules need not to be ss-supplemented.

Example 18. *Let M be the $\mathbb{Z}$-module $\mathbb{Z}_{p^k}$, for p is any prime integer and $k \geq 3$. Note that M is artinian. Since $Soc_s(\mathbb{Z}_{p^k}) = Soc(\mathbb{Z}_{p^k}) \cong \mathbb{Z}_p$ and $Rad(M) = p\mathbb{Z}_{p^k}$, M is not strongly local and so it is not ss-supplemented by Proposition 16.*

Lemma 19. *Let M be a supplemented module and $Rad(M) \subseteq Soc(M)$. Then M is ss-supplemented.*

Proof. Let $U \subseteq M$. Since M is supplemented, there exists a submodule V of M such that $M = U + V$ and $U \cap V << V$. Then $U \cap V \subseteq Rad(V) \subseteq Rad(M)$ and so $U \cap V$ is semisimple by the assumption. Hence V is an ss-supplement of U in M. It means that M is ss-supplemented. □

Theorem 20. *Let M be a module with $Rad(M) << M$. Then the following statements are equivalent:*

(1) *M is ss-supplemented,*
(2) *M is supplemented and $Rad(M)$ has an ss-supplement in M,*
(3) *M is supplemented and $Rad(M) \subseteq Soc(M)$.*

Proof. (1) $\Longrightarrow$ (2) It is clear.
(2) $\Longrightarrow$ (3) It follows from Lemma 13.
(3) $\Longrightarrow$ (1) By Lemma 19. □

Since finitely generated modules have small radical, we have the following result.

Corollary 21. *Let M be a finitely generated module. Then M is ss-supplemented if and only if it is supplemented and $Rad(M) \subseteq Soc(M)$.*

Next, in order to prove that every finite sum of ss-supplemented modules is ss-supplemented, we use the following standard lemma (see, [7, 41.2]).

Lemma 22. *Let M be a module and M_1, U be submodules of M with M_1 ss-supplemented. If M_1+U has an ss-supplement in M, U also has an ss-supplement in M.*

Proof. Suppose that X is an ss-supplement of $M_1 + U$ in M and Y is an ss-supplement of $(X+U)\cap M_1$ in M_1. Then $M = X+Y+U$ and $(X+Y)\cap U << X+Y$. Moreover, $X \cap (Y + U)$ is semisimple as a submodule of the semisimple module $X \cap (M_1 + U)$. Note that $Y \cap [(X + U) \cap M_1] = Y \cap (X + U)$ is semisimple. It follows from [3, 8.1.5] that $(X + Y) \cap U$ is semisimple. Hence $X + Y$ is an ss-supplement of U in M. □

Proposition 23. *Let M_1, M_2 be any submodules of a module M such that $M = M_1 + M_2$. Then if M_1 and M_2 are ss-supplemented, M is ss-supplemented.*

Proof. Let U be any submodule of M. The trivial submodule 0 is ss-supplement of $M = M_1 + M_2 + U$ in M. Since M_1 is ss-supplemented, $M_2 + U$ has an ss-supplement in M by Lemma 22. Again applying Lemma 22, we also have that U has an ss-supplement in M. This shows that M is ss-supplemented. □

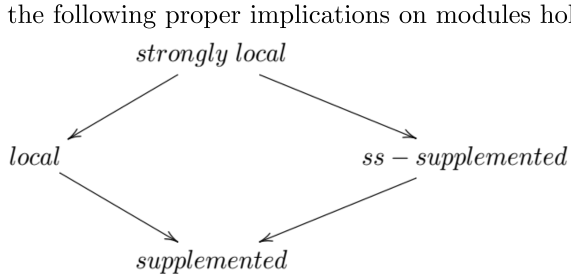
Using this fact we obtain the following corollary.

Corollary 24. *Every finite sum of ss-supplemented modules is ss-supplemented.*

Now we give an example of an ss-supplemented module which is not strongly local.

