Available online: January 6, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 486–510 (2020)
DOI: 10.31801/cfsuasmas.577438
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr/index.php?series=A1

ASYMPTOTIC BEHAVIOUR OF RESONANCE EIGENVALUES OF THE SCHRÖDINGER OPERATOR WITH A MATRIX POTENTIAL

SEDEF KARAKILIÇ, SETENAY AKDUMAN, AND DIDEM COŞKAN

Abstract. We will discuss the asymptotic behaviour of the eigenvalues of a Schrödinger operator with a matrix potential defined by the Neumann boundary condition in $L_2^m(F)$, where F is a d-dimensional rectangle and the potential is an $m \times m$ matrix with $m \geq 2$, $d \geq 2$, when the eigenvalues belong to the resonance domain, roughly speaking they lie near the planes of diffraction.

1. Introduction

In this paper, we consider the Schrödinger operator with a matrix potential $V(x)$ defined by the differential expression

$$L\phi = -\Delta\phi + V\phi \tag{1}$$

and the Neumann boundary condition

$$\frac{\partial\phi}{\partial n}|_{\partial F} \ = 0, \tag{2}$$

in $L_2^m(F)$ where F is the d dimensional rectangle $F = [0, a_1] \times [0, a_2] \times \ldots \times [0, a_d]$, ∂F is the boundary of F, $m \geqslant 2, d \geqslant 2$, $\frac{\partial}{\partial n}$ denotes differentiation along the outward normal of the boundary ∂F, Δ is a diagonal $m \times m$ matrix whose diagonal elements are the scalar Laplace operators $\triangle = \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \ldots + \frac{\partial^2}{\partial {x_d}^2}$, $x = (x_1, x_2, \ldots, x_d) \in R^d$, V is a real valued symmetric matrix $V(x) = (v_{ij}(x)), i, j = 1, 2, \ldots, m, v_{ij}(x) \in L_2(F)$, that is, $V^T(x) = V(x)$.

We denote the operator defined by (1)-(2) by $L(V)$, the eigenvalues and the corresponding eigenfunctions of $L(V)$ by Λ_N and Ψ_N, respectively.

The eigenvalues of the operator $L(0)$ which is defined by the differential expression (1) when $V(x) = 0$ and the boundary condition (2) are $|\gamma|^2$, and the

Received by the editors: June 16, 2019; Accepted: November 27, 2019.

2010 *Mathematics Subject Classification.* 47F05, 35P15.

Key words and phrases. Schrödinger operator, Neumann condition, Resonance eigenvalue, perturbation theory.

corresponding eigenspaces are $E_\gamma = span\{\Phi_{\gamma,1}(x), \Phi_{\gamma,2}(x), \ldots, \Phi_{\gamma,m}(x)\}$, where

$$\begin{aligned} \gamma &= (\gamma^1, \gamma^2, \ldots, \gamma^d) \in \frac{\Gamma^{+0}}{2}, \\ \frac{\Gamma^{+0}}{2} &= \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}) : n_k \in Z^+ \cup \{0\}, k = 1, 2, \ldots, d\}, \\ \Phi_{\gamma,j}(x) &= (0, \ldots, 0, u_\gamma(x), 0, \ldots, 0), j = 1, 2, \ldots, m, \end{aligned}$$

and the non-zero component of $\Phi_{\gamma,j}(x)$ is $u_\gamma(x) = cos\frac{n_1\pi}{a_1}x_1 cos\frac{n_2\pi}{a_2}x_2 \cdots cos\frac{n_d\pi}{a_d}x_d$, which stands in the jth component. In particular, $u_0(x) = 1$ when $\gamma = (0, 0, \ldots, 0)$.

It can be easily calculated that the norm of $u_\gamma(x)$, $\gamma \in \frac{\Gamma^{+0}}{2}$, in $L_2(F)$ is $\sqrt{\frac{\mu(F)}{|A_\gamma|}}$, where $\mu(F)$ is the measure of the d-dimensional parallelepiped F, $|A_\gamma|$ is the number of vectors in $A_\gamma = \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \frac{\Gamma}{2} : \ |\alpha_k| = |\gamma^k|, \ k = 1, 2, \ldots, d\}$, $\frac{\Gamma}{2} = \{(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2} \cdots, \frac{n_d\pi}{a_d}) : \ n_k \in Z, \ k = 1, 2, \ldots, d\}$.

From now on, $\langle .,. \rangle$ and $(.,.)$ will denote the inner products in $L_2^m(F)$ and $L_2(F)$, respectively.

Since $\{u_\gamma(x)\}_{\gamma \in \frac{\Gamma^{+0}}{2}}$ is a complete system in $L_2(F)$, for any $q(x)$ in $L_2(F)$ we have

$$q(x) = \sum_{\gamma \in \frac{\Gamma^{+0}}{2}} \frac{|A_\gamma|}{\mu(F)} (q, u_\gamma) u_\gamma(x). \tag{3}$$

In our study, it is convenient to use the equivalent decomposition (see [9])

$$q(x) = \sum_{\gamma \in \frac{\Gamma}{2}} q_\gamma u_\gamma(x), \tag{4}$$

where $q_\gamma = \frac{1}{\mu(F)}(q(x), u_\gamma(x))$ for the sake of simplicity. That is, the decomposition (3) and (4) are equivalent for any $d \geq 2$. Thus, according to (4), each matrix element $v_{ij}(x) \in L_2(F)$ of the matrix $V(x)$ can be written in its Fourier series expansion

$$v_{ij}(x) = \sum_{\gamma \in \frac{\Gamma}{2}} v_{ij\gamma} u_\gamma(x), \tag{5}$$

$v_{ij\gamma} = \frac{(v_{ij}, u_\gamma)}{\mu(F)}$, $(v_{ij}, u_\gamma) = \frac{1}{\mu(F)} \int_F v_{ij}(x) u_\gamma(x) dx$ and $v_{ij0} = \frac{1}{\mu(F)} \int_F v_{ij}(x) dx$ $i, j = 1, 2, \ldots, m$.

We assume that $l > \frac{(d+20)(d-1)}{2} + d + 3$ and the Fourier coefficients $v_{ij\gamma}$ of $v_{ij}(x)$ satisfy

$$\sum_{\gamma \in \frac{\Gamma}{2}} |v_{ij\gamma}|^2 (1 + |\gamma|^{2l}) < \infty, \tag{6}$$

for each $i, j = 1, 2, \ldots, m$. Let ρ be a large parameter, $\rho \gg 1$ and α be a positive number with $0 < \alpha < \frac{1}{d+20}$ then for $\Gamma(\rho^\alpha) = \{\gamma \in \frac{\Gamma}{2} : 0 \leq |\gamma| < \rho^\alpha\}$ and $p = l - d$

the condition (6) implies that

$$v_{ij}(x) = \sum_{\gamma \in \Gamma(\rho^{\alpha})} v_{ij\gamma} u_{\gamma}(x) + O(\rho^{-p\alpha}). \tag{7}$$

Here $O(\rho^{-p\alpha})$ is a function in $L_2(F)$ with norm of order $\rho^{-p\alpha}$. Furthermore, by (6), we have

$$M_{ij} \equiv \sum_{\gamma \in \frac{\Gamma}{2}} |v_{ij\gamma}| < \infty, \tag{8}$$

for all $i, j = 1, 2, \ldots, m$.

Notice that, if a function $q(x)$ is sufficiently smooth$\big(q(x) \in W_2^l(F)\big)$ and the support of $\nabla q(x) = \left(\frac{\partial q}{\partial x_1}, \frac{\partial q}{\partial x_2}, \ldots, \frac{\partial q}{\partial x_d}\right)$ is contained in the interior of the domain F, then $q(x)$ satisfies condition (6) (See [7]). There is also another class of functions $q(x)$, such that $q(x) \in W_2^l(F)$,

$$q(x) = \sum_{\gamma' \in \Gamma} q_{\gamma'} u_{\gamma'}(x),$$

which is periodic with respect to a lattice

$$\Omega = \{(m_1 a_1, m_2 a_2, \ldots, m_d a_d) : m_k \in \mathbf{Z}, k = 1, 2, \ldots, d\}$$

and thus it also satisfies condition (6).

As in [17]-[22], we divide R^d into two domains: Resonance and Non-resonance domains. In order to define these domains, let us introduce the following sets:

Let $0 < \alpha < \frac{1}{d+20}$, $\alpha_k = 3^k \alpha$, $k = 1, 2, \ldots, d-1$ and

$$\begin{aligned} V_b(\rho^{\alpha_1}) &\equiv \left\{x \in R^d : \quad \big||x|^2 - |x+b|^2\big| < \rho^{\alpha_1}\right\} \\ E_1(\rho^{\alpha_1}, p) &\equiv \bigcup_{b \in \Gamma(p\rho^{\alpha})} V_b(\rho^{\alpha_1}) \\ U(\rho^{\alpha_1}, p) &\equiv R^d \setminus E_1(\rho^{\alpha_1}, p) \\ E_k(\rho^{\alpha_k}, p) &= \bigcup_{\gamma_1, \gamma_2, \ldots, \gamma_k \in \Gamma(p\rho^{\alpha})} \left(\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})\right) \end{aligned}$$

where $b \neq 0$, $\gamma_i \neq 0$, $i = 1, 2, \ldots, k$ and the intersection $\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})$ in E_k is taken over $\gamma_1, \gamma_2, \ldots, \gamma_k$ which are linearly independent vectors and the length of γ_i is not greater than the length of the other vector in $\Gamma \bigcap \gamma_i R$. The set $U(\rho^{\alpha_1}, p)$ is said to be a non-resonance domain, and the eigenvalue $|\gamma|^2$ is called a non-resonance eigenvalue if $\gamma \in U(\rho^{\alpha_1}, p)$. The domains $V_b(\rho^{\alpha_1})$, for $b \in \Gamma(p\rho^{\alpha})$ are called resonance domains and the eigenvalue $|\gamma|^2$ is a resonance eigenvalue if $\gamma \in V_b(\rho^{\alpha_1})$.

As noted in [20]-[21], the domain $V_b(\rho^{\alpha_1}) \setminus E_2$, called a single resonance domain, has asymptotically full measure on $V_b(\rho^{\alpha_1})$, that is,

$$\frac{\mu\left((V_b(\rho^{\alpha_1}) \setminus E_2) \bigcap B(q)\right)}{\mu\left(V_b(\rho^{\alpha_1}) \bigcap B(q)\right)} \to 1, \text{ as } \rho \to \infty,$$

where $B(\rho) = \left\{x \in \boldsymbol{R}^d : |x| = \rho\right\}$, if

$$2\alpha_2 - \alpha_1 + (d+3)\alpha < 1, \quad \alpha_2 > 2\alpha_1, \tag{9}$$

hold. Since $0 < \alpha < \frac{1}{d+20}$, the conditions in (9) hold.

