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MEASURE OF NONCOMPACTNESS FOR NONLINEAR HILFER FRACTIONAL DIFFERENTIAL EQUATION IN BANACH SPACES

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Abstract

This paper deals with nonlinear fractional differential equation with boundary value problem conditions. We investigate the existence of solutions in Banach spaces with Hilfer derivative. To obtain such result we apply Mönch's fixed point theorem and the technique of measures of noncompactness. At the end an example is given.

Keywords: Fractional differential equation; Hilfer fractional derivative; Kuratowski measures of noncompactness; Mönch fixed point theorems; Banach space.

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1 Introduction

In recent years, several papers have been devoted to the study of the existence of solutions for fractional differential equations, among others we refer the readers to the following references: Agarwal et al. [5, 4], Abbas et al. [3, 2], Sandeep et al. [32], Furati et al.[20] , Benchohra et al. [17, 18], Gu et al. [21]. Moreover, it has been proved that differential models involving derivatives of fractional order arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in many fields, for instance, about physics, control theory, rheology, chemistry, and so on (see the monograph of Kilbas and al. [25], Hilfer and al. [22, 23], and Samko and al. [30]).

In this paper we focus on the existence of solutions of the following boundary value problem for a nonlinear fractional differential equation,

$$D^{\alpha,\beta}_{a^+}y(t) = f(t, y(t)), t \in J := [0, T]. \tag{1.1}$$

with the fractional boundary conditions

$$\begin{aligned} &I^{1-\gamma}y(0) = y_0, \ \ I^{3-\gamma-2\beta}y'(0) = y_1, \\ &I^{1-\gamma}y(\eta) = \lambda(I^{1-\gamma}y(T)), \gamma = \alpha + \beta - \alpha\beta. \end{aligned} \tag{1.2}$$

where $D^{\alpha,\beta}_{0^+}$ is the Hilfer fractional derivative, $0 < \alpha < 1$,$0 \leq \beta \leq 1$, $0 < \lambda < 1$, $0 < \eta < T$ and let E be a Banach space space with norm $\|.\|$, $f : J \times E \times E \times E \times E \to E$ is given continuous function and satisfying some assumptions that will be specified later. We will use the technique of measures of noncompactness. which is often used in several branches of nonlinear analysis. Especially , that technique turns out to be a very useful tool in existence for several types of integral equations; details are found in Akhmerov et al. [7], Alvàrez [8], Banaŝ et al. [10, 11, 12, 13, 14, 15, 16], Benchohra et al. [17, 18], Mönch [27], Szufla [31].

The main idea used here is that on the Banach space E, we can not use Ascoli-Arzela theorem to prove the compactness of the operator, so we use the technique of measure of nocompactness to conclude.

Recently, considerable attention has been given to the existence of solutions of initial and boundary value problems for fractional differential equations with Hilfer fractional derivative [2, 3, 19], and other problems with Hilfer-Hadamard fractional derivative; see [1, 2, 33, 34]. Many existence results were established by the use of technics of nonlinear analysis such as Banach fixed point theorem, Schaefer's fixed point theorem, Lerayâ-Schauder nonlinear alternative, etc ..., and the technique of measures of noncompactness, see [4, 5, 6, 18, 15, 16].

In 2008, Benchohra et al. [17], considered the existence of solutions of an initial value problem for a nonlinear fractional differential equation

$$\begin{cases} D^r y(t) = f(t, y), & \text{for each } t \in J = [0, T], 1 < r < 2 \\ y(0) = y_0, y'(0) = y_1, & . \end{cases} \tag{1.3}$$

where D^r is the Caputo fractional derivative, $f : J \times E \to E$ is a given function, and E is a Banach space. They obtained results for solutions by using Mönch's fixed point theorem and the technique of measures of noncompactness.