Example 25. *The $\mathbb{Z}$-module $M = \mathbb{Z}_4 \oplus \mathbb{Z}_4$ is ss-supplemented as a sum of strongly local modules. However, M is not (strongly) local.*

Then we have the following proper implications on modules hold:



Proposition 26. *If M is a (amply) ss-supplemented module, then every factor module of M is (amply) ss-supplemented.*

Proof. Let M be an ss-supplemented module and $\frac{M}{L}$ be a factor module of M. By the assumption, for any submodule U of M which contains L, there exists a submodule V of M such that $M = U + V$, $U \cap V << V$ and $U \cap V$ is semisimple. Let $\pi : M \longrightarrow \frac{M}{L}$ be the canonical projection. Then we have that $\frac{M}{L} = \frac{U}{L} + \frac{V+L}{L}$ and $\frac{U}{L} \cap \frac{V+L}{L} = \frac{(U \cap V)+L}{L} = \pi(U \cap V) << \pi(V) = \frac{V+L}{L}$ by [7, 19.3(4)]. Since $U \cap V$ is semisimple, it follows from [3, 8.1.5] that $\pi(U \cap V) = \frac{(U \cap V)+L}{L} = \frac{U}{L} \cap \frac{V+L}{L}$ is semisimple. That is, $\frac{V+L}{L}$ is an ss-supplement of $\frac{U}{L}$ in $\frac{M}{L}$, as required.

By adapting this argument we can prove similarly that if M is amply ss-supplemented, then so is every factor module of M. □

Recall that a module M is said to be *coatomic* if every proper submodule of M is contained in a maximal submodule of M. It is easy to see that every coatomic module has small radical.

Let p be a prime integer and consider the localization ring $R = \mathbb{Z}_{(p)} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \textit{ and } p \nmid b\}$. Note that R is a local ring. Let M be the left R-module $R^{(\mathbb{N})}$. Then M is the direct sum of local submodules but it is not supplemented. Since R is not perfect, $Rad(M)$ is not small in M and so M is not also coatomic. However, any arbitrary direct sum of strongly local modules is ss-supplemented and coatomic, as the next result shows.

Theorem 27. *Let $M = \bigoplus_{i \in I} M_i$, where each M_i is a strongly local module. Then, M is ss-supplemented and coatomic.*

Proof. Since M_i is strongly local for every $i \in I$, it is local and $Rad(M_i) \subseteq Soc(M_i)$ and so $Rad(M) = \bigoplus_{i \in I} Rad(M_i) \subseteq \bigoplus_{i \in I} Soc(M_i) = Soc(M)$ by [7, 21.6 (5) and 21.2 (5)]. Applying Lemma 1, we get that $Rad(M)$ is a small submodule of M. Since strongly local modules are local, it follows from [10, Theorem 1.4 (A)] that M is supplemented. Hence, M is ss-supplemented by Theorem 20.

Let U be a proper submodule of M. It follows from [7, 41.1 (6)] that U is contained in a maximal submodule of M, that is, M is coatomic. □

Let M be a module. A module N is called *M-generated* if there exists an epimorphism $f : M^{(I)} \longrightarrow N$ for some index set I.

Corollary 28. *Let M be a strongly local module. Then every M-generated module is ss-supplemented and coatomic.*

Proof. Suppose that N is M-generated. Then, there exists an epimorphism $f : M^{(I)} \longrightarrow N$ for some index set I. By Theorem 27, $M^{(I)}$ is ss-supplemented and coatomic. Hence N is ss-supplemented by Proposition 26 and it is coatomic by [10, Lemma 1.5 (a)]. □

Corollary 29. *Let R be a left strongly local ring. Then every left R-module is ss-supplemented.*

Proof. Since all left R-modules are R-generated, the proof follows from Corollary 28. □

A submodule U of a module M is said to be *cofinite* if M/U is finitely generated (see [1]). Note that maximal submodules of M are cofinite.

Theorem 30. *The following statements are equivalent for a module M:*

(1) *M is the sum of all strongly local submodules,*
(2) *M is ss-supplemented and coatomic,*
(3) *M is coatomic and every cofinite submodule of M has an ss-supplement in M,*
(4) *M is coatomic and every maximal submodule of M has an ss-supplement in M.*

Proof. (1) $\Longrightarrow$ (2) Let $M = \sum_{i\in I} M_i$, where each M_i is strongly local submodules. Put $N = \bigoplus_{i\in I} M_i$. Then, by Theorem 27, N is ss-supplemented and coatomic. Now we consider the epimorphism $f : N \longrightarrow M$ via $f((m_i)_{i\in I}) = \sum_{i\in I} m_i$ for all $(m_i)_{i\in I} \in N$. It follows from Proposition 26 and [10, Lemma 1.5 (a)] that M is ss-supplemented and coatomic.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are clear.