In most cases, it is important to know the asymptotic behavior of the eigenvalues of the Schrödinger operator $L(V)$. In this paper, [3] and [8], we construct the asymptotic formulas in the high energy region for eigenvalues of the operator $L(V)$.

In [3], we obtain the asymptotic formulas of arbitrary order for the eigenvalue of $L(V)$ corresponding to the non-resonance eigenvalues $|\gamma|^2$ of $L(0)$ in arbitrary dimension $d \geq 2$.

In [8], we constructed the high energy asymptotics of arbitrary order for the eigenvalue of $L(V)$ corresponding to resonance eigenvalue $|\gamma|^2$ when γ belongs to the special single resonance domains $V_\delta(\rho^{\alpha_1}) \setminus E_2$, where δ is from $\{e_1, e_2, \ldots, e_d\}$ and $e_1 = \left(\frac{\pi}{a_1}, 0, \ldots, 0\right), \ldots, e_d = \left(0, \ldots, \frac{\pi}{a_d}\right)$, $d \geq 2$.

In this paper, we study the case for which $|\gamma|^2$ is a resonance eigenvalue. More precisely, in Theorem (1) and (2) of Section(2), we assume that $\gamma \in (\bigcap\limits_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, \ldots, d-1$ and $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ for $k = 1, 2, \ldots, d$ and prove that the corresponding eigenvalue of $L(V)$ is close to the sum of the eigenvalue of the matrix V_0 and the eigenvalue of the matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ (See (14)).

In Section(3), this time we assume that $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, $\delta \in \frac{\Gamma}{2} \setminus \{e_1, e_2, \ldots, e_d\}$, that is, γ is in a single resonance domain and we prove the main result Theorem (7) which gives a connection between the eigenvalues of $L(V)$ corresponding to a single resonance domain and the eigenvalues of the Sturm-Liouville operators.

Note that, the case $\delta = e_i$, $i = 1, 2, \ldots, d$, was considered in [8], by a different but simpler method and better formulas were obtained.

2. Asymptotic Formulas for the Eigenvalues in the Resonance Domain

We assume that $\gamma \notin V_{e_k}(\rho^{\alpha_1})$ for $k = 1, 2, \ldots, d$, and $|\gamma|^2$ is a resonance eigenvalue of the operator $L(0)$, that is, $\gamma \in (\bigcap\limits_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $k = 1, 2, \ldots, d-1$, such that $\mid \gamma \mid \sim \rho$ where $\mid \gamma \mid \sim \rho$ means that $\mid \gamma \mid$ and ρ are asymptotically equal, that is, there exist c_1, c_2 satisfying the inequality $c_1\rho \leq \mid \gamma \mid \leq c_2\rho$, c_i, $i = 1, 2, 3, \ldots$

are positive real constants which do not depend on ρ. To obtain the asymptotic formulas for the eigenvalues of $L(V)$ corresponding to $|\gamma|^2$ we use the binding formula (see (9) in [3])

$$(\Lambda_N - |\gamma|^2)\langle\Psi_N, \Phi_{\gamma,j}\rangle = \langle\Psi_N, V\Phi_{\gamma,j}\rangle. \tag{10}$$

Now, we decompose $V(x)\Phi_{\gamma,j}(x)$ with respect to the basis $\{\Phi_{\gamma',i}(x)\}_{\gamma'\in\frac{\Gamma}{2},i=1,2,\ldots,m}$. By definition of $\Phi_{\gamma,j}(x)$, it is obvious that

$$V(x)\Phi_{\gamma,j}(x) = (v_{1j}(x)u_\gamma(x), \ldots, v_{mj}(x)u_\gamma(x)). \tag{11}$$

Substituting the decomposition (7) of $v_{ij}(x)$ in (11), we get

$$V(x)\Phi_{\gamma,j}(x) = (\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{1j\gamma'}u_{\gamma'}(x)u_\gamma(x), \ldots, \sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{mj\gamma'}u_{\gamma'}(x)u_\gamma(x)) + O(\rho^{-p\alpha}).$$

Since γ does not belong to the domains $V_{e_k}(\rho^{\alpha_1})$, for each $k = 1, 2, ...d$, we may use the following equation

$$\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}u_{\gamma'}(x)u_\gamma(x) = \sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}u_{\gamma-\gamma'}(x)$$

which is proved in [9] (see equation (18) in [9]), and obtain

$$\begin{aligned} V(x)\Phi_{\gamma,j}(x) &= (\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{1j\gamma'}u_{\gamma-\gamma'}(x), \ldots, \sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{mj\gamma'}u_{\gamma-\gamma'}(x)) + O(\rho^{-p\alpha}) \\ &= \sum_{i=1}^{m}\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}\Phi_{\gamma-\gamma',i}(x) + O(\rho^{-p\alpha}). \end{aligned} \tag{12}$$

Substituting (12) into (10), we obtain

$$\begin{aligned} < \Psi_N, \Phi_{\gamma,j} > &= \frac{< \Psi_N, V\Phi_{\gamma,j} >}{(\Lambda_N - \mid \gamma \mid^2)} \\ &= \sum_{i=1}^{m}\sum_{\gamma'\in\Gamma(\rho^\alpha)} v_{ij\gamma'}\frac{< \Psi_N, \Phi_{\gamma-\gamma',i} >}{(\Lambda_N - \mid \gamma \mid^2)} + O(\rho^{-p\alpha}) \end{aligned} \tag{13}$$

for every vector $\gamma \in \frac{\Gamma}{2}$, satisfying the condition

$$\mid \Lambda_N - \mid \gamma \mid^2 \mid > \frac{1}{2}\rho^{\alpha_1}.$$

Letting $p_1 = [\frac{p+1}{2}]$, that is, p_1 is the integer part of $\frac{p+1}{2}$, we define the following sets

$$B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{b : b = \sum_{i=1}^{k} n_i\gamma_i, n_i \in Z, |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_{k+1}}\},$$

$$B_k(\gamma) = \gamma + B_k(\gamma_1, \gamma_2, \ldots, \gamma_k) = \{\gamma + b : b \in B_k(\gamma_1, \gamma_2, \ldots, \gamma_k)\},$$

$$B_k(\gamma, p_1) = B_k(\gamma) + \Gamma(p_1\rho^\alpha).$$

Let h_τ, $\tau = 1, 2, \ldots, b_k$ denote the vectors of $B_k(\gamma, p_1)$, b_k the number of the vectors in $B_k(\gamma, p_1)$. By its definition, it can easily be obtained that $b_k = O(\rho^{\frac{d}{2}3^d\alpha})$, since $\alpha_k = 3^k\alpha$, $2 \le k \le d$. We define the $mb_k \times mb_k$ matrix $C = C(\gamma, \gamma_1, \ldots, \gamma_k)$ by

$$C = \begin{bmatrix} |h_1|^2 I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & |h_2|^2 I - V_0 & \cdots & V_{h_2-h_{b_k}} \\ \vdots & & & \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & |h_{b_k}|^2 I - V_0 \end{bmatrix}, \tag{14}$$

where $V_{h_\tau-h_\xi}$, $\tau, \xi = 1, 2, \ldots, b_k$ are the $m \times m$ matrices defined by

$$V_{h_\tau-h_\xi} = \begin{bmatrix} v_{11h_\tau-h_\xi} & v_{12h_\tau-h_\xi} & \cdots & v_{1mh_\tau-h_\xi} \\ v_{21h_\tau-h_\xi} & v_{22h_\tau-h_\xi} & \cdots & v_{2mh_\tau-h_\xi} \\ \vdots & & & \\ v_{m1h_\tau-h_\xi} & v_{m2h_\tau-h_\xi} & \cdots & v_{mmh_\tau-h_\xi} \end{bmatrix}. \tag{15}$$

Writing equation (13) for all $h_\tau \in B_k(\gamma, p_1)$, $\tau = 1, 2, \ldots, b_k$ and $j = 1, 2, \ldots, m$, we get

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau,j} >= \sum_{i=1}^{m} \sum_{\gamma\prime \in \Gamma(\rho^\alpha)} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau-\gamma\prime,i} > +O(\rho^{-p\alpha}). \tag{16}$$

Similar system of equations for quasi-periodic boundary condition was investigated in [19], [21] and [22]. More recently, in [22], Lemma 2.2.1. states that for $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$, $h_\tau \in B_k(\gamma, p_1)$ and $\gamma\prime, \gamma_1, \gamma_2, \ldots, \gamma_s \in \Gamma(\rho^\alpha)$, if $h_\tau - \gamma\prime \notin B_k(\gamma, p_1)$ then

$$||\gamma|^2 - |h_\tau - \gamma\prime - \gamma_1 - \ldots - \gamma_s|^2 |> \frac{1}{5}\rho^{\alpha_{k+1}}, \tag{17}$$

for $s = 0, 1, 2, \ldots, p_1 - 1$.