In 2018, S. Abbas et al. [2], studied the existence of solutions for the following coupled system of Hilfer fractional differential equations

$$\begin{cases} D_0^{\alpha_1,\beta_1} u(t) = f_1(t, u(t), v(t)), & t \in J = [0, T] \\ D_0^{\alpha_2,\beta_2} v(t) = f_2(t, u(t), v(t)), \end{cases} \tag{1.4}$$

with the following initial conditions

$$\begin{cases} I_0^{1-\gamma_1} u(0) = \phi_1 \\ I_0^{1-\gamma_2} v(0) = \phi_2, \end{cases} \tag{1.5}$$

where $T > 0$, $\alpha_i \in (0, 1)$, $\beta_i \in [0, 1]$, $\gamma_i = \alpha_i + \beta_i - \alpha_i\beta_i$, $\phi_i \in E$, $f_i : I \times E \times E \to E$; $i = 1, 2$, are given functions, E is a real (or complex) Banach space with a norm $\|.\|$, $I_0^{1\gamma_i}$ is the left- sided mixed Riemann-Liouville integral of order $1 - \gamma_i$, and $D_0^{\alpha_i,\beta_i}$ is the generalized Riemann-Liouville derivative (Hilfer) operator of order α_i and type β_i: $i = 1, 2$. They obtained results for solutions by using the technique of measure of noncompactness and the fixed point theory.

In 2018, D.Vivek et al. [34], studied the existence, uniqueness and stability analysis of Hilfer-Hadamard type fractional neutral pantograph equations with boundary conditions of the form

$$\begin{cases} D_{1^+}^{\alpha,\beta} x(t) = f(t, x(t), x(\lambda t), D_{1^+}^{\alpha,\beta} x(\lambda t)), & t \in J = [0, T]. \\ I_{1^+}^{1-\gamma} x(1) = a, I_{1^+}^{1-\gamma} x(T) = b, & \gamma = \alpha + \beta - \alpha\beta. \end{cases} \tag{1.6}$$

where $D_{1^+}^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative, $0 < \alpha < 1$, $0 \le \beta \le 1$, $0 < \lambda < 1$. Let E be a Banach space, $f : J \times E \times E \times E \to E$ is a given continuous function. They obtained results for solutions by using Schaefer's fixed point theorem.

The principal goal of this paper is to prove the existence of solutions for the problem (1.1)-(1.2) using Mönch's fixed point theorem and its related Kuratowski measure of noncompactness.

2 Preliminaires

In what follows we introduce definitions, notations, and preliminary facts which are used in the sequel.

For more details, we refer to [4, 5, 7, 9, 11, 19, 20, 21, 22, 23, 24, 25, 26, 31, 32].
Denote by $C(J,E)$ the Banach space of continuous functions $y : J \to E$, with the usual supremum norm

$$\|y\|_\infty = \sup\{\|y(t)\|, t \in J\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_J y(t)dt.$$

$AC^1(J,E)$ denotes the space of functions $y : J \to E$, whose first derivative is absolutely continuous.

Definition 2.1. [20] Let $J = [0,T]$ be a finite interval and γ as a real such that $0 \leq \gamma < 1$. We introduce the weighted space $C_{1-\gamma}(J,E)$ of continuous functions f on $(0,T]$ as

$$C_{1-\gamma}(J,E) = \{f : (0,T] \to E : (t-a)^{1-\gamma} f(t) \in C(J,E)\}.$$

In the space $C_{1-\gamma}(J,E)$, we define the norm

$$\|f\|_{C_{1-\gamma}} = \|(t-a)^{1-\gamma} f(t)\|_C, C_0(J,E) = C(J,E).$$

Definition 2.2. [20] Let $0 < \alpha < 1, 0 \leq \beta \leq 1$, the weighted space $C^{\alpha,\beta}_{1-\gamma}(J,E)$ is defined by

$$C^{\alpha,\beta}_{1-\gamma}(J,E) = \{f : (0,T] \to \mathbb{R} : D^{\alpha,\beta}_{0^+} f \in C_{1-\gamma}(J,E)\}, \gamma = \alpha + \beta - \alpha\beta$$

and

$$C^1_{1-\gamma}(J,E) = \{f : (0,T] \to \mathbb{R} : f' \in C_{1-\gamma}(J,E)\}, \gamma = \alpha + \beta - \alpha\beta$$