(4) $\Longrightarrow$ (1) Let S be the sum of all strongly local submodules of M. Assume that $S \neq M$. Since M is coatomic, there exists a maximal submodule K of M with $S \subseteq K$. By (4), K has an ss-supplement, say V, in M. It follows from Proposition 12 that V is strongly local. Therefore, $V \subseteq S \subseteq K$, a contradiction. □

The following fact is a direct consequence of Theorem 30.

Corollary 31. *For a coatomic module M, the following statements are equivalent:*

(1) *M is the sum of all strongly local submodules,*
(2) *M is ss-supplemented,*
(3) *Every cofinite (maximal) submodule of M has an ss-supplement in M.*

A ring R is called *left max* if every non-zero left R-module has a maximal submodule. Note that if R is a left max ring, then every left R-module is coatomic. Using this fact and Corollary 31, we obtain the following result.

Corollary 32. *Let R be a left max ring and M be a non-zero left R-module. Then M is the sum of all strongly local submodules of M if and only if it is ss-supplemented.*

Proposition 33. *Let M be a module. If every submodule of M is ss-supplemented, then M is amply ss-supplemented.*

Proof. Let U and V be two submodules of M such that $M = U + V$. Since V is ss-supplemented, there exists a submodule V' of V such that $V = (U \cap V) + V'$, $U \cap V' << V'$ and $U \cap V'$ is semisimple. Note that $M = U + V = U + ((U \cap V) + V') =$

$U+V^{'}$. It means that U has ample ss-supplements in M. Hence M is amply ss-supplemented. □

Lemma 34. *Let M be amply ss-supplemented module and V be an ss-supplement submodule in M. Then V is amply ss-supplemented.*

Proof. Let V be an ss-supplement of a submodule U of M. Let X and Y be submodules of V such that $V = X + Y$. Then $M = (U + X) + Y$. Since M is amply ss-supplemented, $U + X$ has an ss-supplement $Y^{'} \subseteq Y$ in M. It follows that $X + Y^{'} \subseteq V$. By the minimality of V, we have $V = X + Y^{'}$. In addition, $X \cap Y^{'} \subseteq (U + X) \cap Y^{'} << Y^{'}$, that is, $X \cap Y^{'} << Y^{'}$. Since $(U + X) \cap Y^{'}$ is semisimple, $X \cap Y^{'}$ is also semisimple by [3, 8.1.5]. It means that $Y^{'}$ is an ss-supplement of X in V. Finally, V is amply ss-supplemented. □

The next result gives a useful characterization of amply ss-supplemented modules.

Theorem 35. *Let M be a module. Then, M is amply ss-supplemented if and only if every submodule U of M is of the form $U = X + Y$, where X is ss-supplemented and $Y \subseteq Soc_s(M)$.*

Proof. Let U be a submodule of M. Since M is ss-supplemented, U has an ss-supplement V in M. Then $M = U+V$. By the assumption, there exists a submodule X of U such that X is an ss-supplement of V in M. Put $Y = U \cap V$. Since V is an ss-supplement of U in M, we have that $Y \subseteq Soc_s(V) \subseteq Soc_s(M)$. Applying the modular law, we get $U = U \cap M = U \cap (X + V) = X + U \cap V = X + Y$. Note that X is ss-supplemented by Lemma 34.

Conversely, let U be a submodule of M. By the assumption, there exist submodules X and Y of M such that $U = X + Y$, X ss-supplemented and $Y \subseteq Soc_s(M)$. By Proposition 23, U is ss-supplemented. Hence M is amply ss-supplemented from Proposition 33. □

The next result is crucial.

Corollary 36. *For a module M, the following statements are equivalent:*

(1) *M is amply ss-supplemented,*
(2) *Every submodule of M is ss-supplemented,*
(3) *Every submodule of M is amply ss-supplemented.*

Note that it is not in general true that any submodule of an amply supplemented module is (amply) supplemented. Let R be a local Dedekind domain which is not field. Suppose that $M = R^{(\mathbb{N})}$. Then, M is not (amply) supplemented. The group $F = R \times M$ can be converted to a ring by the following operation: $(x, y) \cdot \left(x^{'}, y^{'}\right) = (xx^{'}, xy^{'} + x^{'}y)$ where $x, x^{'} \in R$ and $y, y^{'} \in M$. Then F is a commutative local ring and so F is amply supplemented. Put $L = \{0\} \times M$. Therefore, L is an ideal of F. Hence the submodule L of F is not a (amply) supplemented F-module.