Thus, if an eigenvalue Λ_N of $L(V)$ satisfies

$$|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{\alpha_1}, \tag{18}$$

then by (17) and (18), we have

$$|\Lambda_N - |h_\tau - \gamma\prime - \gamma_1 - \ldots - \gamma_s|^2| > \frac{1}{6}\rho^{\alpha_{k+1}}. \tag{19}$$

Now, we prove that if (18) holds then

$$O(\rho^{-p\alpha}) = \sum_{i=1}^{m} \sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau-\gamma\prime,i} > \tag{20}$$

for any $j = 1, 2, \ldots, m$. Here we remark that $\gamma\prime \neq 0$. If it were the case, then we would have from $h_\tau - \gamma\prime \notin B_k(\gamma, p_1)$ that $h_\tau \notin B_k(\gamma, p_1)$ which is a contradiction. So, to prove (20), we argue as Theorem 2.2.2 (a) of [22]: Since Λ_N satisfies the inequality (18), by (19) (for $s = 0$) we have $\mid \Lambda_N - \mid h_\tau - \gamma\prime \mid^2 \mid > \frac{1}{6}\rho^{\alpha_{k+1}}$. Using this, in the equation (13) instead of γ we write $h_\tau - \gamma\prime$ to get

$$< \Psi_N, \Phi_{h_\tau - \gamma\prime, j} >= \sum_{i_1=1}^{m} \sum_{\gamma_1 \in \Gamma(\rho^\alpha)} v_{ij\gamma_1} \frac{< \Psi_N, \Phi_{h_\tau - \gamma\prime - \gamma_1, i_1} >}{(\Lambda_N - \mid h_\tau - \gamma\prime \mid^2)} + O(\rho^{-p\alpha}). \quad (21)$$

Substituting this equation (21) into the right hand side of (20), we obtain

$$\sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} > =$$

$$\sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} \frac{v_{ij\gamma\prime}}{\Lambda_N - \mid h_\tau - \gamma\prime \mid^2} \sum_{i_1=1}^{m} \sum_{\substack{\gamma_1 \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{i_1 i \gamma_1} < \Psi_N, \Phi_{h_\tau - \gamma\prime - \gamma_1, i_1} >$$
$$+ O(\rho^{-p\alpha}).$$

In this manner, iterating p_1 times, we get

$$\sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} >= \sum_{i_1, i_2, \ldots, i_{p_1}=1}^{m} \sum_{\substack{\gamma\prime, \gamma_1, \gamma_2, \ldots, \gamma_{p_1} \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}}$$

$$\frac{v_{ij\gamma\prime} v_{i_1 i \gamma_1} \ldots v_{i_{p_1} i_{p_1 - 1} \gamma_{p_1}} < \Psi_N, \Phi_{h_\tau - \gamma\prime - \gamma_1 - \cdots - \gamma_{p_1}, i_{p_1}} >}{(\Lambda_N - \mid h_\tau - \gamma\prime \mid^2)(\Lambda_N - \mid h_\tau - \gamma\prime - \gamma_1 \mid^2) \ldots (\Lambda_N - \mid h_\tau - \gamma\prime - \gamma_1 - \cdots - \gamma_{p_1 - 1} \mid^2)}$$
$$+ O(\rho^{-p\alpha}).$$

Taking norm of both sides of the last equality, using (19), the relation (8) and the fact that $p_1 \alpha_{k+1} \geq p_1 \alpha_2 > p\alpha$, we obtain

$$\mid \sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \notin B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} > \mid = O(\rho^{-p\alpha}),$$

which implies (20). Therefore, the equation (16) becomes

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau, j} >= \sum_{i=1}^{m} \sum_{\substack{\gamma\prime \in \Gamma(\rho^\alpha) \\ h_\tau - \gamma\prime \in B_k(\gamma, p_1)}} v_{ij\gamma\prime} < \Psi_N, \Phi_{h_\tau - \gamma\prime, i} > + O(\rho^{-p\alpha}). \quad (22)$$

Since $h_\tau - \gamma\prime \in B_k(\gamma, p_1)$, using the notation $h_\xi = h_\tau - \gamma\prime$, the decomposition (22) can be written as

$$(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau, j} >= \sum_{i=1}^{m} \sum_{h_\tau - h_\xi \in \Gamma(\rho^\alpha)} v_{ijh_\tau - h_\xi} < \Psi_N, \Phi_{h_\xi, i} > + O(\rho^{-p\alpha}). \quad (23)$$

Isolating the terms where $h_\tau - h_\xi = 0$ in (23), we get

$$\begin{aligned}(\Lambda_N - |h_\tau|^2) < \Psi_N, \Phi_{h_\tau,j} > &= \sum_{i=1}^{m} v_{ij0} < \Psi_N, \Phi_{h_\tau,i} > \\ &+ \sum_{i=1}^{m} \sum_{\substack{h_\tau - h_\xi \in \Gamma(\rho^\alpha) \\ h_\tau - h_\xi \neq 0}} v_{ijh_\tau - h_\xi} < \Psi_N, \Phi_{h_\xi,i} > \\ &+ O(\rho^{-p\alpha}). \end{aligned} \tag{24}$$

Writing the equation (24) for all $j = 1, 2, \ldots, m$ and for any $\tau = 1, 2, \ldots, b_k$, , we get the system of equations

$$[(\Lambda_N - |h_\tau|^2)I - V_0]A(N, h_\tau) = \sum_{\substack{\xi=1 \\ \xi\neq\tau}}^{b_k} V_{h_\tau - h_\xi} A(N, h_\xi) + O(\rho^{-p\alpha}), \tag{25}$$

where I is an $m \times m$ identity matrix, $V_{h_\tau - h_\xi}$ is given by (15),

$$O(\rho^{-p\alpha}) = (O(\rho^{-p\alpha}), \ldots, O(\rho^{-p\alpha}))$$

is an $m \times 1$ vector and $A(N, h_\xi)$ is the $m \times 1$ vector

$$A(N, h_\xi) = (< \Psi_N, \Phi_{h_\xi,1} >, < \Psi_N, \Phi_{h_\xi,2} >, \ldots, < \Psi_N, \Phi_{h_\xi,m} >) \tag{26}$$

for any $\xi = 1, 2, \ldots, b_k$. Letting $\lambda_{N,\tau} = \Lambda_N - |h_\tau|^2$, we have

$$\begin{bmatrix} \lambda_{N,1}, I - V_0 & -V_{h_1-h_2} & \cdots & -V_{h_1-h_{b_k}} \\ -V_{h_2-h_1} & \lambda_{N,2} I - V_0 & \cdots & -V_{h_2-h_{b_k}} \\ \vdots & & & \\ -V_{h_{b_k}-h_1} & -V_{h_{b_k}-h_2} & \cdots & \lambda_{N,b_k} I - V_0 \end{bmatrix} \begin{bmatrix} A(N,h_1) \\ A(N,h_2) \\ \vdots \\ A(N,h_{b_k}) \end{bmatrix} = \begin{bmatrix} O(\rho^{-p\alpha}) \\ O(\rho^{-p\alpha}) \\ \vdots \\ O(\rho^{-p\alpha}), \end{bmatrix}. \tag{27}$$

We may write the system (27) as

$$[\Lambda_N I - C]\mathcal{A}(N, h_1, h_2, \ldots, h_{b_k}) = \mathcal{O}(\rho^{-p\alpha}), \tag{28}$$

where I is an $mb_k \times mb_k$ identity matrix, C is given by (14), $A(N, h_1, h_2, \ldots, h_{b_k})$ is the $mb_k \times 1$ vector

$$\mathcal{A}(N, h_1, h_2, \ldots, h_{b_k}) = (A(N, h_1), A(N, h_2), \ldots, A(N, h_{b_k})) \tag{29}$$

and the right side of the system (28) is the $mb_k \times 1$ vector whose norm is

$$|\mathcal{O}(\rho^{-p\alpha})| = O(\sqrt{b_k}\rho^{-p\alpha}). \tag{30}$$

Theorem 1. *Let* $| \gamma |^2$ *be a resonance eigenvalue of the operator* $L(0)$*, that is,* $\gamma \in (\bigcap_{i=1}^{k} V_{\gamma_i}(\rho^{\alpha_k})) \setminus E_{k+1}$*,* $k = 1, 2, \ldots, d-1$ *where* $| \gamma |\sim \rho$*, and* Λ_N *an eigenvalue* *of the operator $L(V)$ for which* (18) *holds and its corresponding eigenfunction Ψ_N satisfies*

$$|<\Phi_{\gamma,j},\Psi_N>|>c_4\rho^{-c\alpha}. \tag{31}$$

Then there exists an eigenvalue $\eta_s(\gamma)$, $1\le s\le mb_k$ of the matrix C such that

$$\Lambda_N=\eta_s(\gamma)+O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}).$$

Proof. Since (18) is satisfied, (28) holds. Then multiplying both sides of the equation (28) by $[\Lambda_N I-C]^{-1}$, then taking norm of both sides and by (30), we get

$$|\mathcal{A}(N,h_1,h_2,\ldots,h_{b_k})|\le\|[\Lambda_N I-C]^{-1}\|\,O(\sqrt{b_k}\rho^{-p\alpha}). \tag{32}$$

Using the fact that γ is one of $h_1,h_2,\ldots,h_\tau$ (See definition of $B_k(\gamma,p_1)$) and hence by (31) and (32), we obtain

$$c_5\rho^{-c\alpha}<|\mathcal{A}(N,h_1,h_2,\ldots,h_{b_k})|\le\|[\Lambda_N I-C]^{-1}\|\sqrt{b_k}c_6\rho^{-p\alpha}.$$

Since $[\Lambda_N I-C]^{-1}$ is symmetric matrix with the eigenvalues $\frac{1}{\Lambda_N-\eta_s(\gamma)}$, $s=1,\ldots,mb_k$, we have

$$\max_{s=1,\ldots,mb_k}|\Lambda_N-\eta_s(\gamma)|^{-1}=\|[\Lambda_N I-C]^{-1}\|>c_7c_8^{-1}b_k^{-\frac{1}{2}}\rho^{-c\alpha+p\alpha},$$

where $b_k=O(\rho^{\frac{d}{2}3^d\alpha})$, thus

$$\min_{s=1,2,\ldots,mb_k}|\Lambda_N-\eta_s(\gamma,\lambda_i)|\le c_9\rho^{-(p-c-\frac{d}{4}3^d)\alpha},$$

and

$$\Lambda_N=\eta_s(\gamma,\lambda_i)+O(\rho^{-(p-c-\frac{d}{4}3^d)\alpha}).$$

□

Theorem 2. *Let $|\gamma|^2$ be a resonance eigenvalue of the operator $L(0)$, that is, $\gamma\in(\bigcap_{i=1}^{k}V_{\gamma_i}(\rho^{\alpha_k}))\setminus E_{k+1}$, $k=1,2,\ldots,d-1$ where $|\gamma|\sim\rho$, $\eta_s(\gamma)$ an eigenvalue of the matrix C such that $|\eta_s(\gamma)-|\gamma|^2|<\frac{3}{8}\rho^{\alpha_1}$. Then there is an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N=\eta_s(\gamma)+O(\rho^{-p\alpha+\frac{d}{4}3^d\alpha+\frac{d-1}{2}}). \tag{33}$$

Proof. By the general perturbation theory, there is an eigenvalue Λ_N of the operator $L(V)$ such that $|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}$ holds. Thus one can use the system (28) and we prove the theorem for this eigenvalue Λ_N:

Let η_s, $s=1,2,\ldots,mb_k$ be an eigenvalue of the matrix C and $\theta_s=(\theta_s^1,\theta_s^2,\ldots,\theta_s^{b_k})_{mb_k\times1}$ the corresponding normalized eigenvector, where $\theta_s^\tau=(\theta_s^{\tau1},\theta_s^{\tau2},\ldots,\theta_s^{\tau m})_{m\times1}$, $\tau=1,2,\ldots,b_k$. Multiplying the equation (28) by θ_s, since C is symmetric (see (14) and (15)), we get

$$|\Lambda_N-\eta_s||\mathcal{A}(N,h_1,h_2,\ldots,h_{b_k})\cdot\theta_s|=|\mathcal{O}(\rho^{-p\alpha})\cdot\theta_s|. \tag{34}$$

By using $b_k = O(\rho^{\frac{d}{2}3^d\alpha})$, (30) and the Cauchy Schwartz Inequality for the right hand side of (34), we have

$$|\Lambda_N - \eta_s||\mathcal{A}(N, h_1, h_2, \dots, h_{b_k}) \cdot \theta_s| = O(\rho^{-p\alpha + \frac{d}{4}3^d\alpha}). \tag{35}$$

So we need to prove that

$$|\mathcal{A}(N, h_1, h_2, \dots, h_{b_k}) \cdot \theta_s| > c_{10}\rho^{-\frac{d-1}{2}}, \tag{36}$$

from which the theorem follows.