with the norm

$$\|f\|_{C^1_{1-\gamma}} = \|f\|_C + \|f'\|_{C_{1-\gamma}}. \tag{2.1}$$

One have, see [20], $D^{\alpha,\beta}_{0^+} f = I^{\beta(1-\alpha)}_{0^+} D^{\gamma}_{0^+} f$ and $C^{\gamma}_{1-\gamma}(J,E) \subset C^{\alpha,\beta}_{1-\gamma}(J,E), \gamma = \alpha + \beta - \alpha\beta, 0 < \alpha < 1, 0 \leq \beta \leq 1$. Moreover, $C_{1-\gamma}(J,E)$ is complete metric space of all continuous functions mapping J into E with the metric d defined by

$$d(y_1,y_2) = \|y_1 - y_2\|_{C_{1-\gamma}(J,E)} := \max_{t \in J} |(t-a)^{1-\gamma}[y_1(t) - y_2(t)]|$$

for details see [20].

Notation 2.3. For a given set V of functions $v : J \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, t \in J,$$

and

$$V(J) = \{v(t) : v \in V, t \in J\}.$$

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Defininon 2.4. ([7, 11]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0,\infty]$ defined by

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } diam(B_i) \leq \epsilon\}; \text{ here } B \in \Omega_E.$$

This measure of noncompactness satisfies some important properties [7, 11]:
(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).
(b) $\mu(B) = \mu(\overline{B})$.
(c) $A \subset B \Rightarrow \mu(A) \leq \mu(B)$.
(d) $\mu(A + B) \leq \mu(A) + \mu(B)$
(e) $\mu(cB) = |c|\mu(B); c \in \mathbb{R}$.
(f) $\mu(convB) = \mu(B)$.

Now, we give some results and properties of fractional calculus. Definition 2.5. [26] Let $(0, T]$ and $f : (0, \infty) \to \mathbb{R}$ is a real valued continuous function. The Riemann-Liouville fractional integral of a function f of order $\alpha \in \mathbb{R}^+$ is denoted as $I^{\alpha}_{0+} f$ and defined by

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s)ds, t > 0. \tag{2.2}$$

where $\Gamma(\alpha)$ is the Euler's Gamma function.

Definion 2.6. [25] Let $(0, T]$ and $f : (0, \infty) \to \mathbb{R}$ is a real valued continuous function. The Riemann-Liouville fractional derivative of a function f of order $\alpha \in \mathbb{R}^+_0 = [0, +\infty)$ is denoted as $D^{\alpha}_{0+} f$ and defined by

$$D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t (t - s)^{n-\alpha-1} f(s)ds. \tag{2.3}$$

where $n = [\alpha] + 1$, and $[\alpha]$ means the integral part of α, provided the right hand side is pointwise defined on $(0, \infty)$.

Definion 2.7. [25] The Caputo fractional derivative of function f with order $\alpha > 0, n - 1 < \alpha < n, n \in \mathbb{N}$ is defined by

$$^C D^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - s)^{n-\alpha-1} f^{(n)}(s)ds, t > 0. \tag{2.4}$$

In [22], R. Hilfer studied applications of a generalized fractional operator having the Riemann-Liouville and Caputo derivatives as specific cases (see also [23, 24]).

Definion 2.8. [22] The Hilfer fractional derivative $D^{\alpha,\beta}_{0+}$ of order α $(n - 1 < \alpha < n)$ and type β $(0 \leq \beta \leq 1)$ is defined by

$$D^{\alpha,\beta}_{0+} = I^{\beta(n-\alpha)}_{0+} D^n I^{(1-\beta)(n-\alpha)}_{0+} f(t) \tag{2.5}$$

where I^{α}_{0+} and D^{α}_{0+} are Riemann-Liouville fractional integral and derivative defined by 2.2 and 2.3, respectively.

Remark 2.9. (See [19]) Hilfer fractional derivative interpolates between the R-L (2.3, if $\beta = 0$) and Caputo (2.4, if $\beta = 1$) fractional derivatives since

$$D^{\alpha,\beta}_{0+} = \begin{cases} DI^{1-\alpha} = D^{\alpha}_{0+}, \beta = 0, \quad I^{1-\alpha}D =^C D^{\alpha}_{0+}, \beta = 1, \\ I^{1-\alpha}D =^C D^{\alpha}_{0+}, \beta = 1, \end{cases}$$