A module M is said to be *π-projective* if whenever U and V are submodules of M such that $M = U + V$, there exists an endomorphism f of M such that $f(M) \subseteq U$ and $(1 - f)(M) \subseteq V$. Hollow (local) modules and self-projective modules are π-projective and π-projective supplemented modules are amply supplemented. Similarly, we show that π-projective ss-supplemented modules are amply ss-supplemented. The proof is virtually the same that of [7, 41.15], but we give it for completeness.

Proposition 37. *Let M be a π-projective and ss-supplemented module. Then M is amply ss-supplemented.*

Proof. Let U and V be submodules of M such that $M = U + V$. Since M is π-projective, there exists an endomorphism f of M such that $f(M) \subseteq U$ and $(1 - f)(M) \subseteq V$. Note that $(1 - f)(U) \subseteq U$. Let $V^{'}$ be an ss-supplement of U in M. Then $M = f(M) + (1 - f)(M) = f(M) + (1 - f)(U + V^{'}) \subseteq U + (1 - f)(V^{'})$, so that $M = U + (1 - f)(V^{'})$. Note that $(1 - f)(V^{'})$ is a submodule of V. Let $y \in U \cap (1 - f)(V^{'})$. Then, $y \in U$ and $y = (1 - f)(x) = x - f(x)$ for some $x \in V^{'}$. Next $x = y + f(x) \in U$ so that $y \in (1 - f)(U \cap V^{'})$. Since $U \cap V^{'} << V^{'}$, $U \cap (1 - f)(V^{'}) = (1 - f)(U \cap V^{'}) << (1 - f)(V^{'})$ by [7, 19.3(4)]. By [3, 8.1.5], $U \cap (1 - f)(V^{'}) = (1 - f)(U \cap V^{'})$ is semisimple because $U \cap V^{'}$ is semisimple. Thus $(1 - f)(V^{'})$ is an ss-supplement of U in M. Therefore M is amply ss-supplemented module. □

Since every projective module is π-projective, the following result follows from Proposition 37 and Corollary 36.

Corollary 38. *Any submodule of a projective ss-supplemented module is ss-supplemented.*

Now, we characterize the rings whose modules are ss-supplemented. Firstly, we need the following lemmas.

Lemma 39. *Let M be a projective module. Then M is ss-supplemented if and only if it is supplemented and $Rad(M) \subseteq Soc(M)$.*

Proof. Suppose that M is projective supplemented module. Therefore we have $Rad(M) << M$ by [7, 42.5]. Then the proof is obvious from Theorem 20. □

Lemma 40. *Let R be a ring. Then every left R-module is ss-supplemented if and only if every left R-module is the sum of all strongly local submodules.*

Proof. Assume that every left R-module M is ss-supplemented. Then, by [7, 43.9], R is left perfect. This implies that R is a left max ring. Applying Corollary 32, M is the sum of all strongly local submodules of M. The converse follows from Theorem 30. □

Theorem 41. *The following statements are equivalent for a ring R:*

(1) $_RR$ *is ss-supplemented,*
(2) R *is semiperfect and* $Rad(R) \subseteq Soc(_RR)$,
(3) R *is semilocal and* $Rad(R) \subseteq Soc(_RR)$,
(4) *Every projective left R-module is (amply) ss-supplemented,*
(5) *Every left R-module is (amply) ss-supplemented,*
(6) *Every left R-module is the sum of all strongly local submodules,*
(7) $_RR$ *is a finite sum of strongly local submodules,*
(8) *Every maximal left ideal of R has an ss-supplement in R.*

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) By Corollary 21 and [7, 42.6].

(3) $\Longrightarrow$ (4) Let M be a projective R-module. Then, by [7, 21.17 (2)], we can write $Rad(M) = Rad(R)M \subseteq Soc(_RR)M = Soc(M)$. From [7, 43.9] and Lemma 39, the proof is completed.

(4) $\Longrightarrow$ (5) follows [7, 18.6] and Proposition 26.

(5) $\Longrightarrow$ (6) By Lemma 40.

(6) $\Longrightarrow$ (7) is obvious.

(7) $\Longrightarrow$ (8) By Theorem 30.

(8) $\Longrightarrow$ (1) By Corollary 31. □

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