For this purpose,we first consider the decomposition of the matrix C as $C = A + B$, where

$$A = \begin{bmatrix} |h_1|^2 I & & 0 \\ & \ddots & \\ 0 & & |h_{b_k}|^2 I \end{bmatrix}, \quad B = \begin{bmatrix} V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_k}} \\ V_{h_2-h_1} & V_0 & \cdots & V_{h_2-h_{b_k}} \\ \vdots & & \ddots & \vdots \\ V_{h_{b_k}-h_1} & V_{h_{b_k}-h_2} & \cdots & V_0 \end{bmatrix}. \tag{37}$$

The eigenvalues and the corresponding eigenspaces of the matrix A are $|h_\tau|^2$ and $E_\tau = span\{e_j : (\tau-1)m + 1 \le j \le \tau m\}$, respectively, where

$$\{e_j = (0, \dots, 0, 1, 0, \dots, 0)\}_{j=1}^{mb_k}$$

is the standard basis of R^{mb_k}. Now, we use the following notation

$$\theta_s(h_{\tau,j}) \equiv \theta_s \cdot e_j = \theta_s^{\tau j}, \quad \text{if} \quad (\tau-1)m + 1 \le j \le \tau m, \tag{38}$$

for $\tau = 1, 2, \cdots, b_k$.

Multiplying $(A + B)\theta_s = \eta_s\theta_s$ by e_j, since A and B are symmetric, we get

$$(\eta_s - |h_\tau|^2)\theta_s(h_{\tau,j}) = \theta_s \cdot Be_j \tag{39}$$

and $(\tau-1)m + 1 \le j \le \tau m$, and $\tau = 1, 2, \cdots, b_k$.

On the other hand, if we consider the sum of the elements in the i-th row of the matrix B, by (8)

$$\sum_{\substack{\tau=1 \\ \tau \ne i}}^{b_k} \sum_{j=1}^{m} v_{ijh_i - h_\tau} < \sum_{j=1}^{m} M_{ij}, \tag{40}$$

for all $i = 1, 2, \dots, m$. Since B is a symmetric matrix and by (40), the sum of elements in each row of B is less then $M = \max_{i=1,2,\dots,m}\{\sum_{j=1}^{m} M_{ij}\}$, the eigenvalues of B are also less then M from which we have $\| B \| \le M$.

Thus, by (26), (36), (38), we have

$$|\mathcal{A}(N, h_1, \dots, h_{b_k}) \cdot \theta_s| = |\langle \psi_N, \sum_{\tau=1}^{b_k} \sum_{j=1}^{m} \theta_s(h_{\tau,j})\phi_{h_{\tau,j}} \rangle|, \tag{41}$$

which, together with Parseval's relation, imply

$$\begin{aligned}1 &= \| \sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i})\Phi_{h_\tau,i} \|^2 \\ &= \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &\quad + \sum_{N:|\Lambda_N-|\gamma|^2|<\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2. \qquad (42)\end{aligned}$$

Now we estimate the first summation in the expression (42):

$$\begin{aligned}&\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &= \sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > \\ &+ \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &< 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|<\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &+ 2\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2. \qquad (43)\end{aligned}$$

Using Bessel's inequality, Parseval's relation, orthogonality of the functions $\Phi_{h_\tau,i}(x)$, $\tau = 1, 2, \ldots, b_k$, $i = 1, 2, \ldots, m$, the binding formula (39) and $\| B \| \leq M$, we have

$$\begin{aligned}&\sum_{N:|\Lambda_N-|\gamma|^2|\geq\frac{1}{2}\rho^{2\alpha_1}} | \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}\theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\ &\leqslant \| \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}\theta_s(h_{\tau,i})\Phi_{h_\tau,i} \|^2 \\ &= \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}|\theta_s(h_{\tau,i})|^2 \| \Phi_{h_\tau,i} \|^2 \\ &= \sum_{\tau:|\eta_s-|h_\tau|^2|\geq\frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m}\frac{|\theta_s \cdot Be_i|^2}{|\eta_s-|h_\tau|^2|^2} = O(\rho^{-2\alpha_1}). \qquad (44)\end{aligned}$$

The assumption $|\eta_s - |\gamma|^2| < \frac{3}{8}\rho^{\alpha_1}$ of the theorem and $|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}$ imply that $||\gamma|^2 - |h_\tau|^2| < \frac{1}{2}\rho^{\alpha_1}$. So by the well-known formula

$$\frac{1}{\Lambda_N - |h_\tau|^2} = \frac{1}{\Lambda_N - |\gamma|^2}\{\sum_{n=0}^{k}(\frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2})^n + O(\rho^{-(k+1)\alpha_1})\},$$

for $|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}$, and $||\gamma|^2 - |h_\tau|^2| < \frac{1}{2}\rho^{2\alpha_1}$, using (39), we have

$$\begin{aligned}
&\sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \\
&= \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) \frac{< \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |h_\tau|^2} |^2 \\
&\leq \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1) |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |\gamma|^2} |^2 \\
&+ \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1) |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |\gamma|^2} \frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2} |^2 \\
&\qquad\vdots \\
&+ \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1) |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \frac{\theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} >}{\Lambda_N - |\gamma|^2} [\frac{|h_\tau|^2 - |\gamma|^2}{\Lambda_N - |\gamma|^2}]^k |^2 \\
&+ \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1) |\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > O(\rho^{-(k+1)\alpha_1}) |^2.
\end{aligned} \tag{45}$$

To calculate the order of each term in (44), we use Bessel's inequality and the orthogonality of $\Phi_{h_\tau,i}$. So we have

$$\begin{aligned}
&2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1) \\
&\times \left| \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > \frac{(|h_\tau|^2 - |\gamma|^2)^r}{(\Lambda_N - |\gamma|^2)^{r+1}} \right|^2 \\
= \; &2 \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \frac{(k+1)}{|\Lambda_N - |\gamma|^2|^{2(r+1)}}
\end{aligned}$$

$$\times \left| \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > (|h_\tau|^2 - |\gamma|^2)^r \right|^2$$

$$\leq c_{11}(\rho^{2\alpha_1})^{-2(r+1)}(k+1)$$

$$\times \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} \left| < \Psi_N, \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V\Phi_{h_\tau,i} > \right|^2$$

$$\leq c_{12}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \left\| \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V\Phi_{h_\tau,i} \right\|^2$$

$$\leq c_{13}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \left(\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \left\| \theta_s(h_{\tau,i})(|h_\tau|^2 - |\gamma|^2)^r V\Phi_{h_\tau,i} \right\| \right)^2$$

$$= c_{14}(\rho^{2\alpha_1})^{-2(r+1)}(k+1) \left(\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} |\theta_s(h_{\tau,i})| ||h_\tau|^2 - |\gamma|^2|^r \parallel V\Phi_{h_\tau,i} \parallel \right)^2$$

$$\leq c_{15}(\rho^{2\alpha_1})^{-2(r+1)}(\frac{1}{2}\rho^{\alpha_1})^{2r}(k+1)(\sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \parallel V\Phi_{h_\tau,i} \parallel)^2 = O(\rho^{-2(r+1)\alpha_1}), \tag{46}$$

for $r = 0, 1, 2, \ldots, k$. Now let K be the number of h_τ satisfying $|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}$, then the order of the last summation in (46) is:

$$\sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1)$$

$$\times \left| \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, V\Phi_{h_\tau,i} > O(\rho^{-(k+1)\alpha_1}) \right|^2$$

$$\leq K \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} (k+1)$$

$$\times \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} |O(\rho^{-(k+1)\alpha_1})|^2 \cdot |\theta_s(h_{\tau,i})|^2 \cdot | < \Psi_N, V\Phi_{h_\tau,i} > |^2$$

$$\leq c_{16} \cdot K \cdot \rho^{-2(k+1)\alpha_1} \cdot \sum_{\tau:|\eta_s - |h_\tau|^2| < \frac{1}{8}\rho^{\alpha_1}} \parallel V(x)\Phi_{h_\tau,i} \parallel^2$$

$$\leq c_{17} \cdot K^2 \cdot M^2 \cdot \rho^{-2(k+1)\alpha_1} = K^2 \cdot 0(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}),$$

since $K = O(\rho^{\frac{d}{2}\alpha_d})$ and we can always choose k in $O(\rho^{-2(k+1)\alpha_1})$ such that

$$K^2 \cdot O(\rho^{-2(k+1)\alpha_1}) = O(\rho^{-2\alpha_1}), \tag{47}$$

which together with the estimations (44), , (45) and (46) imply

$$O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N - |\gamma|^2| \geq \frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2.$$

Therefore, from the decomposition (42) we have

$$1 - O(\rho^{-2\alpha_1}) = \sum_{N:|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}} |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2.$$

Since the number of indexes N satisfying $|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}$ is less then ρ^{d-1}, we have

$$1 - O(\rho^{-2\alpha_1}) \leq \rho^{d-1} \max_{N:|\Lambda_N - |\gamma|^2| < \frac{1}{2}\rho^{2\alpha_1}} \left\{ |\sum_{\tau=1}^{b_k}\sum_{i=1}^{m} \theta_s(h_{\tau,i}) < \Psi_N, \Phi_{h_\tau,i} > |^2 \right\}$$

which implies together with the relation (41) that

$$|A(N, h_1, h_2, \ldots, h_{b_k}) \cdot \theta_s|^2 \geq \frac{1 - O(\rho^{-2\alpha_1})}{\rho^{d-1}}. \tag{48}$$

It follows from the equation (35) and the estimation (48) that

$$\Lambda_N = \eta_s + \frac{O(\rho^{-p\alpha + \frac{d}{4}3^d\alpha})}{O(\rho^{-\frac{d-1}{2}})},$$

that is, (36) holds. □

3. Asymptotic Formulas for the Eigenvalues in a Single Resonance Domain

Now, we investigate in detail the eigenvalues of $L(V)$ in a single resonance domain. In order the inequalities

$$0 < \alpha < \frac{1}{d+20}, \quad 2\alpha_2 - \alpha_1 + (d+3)\alpha < 1 \tag{49}$$

and

$$\alpha_2 > 2\alpha_1, \tag{50}$$

to be satisfied, we can choose $\alpha,\ \alpha_1$ and α_2 as follows

$$\alpha = \frac{1}{d+p}, \ \ \alpha_1 = \frac{p_2}{d+p}, \ \ \alpha_2 = \frac{2p_2+1}{d+p},$$

where $p_2 = [\frac{p-5}{3}] - 1$. Let $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$, $\delta \in \frac{\Gamma}{2} \backslash \{e_i\}$, where δ is minimal in its direction. Consider the following sets :

$$B_1(\delta) = \{b : b = n\delta\,, n \in Z\,,\ |b| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}\},$$
$$B_1(\gamma) = \gamma + B_1(\delta) = \{\gamma + b : b \in B_1(\delta)\},$$
$$B_1(\gamma, p_1) = B_1(\gamma) + \Gamma(p_1\rho^{\alpha}).$$

As before, denote by h_τ, $\tau = 1, 2, ..., b_1$ the vectors of $B_1(\gamma, p_1)$, where b_1 is the number of vectors in $B_1(\gamma, p_1)$. Then the matrix $C(\gamma, \delta) = (c_{ij})$, $i, j = 1, 2, \ldots, mb_1$ is defined by

$$C(\gamma,\delta) = \begin{bmatrix} |h_1|^2 I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{b_1}} \\ V_{h_2-h_1} & |h_2|^2 I - V_0 & \cdots & V_{h_2-h_{b_1}} \\ \vdots & & & \\ V_{h_{b_1}-h_1} & V_{h_{b_1}-h_2} & \cdots & |h_{b_1}|^2 I - V_0 \end{bmatrix}, \tag{51}$$

where $V_{h_\tau - h_\xi}$, $\tau, \xi = 1, 2, \ldots, b_1$ are the $m \times m$ matrices defined by (15).
Also we define the matrix $D(\gamma, \delta) = (c_{ij})$ for $i, j = 1, 2, ..., ma_1$, where $h_1, h_2, ..., h_{a_1}$ are the vectors of $B_1(\gamma, p_1) \bigcap \{\gamma + n\delta : n \in Z\}$, and a_1 is the number of vectors in $B_1(\gamma, p_1) \bigcap \{\gamma + n\delta : n \in Z\}$. Clearly $a_1 = O(\rho^{\frac{1}{2}\alpha_2})$.

Lemma 3. *a) If η_{j_s} is an eigenvalue of the matrix $C(\gamma, \delta)$ such that $|\eta_{j_s} - |h_s|^2| < M$ for $s = 1, 2, ..., a_1$, $1 + (s-1)m \le j_s \le ms$, then*

$$|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4}\rho^{\alpha_2}\ ,\ \ \forall \tau = a_1 + 1, a_1 + 2, ..., b_1.$$

b) If η_{j_s} is an eigenvalue of the matrix $C(\gamma, \delta)$ such that $|\eta_{j_s} - |h_s|^2| < M$ for $s = a_1 + 1, a_1 + 2, ..., b_1$ and $1 + (s-1)m \le j_s \le ms$, then

$$|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4}\rho^{\alpha_2}\ ,\ \ \forall \tau = 1, 2, ..., a_1.$$

Proof. First we prove

$$||h_\tau|^2 - |h_s|^2| \ge \frac{1}{3}\rho^{\alpha_2}, \qquad \forall s \le a_1, \qquad \forall \tau > a_1. \tag{52}$$

By definition, if $s \le a_1$ then $h_s = \gamma + n\delta$, where $|n\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2} + p_1\rho^{\alpha}$. If $\tau > a_1$ then $h_\tau = \gamma + s^{'}\delta + a$, where $|s^{'}\delta| < \frac{1}{2}\rho^{\frac{1}{2}\alpha_2}$, $a \in \Gamma(p_1\rho^{\alpha}) \setminus \delta R$. Therefore

$$|h_\tau|^2 - |h_s|^2 = 2\gamma \cdot a + 2s^{'}\delta \cdot a + 2s^{'}\gamma \cdot \delta + |s^{'}\delta|^2 + |a|^2 - 2n\gamma \cdot \delta - |n\delta|^2.$$

Since $\gamma \notin V_a(\rho^{\alpha_2})$, $|a| < p_1\rho^{\alpha}$, we have

$$|2\gamma \cdot a| > \rho^{\alpha_2} - c_0\rho^{2\alpha}.$$

The relation $\gamma \in V_\delta(\rho^{\alpha_1})$ and the inequalities for $s^{'}$ and n imply that

$$2s^{'}\gamma \cdot \delta + 2s^{'}\gamma \cdot a + |a|^2 - 2n\gamma \cdot \delta \quad = \quad O(\rho^{\frac{1}{2}\alpha_2 + \alpha_1}),$$

$$||s^{'}\delta|^2 - |n\delta|^2| \quad < \quad \frac{1}{4}\rho^{\alpha_2} + c_0\rho^{\frac{1}{2}\alpha_2+\alpha}.$$

Thus (52) follows from these relations, since $\frac{1}{2}\alpha_2 + \alpha_1 < \alpha_2$ and $\frac{1}{2}\alpha_2 + \alpha < \alpha_2$.

The eigenvalues of $D(\gamma,\delta)$ and $C(\gamma,\delta)$ lay in M-neighborhood of the numbers $|h_k|^2$ for $k = 1,2,...,a_1$ and for $k = 1,2,...,b_1$, respectively. The inequality (52) shows that one can enumerate the eigenvalues η_j $(j = 1,2,\ldots,mb_1)$ of C in the following way:

$$\eta_j \equiv \eta_{j_s}, \quad j_s \leq ma_1, \quad 1 + (s-1)m \leq j_s \leq sm$$

when for $s \leq a_1$, η_j lay in M-neighborhood of $|h_s|^2$ and

$$\eta_j \equiv \eta_{j_\tau}, \quad j_\tau \geq ma_1, \quad 1 + (\tau-1)m \leq j_\tau \leq \tau m$$

when for $\tau > a_1$, η_j lay in M-neighborhood $|h_\tau|^2$. Then by (52), we get

$$|\eta_{j_s} - |h_\tau|^2| > \frac{1}{4}\rho^{\alpha_2}, \tag{53}$$

for $s \leq a_1$, $\tau > a_1$ and $s > a_1$, $\tau \leq a_1$. □

Now, using the notation $h_s = \gamma - (\frac{s}{2})\delta$ if s is even, $h_s = \gamma + (\frac{s-1}{2})\delta$ if s is odd, for $s = 1,2,...,a_1$, (without loss of generality assume that a_1 is even) and using the orthogonal decomposition of $\gamma \in \frac{\Gamma}{2}$, $\gamma = \beta + (l + v(\beta))\delta$, where $\beta \in H_\delta \equiv \{x \in R^d : x\cdot\delta = 0\}$, $l \in Z$, $v \in [0,1)$ we can write the matrix $D(\gamma,\delta)$ as

$$D(\gamma,\delta) = |\beta|^2 I + E(\gamma,\delta), \tag{54}$$

where I is a maximal identity matrix and $E(\gamma,\delta)$ is

$$E(\gamma,\delta) = \begin{bmatrix} \left((l+v)^2|\delta|^2\right)I + V_0 & V_\delta & V_{-\delta} & \cdots & V_{\frac{a_1}{2}\delta} \\ V_{-\delta} & \left((l-1+v)^2|\delta|^2\right)I + V_0 & V_{-2\delta} & \cdots & V_{\left(\frac{a_1}{2}-1\right)\delta} \\ V_\delta & V_{2\delta} & \left((l+1+v)^2|\delta|^2\right)I + V_0 & \cdots & V_{\left(\frac{a_1}{2}+1\right)\delta} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{-\frac{a_1}{2}\delta} & \cdot & \cdot & \cdots & \left((l-\frac{a_1}{2}+v)^2|\delta|^2\right)I + V_0 \end{bmatrix}$$

Denote $n_k = -\frac{k}{2}$ if k is even, $n_k = \frac{k-1}{2}$ if k is odd. The system $\{e^{i(n_k+v)t} : k = 1,2,...\}$ is a basis in $L_2^m[0,2\pi]$. Let $T(\gamma,\delta) \equiv T(P(t),\beta)$ be the operator in ℓ_2 corresponding to the Sturm-Liouville operator T, generated by

$$-|\delta|^2 Y''(t) + P(t)Y(t) = \mu Y(t), \tag{55}$$

$$Y(t+2\pi) = e^{i2\pi v(\beta)}Y(t),$$

where $P(t) = (p_{ij}(t))$, $p_{ij}(t) = \sum\limits_{k=1}^{\infty} v_{ijn_k\delta}e^{in_kt}$, $v_{ijn_k\delta} = (v_{ij}(x), \frac{1}{|A_{n_k\delta}|}\sum\limits_{\alpha\in A_{n_k\delta}} e^{i(\alpha\cdot x)})$, $t = x\cdot\delta$. It means that $T(\gamma,\delta)$ is the infinite matrix $(Te^{i(l+n_k+v)t}, e^{i(l+n_m+v)t})$, $k,m = 1,2,\ldots.$

To find the relation between the eigenvalues of $L(V)$ in a single resonance domain and the eigenvalues of the Sturm-Liouville operators defined by (55), we need the following theorems.

Theorem 4. *Let $\gamma \in V_\delta(\rho^{\alpha_1}) \setminus E_2$ and $|\gamma| \sim \rho$. Then, for any eigenvalue $\eta_{j_s}(\gamma)$ of the matrix $C(\gamma,\delta)$ satisfying*

$$|\eta_{j_s} - |h_s|^2| < M, \quad 1 + (s-1)m \leq j_s \leq sm, \quad s = 1, 2, ..., a_1 \tag{56}$$

there exists an eigenvalue $\widetilde{\eta}_{k(j_s)}$ of the matrix $D(\gamma,\delta)$ such that

$$\eta_{j_s} = \widetilde{\eta}_{k(j_s)} + O(\rho^{-\frac{3}{4}\alpha_2}).$$

Proof. Let η_{j_s} be an eigenvalue of the matrix $C(\gamma,\delta)$ satisfying (56) and $\theta_{j_s} = (\theta^1_{j_s}, \theta^2_{j_s}, ..., \theta^{b_1}_{j_s})_{mb_1\times 1}$ be the corresponding normalized eigenvector, $|\theta_{j_s}| = 1$. Now, we consider the decomposition $C = A + B$ and the matrices A, B which are defined in (37). Writing the binding formula (39) for η_{j_s} and using (38), we get

$$(\eta_{j_s} - |h_\tau|^2)\theta_{j_s}(h_{\tau,i}) = \theta_{j_s} \cdot Be_i, \tag{57}$$

$\tau = 1, 2, \ldots, b_1, \;\; 1 + (\tau - 1)m \leq i \leq \tau m.$
For simplicity, we use the following notation in the sequel:

$$e_{\zeta,k} = e_k \quad \text{if} \quad 1 + (\zeta-1)m \leq k \leq \zeta m, \quad \zeta = 1, \ldots, b_1,$$
$$Be_i \cdot e_{k_1} = Be_{\tau,i} \cdot e_{\xi,k_1} = b(\tau, i, \xi, k_1).$$

Thus, substituting the orthogonal decomposition

$$Be_i = Be_{\tau,i} = \sum_{\substack{\xi=1,2,\ldots,b_1 \\ 1+(m-1)\xi \leq k_1 \leq m\xi}} b(\tau, i, \xi, k_1) e_{\xi,k_1}$$

into the formula (57), we get

$$\begin{aligned}
(\eta_{j_s} - |h_\tau|^2)\theta_{j_s}(h_{\tau,i}) &= \theta_{j_s} \cdot \sum_{\substack{\xi=1,2,\ldots,b_1 \\ 1+(m-1)\xi \leq k_1 \leq m\xi}} b(\tau, i, \xi, k_1) e_{\xi,k_1} \\
&= \sum_{\substack{\xi=1,2,\ldots,b_1 \\ 1+(m-1)\xi \leq k_1 \leq m\xi}} b(\tau, i, \xi, k_1) \theta_{j_s} \cdot e_{\xi,k_1} \\
&= \sum_{\substack{\xi=1,2,\ldots,b_1 \\ 1+(m-1)\xi \leq k_1 \leq m\xi}} b(\tau, i, \xi, k_1) \theta_{j_s}(h_\xi, k_1).
\end{aligned}$$

It is clear that

$$b(\tau, i, \xi, k_1) = \begin{cases} 0 & \text{if} \quad \xi = \tau, \\ v_{k_1 i h_\xi - h_\tau} & \text{if} \quad \xi \neq \tau, \end{cases}$$

which implies

$$\sum_{\substack{\xi=1,2,\ldots,b_1 \\ 1+(m-1)\xi \leq k_1 \leq m\xi}} b(\tau, i, \xi, k_1) = \sum{}^{\xi=1,2,\ldots,b_1} v_{\quad k_1 i h_\xi - h_\tau}.$$

Thus one has

$$\begin{aligned}(\eta_{j_s}-|h_\tau|^2)\theta_{j_s}(h_\tau,i) &= \sum{}^{\xi=1,2,\ldots,b_1} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1)\\ &= \sum{}^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1)\\ &\quad+\sum{}^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1). \qquad (58)\end{aligned}$$

Now, writing the equation (58) for all h_τ , $\tau = 1, 2, ..., a_1$, we get the system of linear algebraic equations:

$$\begin{aligned}&(\eta_{j_s}-|h_1|^2)\theta_{j_s}(h_1,i)-\sum{}^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_1}\theta_{j_s}(h_\xi,k_1)\\ =\;&\sum{}^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_1}\theta_{j_s}(h_\xi,k_1)\\[1em] &(\eta_{j_s}-|h_2|^2)\theta_{j_s}(h_2,i)-\sum{}^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_2}\theta_{j_s}(h_\xi,k_1)\\ =\;&\sum{}^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_2}\theta_{j_s}(h_\xi,k_1)\\ &\vdots\\ &(\eta_{j_s}-|h_{a_1}|^2)\theta_{j_s}(h_{a_1},i)-\sum{}^{\xi=1,2,\ldots,a_1} v_{k_1ih_\xi-h_{a_1}}\theta_{j_s}(h_\xi,k_1)\\ =\;&\sum{}^{\xi=a_1+1,\ldots,b_1} v_{k_1ih_\xi-h_{a_1}}\theta_{j_s}(h_\xi,k_1) \qquad (59)\end{aligned}$$

Using the binding formula (57), the relation (53), and $\parallel B \parallel\leq M$, for any $\tau = 1, 2, \ldots, a_1$, we find

$$\begin{aligned}|\sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} v_{k_1ih_\xi-h_\tau}\theta_{j_s}(h_\xi,k_1)| &= |\sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} v_{k_1ih_\xi-h_\tau}\frac{\theta_{j_s}\cdot Be_{\xi,k_1}}{(\eta_{j_s}-|h_\xi|^2)}|\\ &\leq \sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} |v_{k_1ih_\xi-h_\tau}|\frac{|\theta_{j_s}|\|B\||e_{\xi,k_1}|}{(\eta_{j_s}-|h_\xi|^2)}\\ &\leq 4\rho^{-\alpha_2}M\sum_{\substack{\xi=a_1+1,\ldots,b_1\\ k_1=1,2,\ldots,m\\ \xi\neq\tau}} |v_{k_1ih_\xi-h_\tau}|\\ &\leq 4\rho^{-\alpha_2}M^2\\ &= O(\rho^{-\alpha_2}) \qquad (60)\end{aligned}$$

and

$$\begin{aligned}\sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}} |\theta_{j_s}(h_\tau,i)|^2 &= \sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}} \left|\frac{\theta_{j_s}\cdot Be_{\tau,i}}{(\eta_{j_s}-|h_\tau|^2)}\right|^2 \\ &= \sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}} \frac{|B\theta_{j_s}\cdot e_{\tau,i}|^2}{(\eta_{j_s}-|h_\tau|^2)^2} \\ &\leq 16M^2\rho^{-2\alpha_2} \\ &= O(\rho^{-2\alpha_2}). \end{aligned} \tag{61}$$

By (60) and (54), (59) becomes

$$[\theta^1_{j_s},\theta^2_{j_s},\ldots,\theta^{a_1}_{j_s}]^t = (D(\gamma,\delta)-\eta_{j_s}I)^{-1}[O(\rho^{-\alpha_2}),O(\rho^{-\alpha_2}),...,O(\rho^{-\alpha_2})]^t. \tag{62}$$

By the Parseval's identity and (61), we get

$$\begin{aligned}\sum_{\substack{\tau=1,2,\ldots,a_1\\ i=1,2,\ldots,m}} |\theta_{j_s}(h_\tau,i)|^2 &= \sum_{\substack{\tau=1,2,\ldots,b_1\\ i=1,2,\ldots,m}} |\theta_{j_s}(h_\tau,i)|^2 - \sum_{\substack{\tau=a_1+1,\ldots,b_1\\ i=1,2,\ldots,m}} |\theta_{j_s}(h_\tau,i)|^2 \\ &\geq 1-O(\rho^{-2\alpha_2}).\end{aligned}$$

Now, taking norm of both sides in (62) and using the above inequality we have

$$\sqrt{1-O(\rho^{-2\alpha_2})} < \Big(\sum_{\substack{\tau=1,2,\ldots,a_1\\ i=1,2,\ldots,m}} |\theta_{j_s}(h_\tau,i)|^2\Big)^{\frac{1}{2}} \leq \|(D(\gamma,\delta)-\eta_{j_s}I)^{-1}\| O(\sqrt{a_1}\rho^{-\alpha_2}).$$

Thus

$$max|\eta_{j_s}-\widetilde{\eta}_{k(j_s)}|^{-1} > \frac{\sqrt{1-O(\rho^{-2\alpha_2})}}{\sqrt{a_1}\rho^{-\alpha_2}},$$

or

$$min|\eta_{j_s}-\widetilde{\eta}_{k(j_s)}| = O(\sqrt{a_1}\rho^{-\alpha_2}) = O(\rho^{-\frac{3}{4}\alpha_2}),$$

where the maximum (minimum) is taken over all $\widetilde{\eta}_{k(j_s)}$, $s=1,2,...,a_1$. So the result follows. □

Theorem 5. *For any eigenvalue $\widetilde{\eta}_\tau$ of the matrix $D(\gamma,\delta)$, there exists an eigenvalue $\eta_{j_s(\tau)}$ of the matrix $C(\gamma,\delta)$ such that*

$$\eta_{j_s(\tau)} = \widetilde{\eta}_\tau + O(\rho^{-\frac{1}{2}\alpha_2})$$

Proof. Define the matrix $D' = D'(\gamma, \delta)$ by

$$D' = \begin{bmatrix} |h_1|^2 I - V_0 & V_{h_1-h_2} & \cdots & V_{h_1-h_{a_1}} & 0 & 0 & \cdots & 0 \\ V_{h_2-h_1} & |h_2|^2 I - V_0 & \cdots & V_{h_2-h_{a_1}} & 0 & 0 & \cdots & 0 \\ \vdots & & & & & & & \\ V_{h_{a_1}-h_1} & V_{h_{a_1}-h_2} & \cdots & |h_{a_1}|^2 I - V_0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & |h_{a_1+1}|^2 I & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & |h_{b_1-1}|^2 I & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & |h_{b_1}|^2 I \end{bmatrix} \tag{63}$$

So that the spectrum of the matrix D' is

$$\begin{aligned} spec(D') &= spec(D(\gamma,\delta)) \bigcup \{|h_{a_1+1}|^2, |h_{a_1+2}|^2, ..., |h_{b_1}|^2\} \\ &\equiv \{\widetilde{\eta}_1, \widetilde{\eta}_2, ..., \widetilde{\eta}_{ma_1}, |h_{a_1+1}|^2, |h_{a_1+2}|^2, ..., |h_{b_1}|^2\}. \end{aligned}$$

Let us denote by $\Upsilon_\tau = (\Upsilon_\tau^1, \Upsilon_\tau^2, ..., \Upsilon_\tau^{a_1}, 0, ..., 0)_{mb_1 \times 1}$, $\Upsilon_\tau^i = (\Upsilon_\tau^{i1}, \Upsilon_\tau^{i2}, \ldots, \Upsilon_\tau^{im})_{m\times 1}$ the normalized eigenvector corresponding to the τ-th eigenvalue of the matrix D', for $\tau = 1, 2, ..., ma_1$ and by $\{e_{k,i}\}_{i=1,2,\cdots,m}$ the eigenvector corresponding to the k-th eigenvalue $|h_k|^2$ of D', for $k = a_1+1, a_1+2, ..., b_1$.

Now, using (62) from the previous theorem, we have

$$\begin{aligned} & (D' - \eta_{j_s} I)[\theta_{j_s}^1, \theta_{j_s}^2, \ldots, \theta_{j_s}^{b_1}]^t \\ = & [(D(\gamma,\delta) - \eta_{j_s} I)[\theta_{j_s}^1, \theta_{j_s}^2, \ldots, \theta_{j_s}^{a_1}]^t, (|h_{a_1+1}|^2 - \eta_{j_s})\theta_{j_s}^{a_1+1}, ..., (|h_{b_1}|^2 - \eta_{j_s})\theta_{j_s}^{b_1}] \\ = & [O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2}), (|h_{a_1+1}|^2 - \eta_{j_s})\theta_{j_s}^{a_1+1}, ..., (|h_{b_1}|^2 - \eta_{j_s})\theta_{j_s}^{b_1}]. \end{aligned}$$

Taking inner product of both sides of the last equality by Υ_τ for $\tau = 1, 2, ..., ma_1$, using that D' is symmetric and $D'\Upsilon_\tau = \widetilde{\eta}_\tau \Upsilon_\tau$ we have

$$(\eta_{j_s(\tau)} - \widetilde{\eta}_\tau) \sum_{k=1}^{a_1} \theta_{j_s}^k \cdot \Upsilon_\tau^k = \sum_{k=1}^{a_1} O(\rho^{-\alpha_2}) \Upsilon_\tau^k, \tag{64}$$

For the right hand side of the equation (64) using the Cauchy-Schwarz inequality, we get

$$|\sum_{k=1}^{a_1} O(\rho^{-\alpha_2})\Upsilon_\tau^k| \le \sqrt{\sum_{k=1}^{a_1} O(\rho^{-\alpha_2})^2} \sqrt{\sum_{k=1}^{a_1} |\Upsilon_\tau^k|^2} \le \sqrt{a_1(\rho^{-\alpha_2})^2} = O(\sqrt{a_1}\rho^{-\alpha_2}),$$

where $a_1 = O(\rho^{\frac{1}{2}\alpha_2})$. Thus, the equation (64) can be written as

$$(\eta_{j_s(\tau)} - \widetilde{\eta}_\tau) \sum_{k=1}^{a_1} \theta_{j_s}^k \cdot \Upsilon_\tau^k = O(\rho^{-\frac{3}{4}\alpha_2}). \tag{65}$$

In order to get the result, we need to show that for any $\tau = 1, 2, ..., ma_1$ there exists $\theta_{j_s(\tau)}$ such that

$$|\sum_{k=1}^{a_1} \theta^k_{j_s(\tau)} \cdot \Upsilon^k_\tau| = |\theta_{j_s(\tau)} \cdot \Upsilon_\tau| > \sqrt{\frac{1 - O(\rho^{-\frac{3}{2}\alpha_2})}{ma_1}} > c_{18}\rho^{-\frac{1}{4}\alpha_2}. \qquad (66)$$

For this, we consider the orthogonal decomposition $\Upsilon_\tau = \sum_{s=1}^{mb_1} (\Upsilon_\tau \cdot \theta_{j_s})\theta_{j_s}$ and the Parseval's identity

$$1 = \sum_{s=1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 = \sum_{s=1}^{ma_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 + \sum_{s=ma_1+1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2.$$

First, let us show that

$$\sum_{s=ma_1+1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 = O(\rho^{-\frac{3}{2}\alpha_2}). \qquad (67)$$

Using the decomposition $\Upsilon_\tau = \sum_{\substack{k=1,2...,a_1 \\ i=1,2,...,m}} (\Upsilon_\tau \cdot e_{k,i})e_{k,i}$, the binding formula (57) for $C(\gamma, \delta)$ and A, the relation (53), and the Bessel's inequality we obtain the estimation

$$\begin{aligned}
& \sum_{s=ma_1+1}^{mb_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 \\
= & \sum_{s=ma_1+1}^{mb_1} |(\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} \Upsilon^{ki}_\tau e_{k,i}) \cdot \theta_{j_s}|^2 \\
= & \sum_{s=ma_1+1}^{mb_1} |\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} \Upsilon^{ki}_\tau (e_{k,i} \cdot \theta_{j_s})|^2 = \sum_{s=ma_1+1}^{mb_1} |\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} \Upsilon^{ki}_\tau \frac{\theta_{j_s} \cdot Be_{k,i}}{(\eta_{j_s} - |h_k|^2)}|^2 \\
\leq & \; 16 \sum_{s=ma_1+1}^{mb_1} \rho^{-2\alpha_2} (\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} |\Upsilon^{ki}_\tau||\theta_{j_s} \cdot Be_{k,i}|)^2 \\
\leq & \sum_{s=ma_1+1}^{mb_1} 16|a_1|m\rho^{-2\alpha_2} \left(\sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} |\Upsilon^{ki}_\tau|^2 |\theta_{j_s} \cdot Be_{k,i}|^2 \right) \\
\leq & \; 16\rho^{-2\alpha_2}|a_1|m \sum_{\substack{k=1,2,...,a_1 \\ i=1,2,...,m}} |\Upsilon^{ki}_\tau|^2 \sum_{s=ma_1+1}^{mb_1} |\theta_{j_s} Be_{k,i}|^2
\end{aligned}$$

$$\leq \quad 16\rho^{-2\alpha_2}|a_1|m \sum_{\substack{k=1,\dots,a_1 \\ i=1,2,\dots,m}} |\Upsilon_\tau^{ki}|^2 |Be_{k,i}|^2 \leq 16\rho^{-2\alpha_2}|a_1|mM^2 \sum_{\substack{k=1,2,\dots,a_1 \\ i=1,2,\dots,m}} |\Upsilon_\tau^{ki}|^2$$

$$\leq \quad 16|a_1|m\rho^{-2\alpha_2}M^2 = O(\rho^{-\frac{3}{2}\alpha_2}).$$

Therefore one has

$$\sum_{s=1}^{ma_1} |\Upsilon_\tau \cdot \theta_{j_s}|^2 = 1 - O(\rho^{-\frac{3}{2}\alpha_2})$$

from which it follows that there exists an eigenvector $\theta_{j_s(\tau)}$ such that (66) holds. Dividing both sides of (65) by (66) we get the result

$$\eta_{j_s(\tau)} = \widetilde{\eta_\tau} + O(\rho^{-\frac{1}{2}\alpha_2}).$$

□

Theorem 6. *For every eigenvalue ς_s of the Sturm-Liouville operator $T(\gamma,\delta)$, there exists an eigenvalue $\widetilde{\varsigma_s}$ of the matrix $E(\gamma,\delta)$ such that*

$$\varsigma_s = \widetilde{\varsigma_s} + O(\rho^{-\frac{3}{4}\alpha_2}).$$

Proof. Decompose the infinite matrix $T(\gamma,\delta)$ as $T(\gamma,\delta) = \widetilde{A} + \widetilde{B}$ where the matrix $\widetilde{A}$ is defined by

$$\widetilde{A} = \begin{bmatrix} \left((l+v)^2|\delta|^2\right) I + V_0 & & & 0 \\ & \left((l-1+v)^2|\delta|^2\right) I + V_0 & & \\ & & \ddots & \\ 0 & & & \left((l-\frac{a_1}{2}+v)^2|\delta|^2\right) I + V_0 \\ & & & \end{bmatrix} \tag{68}$$

and $\widetilde{B} = T(\gamma,\delta) - \widetilde{A}$. Let ς_s be an eigenvalue of $T(\gamma,\delta)$, and $\Theta_s = (\Theta_s^1, \Theta_s^2, \Theta_s^3, ...)$, $\Theta_s^\tau = (\Theta_s^{\tau 1}, \dots, \Theta_s^{\tau m})$ be the corresponding normalized eigenvector, that is, $T\Theta_s = \varsigma_s \Theta_s$. $span\{e_i : (\tau-1)m+1 \leq i \leq \tau m\}$ is the eigenspace of the matrix $\tilde{A}$ which corresponds to the eigenvalue $|(\tau'+v)\delta|^2$, where $\tau' = l - \frac{\tau}{2}$ if τ is even, $\tau' = l + \frac{\tau-1}{2}$ if τ is odd, for $\tau = 1, 2, ...$ and $\{e_i\}$ is the standard basis for l_2.
One can easily verify that

$$\left(\varsigma_s - |(\tau'+v)\delta|^2\right)\Theta_s^\tau = \Theta_s \cdot \widetilde{B}e_{\tau,i}, \tag{69}$$

where $e_{\tau,i} \equiv e_i$, if $(m-1)\tau + 1 \leq i \leq m\tau$.

Using the orthogonal decomposition $\widetilde{B}e_{\tau,i} = \sum_{j=1}^{m}\sum_{k=1}^{\infty}(\widetilde{B}e_{\tau,i}\cdot e_{k,j})e_{k,j}$, (69) reduces to

$$\left(\varsigma_s - |(\tau'+v)\delta|^2 - |v_{ii0}|^2\right)\Theta_s^{\tau i} = \sum_{j=1}^{m}\sum_{k=1}^{\infty}(\widetilde{B}e_{\tau,i}\cdot e_{k,j})\Theta_s^{kj}$$

and since $\widetilde{B}e_{\tau,i}\cdot e_{k,j}=v_{ji(n_k-n_\tau)\delta}$ for $k\neq\tau$,

$$(\varsigma_s-(\tau'+v)\delta^2)\Theta_s^{\tau i}-\sum_{j=1}^{m}\sum_{k=1}^{a_1}v_{ji(n_k-n_\tau)\delta}\Theta_s^{kj}=\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}v_{ji(n_k-n_\tau)\delta}\Theta_s^{kj}. \quad (70)$$

Now take any eigenvalue ς_s of $T(\gamma,\delta)$, satisfying $\left|\varsigma_s-|(i'+v)\delta|^2\right|<sup|P(t)|$ for $s=1,2,...,\frac{ma_1}{2}$, where $i'=l-\frac{s}{2}$ if s is even, $i'=l+\frac{s-1}{2}$ if s is odd. The relations $\gamma\in V_\delta(\rho^{\alpha_1})$ $(\delta\neq e_i)$ and $\gamma=\beta+(l+v)\delta$, $\beta\cdot\delta=0$ imply

$$|2\gamma\cdot\delta+|\delta|^2|=|(l+v)|\delta|^2+|\delta|^2|<\rho^{\alpha_1},\qquad |l|<c_{19}\rho^{\alpha_1}.$$

Therefore, using the definition of i' and τ', we have

$$|(i'+v)\delta|<\frac{|a_1\delta|}{4}+c_{20}\rho^{\alpha_1}$$

for $s=1,2,...\frac{a_1}{2}$ and

$$|(\tau'+v)\delta|>\frac{|a_1\delta|}{2}-c_{21}\rho^{\alpha_1}$$

for $\tau>a_1$. Since $|a_1|>c_{22}\rho^{\frac{\alpha_2}{2}}$ and $\alpha_2>2\alpha_1$, we have

$$\left||(i'+v)\delta|^2-|(\tau'+v)\delta|^2\right|>c_{23}\rho^{\alpha_2} \quad (71)$$

for $s\leq\frac{a_1}{2}$, $\tau>a_1$, which implies

$$\left|\varsigma_s-|(\tau'+v)|\,\delta^2\right|=||\varsigma_s-|\,|(i'+v)\delta|^2\,|-|\,|(\tau'+v)|\,\delta^2|-|\,|(i'+v)\delta|^2\,||>c_{24}\rho^{\alpha_2}, \quad (72)$$

for $s=1,2,...\frac{a_1}{2}$, $\tau>a_1$.

Since $\widetilde{B}$ corresponds to the operator $P:Y\to P(t)Y$ in $L_2^m[0,2\pi]$, which has norm $sup|P(t)|\leq M$. Using this, equation (69) and (72), we have for the right hand side of (70) that

$$\begin{aligned}|\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}v_{ij(n_k-n_\tau)\delta}\Theta_s^{kj}|&\leqslant\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}|v_{ij(n_k-n_\tau)\delta}|\left|\frac{\Theta_s\cdot\widetilde{B}e_{kj}}{\varsigma_s-|(k'+v)\delta|^2}\right|\\ \leq\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}|v_{ij(n_k-n_\tau)\delta}|\frac{\|\Theta_s\|\|\widetilde{B}\|\|e_{kj}\|}{|\varsigma_s-|(k'+v)\delta|^2|}&\leq M\rho^{-\alpha_2}\sum_{j=1}^{m}\sum_{k=a_1+1}^{\infty}|v_{ij(n_k-n_\tau)\delta}|\\ &\leq c_{25}\rho^{-\alpha_2},\end{aligned} \quad (73)$$

Therefore writing the equation (70) for all $\tau=1,2,...,a_1$, and using (73) we get the following system

$$(E(\gamma,\delta)-\varsigma_sI)[\Theta_s^1,\Theta_s^2,...,\Theta_s^{a_1}]=[O(\rho^{-\alpha_2}),O(\rho^{-\alpha_2}),...,O(\rho^{-\alpha_2})], \quad (74)$$

where I is an $ma_1 \times ma_1$ identity matrix. Using $\Theta_s = \sum_{\tau=1}^{\infty} \Theta_s^{\tau} e_{\tau,i}$, the formula (69) and the inequality (72), we have

$$\sum_{\tau=a_1+1}^{\infty} |\Theta_s^{\tau}|^2 = \sum_{\tau=a_1+1}^{\infty} |\frac{\Theta_s \cdot \widetilde{B} e_{\tau,i}}{\varsigma_s - |(\tau' + v)\delta|^2}|^2 = O(\rho^{-2\alpha_2})$$

and thus

$$\sum_{\tau=1}^{a_1} |\Theta_s^{\tau}|^2 = 1 - O(\rho^{-2\alpha_2}). \tag{75}$$

Multiplying both sides of (74) by $(E(\gamma,\delta) - \varsigma_s I)^{-1}$,

$$[\Theta_s^1, \Theta_s^2, ..., \Theta_s^{a_1}] = (E(\gamma,\delta) - \varsigma_s I)^{-1} [O(\rho^{-\alpha_2}), ..., O(\rho^{-\alpha_2})],$$

then taking norm of both sides and using (75), we get

$$\sqrt{\frac{1 - O(\rho^{-2\alpha_2})}{m}} = \|(E(\gamma,\delta) - \varsigma_s I)^{-1}\| O(\sqrt{a_1}\rho^{-\alpha_2})$$

or

$$\min_{\tau} |\varsigma_s - \widetilde{\varsigma}_{\tau}| = \frac{O(\sqrt{a_1}\rho^{-\alpha_2}) \cdot \sqrt{m}}{\sqrt{1 - O(\rho^{-2\alpha_2})}} = O(\rho^{-\frac{3}{4}\alpha_2}),$$

where the minimum is taken over all eigenvalues $\widetilde{\varsigma}_{\tau}$ of the matrix $E(\gamma,\delta)$. Thus, the result follows. □

Theorem 7. *(Main result) For every $\beta \in H_\delta$, $|\beta| \sim \rho$ and for every eigenvalue $\varsigma_s(v(\beta))$ of the Sturm-Liouville operator $T(\gamma,\delta)$, there is an eigenvalue Λ_N of the operator $L(V)$ satisfying*

$$\Lambda_N = |\beta|^2 + \varsigma_s + O(\rho^{-\frac{1}{2}\alpha_2}).$$

Proof. From Theorem 6 and the definition of $E(\gamma,\delta)$, there exists an eigenvalue $\widetilde{\eta}_{\tau(s)}$ of the matrix $D(\gamma,\delta)$, where γ has a decomposition $\gamma = \beta + (\tau + v(\beta))\delta$, satisfying $\widetilde{\eta}_{\tau(s)} = |\beta|^2 + \varsigma_s + O(\rho^{-\frac{3}{4}\alpha_2})$. Therefore, the result follows from Theorem 5 and Theorem 2. □

References

[1] Atılgan, Ş., Karakılıç, S. and Veliev, O. A., Asymptotic Formulas for the Eigenvalues of the Schrödinger Operator, *Turk J Math.*, 26 (2002) 215–227.

[2] Berezin, F. A. and Shubin, M. A., The Schrödinger Equation, Kluwer Academic Publishers, Dordrecht, 1991.

[3] Coşkan, D. and Karakılıç, S., High energy asymptotics for eigenvalues of the Schrödinger operator with a matrix potential, *Mathematical Communications,* 16(2) (2011).

[4] Feldman, J., Knoerrer, H. and Trubowitz, E., The Perturbatively Stable Spectrum of the Periodic Schrödinger Operator, Invent. Math., 100 (1990) 259–300.

[5] Feldman, J., Knoerrer, H. and E. Trubowitz, The Perturbatively Unstable Spectrum of the Periodic Schrödinger Operator, *Comment. Math. Helvetica*, 66(1991) 557–579.

[6] Friedlanger, L., On the Spectrum for the Periodic Problem for the Schrödinger Operator, *Communications in Partial Differential Equations,* 15(1990) 1631–1647.

[7] Hald, O. H. and McLaughlin, J.R., Inverse Nodal Problems: Finding the Potential from Nodal Lines, *Memoirs of AMS*, 572, 119 (1996) 0075–9266.
[8] Karakılıç, S. and Akduman, S., Eigenvalue Asymptotics for the Schrödinger Operator with a Matrix Potential in a Single Resonance Domain, *Filomat,* 29(1) (2015) 21–38.
[9] Karakılıç, S., Atılgan, Ş. and Veliev, O. A., Asymptotic Formulas for the Eigenvalues of the Schrödinger Operator with Dirichlet and Neumann Boundary Conditions, *Reports on Mathematical Physics* (ROMP), 55(2) (2005) 221–239.
[10] Karakılıç, S., Veliev, O. A. and Atılgan, Ş., Asymptotic Formulas for the Resonance Eigenvalues of the Schrödinger Operator, *Turkish Journal of Mathematics*, 29(4) (2005) 323–347.
[11] Karpeshina, Y., Perturbation Theory for the Schrödinger Operator with a non-smooth Periodic Potential, *Math. USSR-Sb*, 71 (1992) 701–123.
[12] Karpeshina, Y., Perturbation series for the Schrödinger Operator with a Periodic Potential near Planes of Diffraction, *Communication in Analysis and Geometry,* 4(3) (1996) 339–413.
[13] Karpeshina, Y., On the Spectral Properties of Periodic Polyharmonic Matrix Operators, Indian Acad. Sci. (Math. Sci.), 112(1) (2002) 117–130.
[14] Kato, T., Perturbation Theory for Linear Operators, Springer Berlin, 1980.
[15] Reed, M. and Simon, B., Methods of Modern Mathematical Physics, 3rd ed., New York, San Francisco, London: Academic Press, vol. IV 1987.
[16] Naimark, M. A., Dawson, E. R. and Everitt, W. N., Linear Differential Operators, Part I, Elementary Theory of Linear Differential Operators, with additional material by the author, Frederick Ungar Publ., Co., New York, 196–6 1967.
[17] Veliev, O. A., On the spectrum of the Schrödinger operator with periodic potential, *Dokl. Akad. Nauk SSSR.*, Vol. 268, No. 6 (1983).
[18] Veliev, O. A., Asymptotic Formulas for the Eigenvalues of the Periodic Schrödinger Operator and the Bethe-Sommerfeld Conjecture, *Functsional Anal. i Prilozhen*, 21(2) (1987) 1–15.
[19] Veliev, O. A., The Spectrum of Multidimensional Periodic Operators, T*eor. Functional Anal. i Prilozhen,* 49 (1988) 17–34.
[20] Veliev, O. A., Asymptotic Formulas for the Bloch Eigenvalues Near Planes of Diffraction, *Reports on Mathematical Physics(ROMP)*, 58(3) (2006) 445–464.
[21] Veliev, O. A., Perturbation Theory for the Periodic Multidimensional Schrödinger Operator and the Bethe-Sommerfeld Conjecture, *International Journal of Contemporary Mathematical Sciences*, 2(2) (2007) 19–87.
[22] Veliev, O. A., Multidimensional periodic Schrödinger operator: Perturbation theory and applications.,Springer, Vol. 263 2015.

Current address: Sedef Karakılıç: Dokuz Eylül University, İzmir Turkey.
E-mail address: sedef.erim@deu.edu.tr
ORCID Address: http://orcid.org/0000-0002-0407-0271
Current address: Setenay Akduman: İzmir Demokrasi University, İzmir Turkey.
E-mail address: setenay.akduman@idu.edu.tr
ORCID Address: http://orcid.org/0000-0003-2492-3734
Current address: Didem Coşkan: Dokuz Eylül University, İzmir Turkey.
E-mail address: coskan.didem@gmail.com
ORCID Address: http://orcid.org/0000-0003-2358-198X