Lemma 2.10. Let $0 < \alpha < 1, 0 \leq \beta \leq 1, \gamma = \alpha + \beta - \alpha\beta$, and $f \in L^1(J, E)$.
The operator $D^{\alpha,\beta}_{0+}$ can be written as

$$\begin{aligned} D^{\alpha,\beta}_{0+} f(t) &= \left(I^{\beta(1-\alpha)}_{0+} \frac{d}{dt} I^{(1-\gamma)}_{0+} f \right)(t) \\ &= I^{\beta(1-\alpha)}_{0+} D^{\gamma} f(t), \quad t \in J. \end{aligned}$$

Moreover, the parameter γ satisfies

$$0<\gamma\leq 1,\gamma\geq\alpha,\gamma>\beta,1-\gamma<1-\beta(1-\alpha).$$

Lemma 2.11. Let $0<\alpha<1, 0\leq\beta\leq 1, \gamma=\alpha+\beta-\alpha\beta$, If $D_{0^+}^{\beta(1-\alpha)}f$ exists and in $L^1(J,E)$, then

$$D_{0^+}^{\alpha,\beta}I_{0^+}^{\alpha}f(t)=I_{0^+}^{\beta(1-\alpha)}D_{0^+}^{\beta(1-\alpha)}f(t),\ \text{for a.e. } t\in J.$$

Furthermore, if $f\in C_{1-\gamma}(J,E)$ and $I_{0^+}^{1-\beta(1-\alpha)}f\in C_{1-\gamma}^{1}(J,E)$, then

$$D_{0^+}^{\alpha,\beta}I_{0^+}^{\alpha}f(t)=f(t),\ \text{for a.e. } t\in J.$$

Lemma 2.12. Let $0<\alpha<1, 0\leq\beta\leq 1, \gamma=\alpha+\beta-\alpha\beta$, and $f\in L^1(J,E)$. If $D_{0^+}^{\gamma}f$ exists and in $L^1(J,E)$, then

$$\begin{aligned}I_{0^+}^{\alpha}D_{0^+}^{\alpha,\beta}f(t)&=I_{0^+}^{\gamma}D_{0^+}^{\gamma}f(t)\\&=f(t)-\frac{I_{0^+}^{1-\gamma}f(0^+)}{\Gamma(\gamma)}t^{\gamma-1},\quad t\in J.\end{aligned}$$

Lemma 2.13. [25] For $t>a$, we have

$$\begin{aligned}I_{0^+}^{\alpha}(t-a)^{\beta-1}(t)&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta+\alpha-1}\\D_{0^+}^{\alpha}(t-a)^{\beta-1}(t)&=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1},\end{aligned}\tag{2.6}$$

Lemma 2.14. Let $\alpha>0, 0\leq\beta\leq 1$, so the homogeneous differential equation with Hilfer fractional order

$$D_{0^+}^{\alpha,\beta}h(t)=0\tag{2.7}$$

has a solution

$$h(t)=c_0t^{\gamma-1}+c_1t^{\gamma+2\beta-2}+c_2t^{\gamma+2(2\beta)-3}+...+c_nt^{\gamma+n(2\beta)-(n+1)}.$$

Definion 2.15. A map $f:J\times E\to E$ is said to be Caratheodory if
(i) $t\mapsto f(t,u)$ is measurable for each $u\in E$;
(ii) $u\mapsto F(t,u)$ is continuous for almost all $t\in J$.

The following theorems will play a major role in our analysis.

Theorem 2.16. ([5, 32]). Let D be a bounded, closed and convex subset of a Banach space such that $0\in D$, and let N be a continuous mapping of D into itself. If the implication
$V=\overline{conv}N(V)$ or $V=N(V)\cup 0\Rightarrow\mu(V)=0$
holds for every subset V of D, then N has a fixed point.

Lemma 2.17. ([32]). Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$, G a continuous function on $J\times J$ and f a function from $J\times E\longrightarrow E$ which satisfies the Caratheodory conditions, and suppose there exists $p\in L^1(J,\mathbb{R}^+)$ such that, for each $t\in J$ and each bounded set $B\subset E$, we have

$$\lim_{h\to 0^+}\mu(f(J_{t,h}\times B))\leq p(t)\mu(B);\ \text{here } J_{t,h}=[t-h,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t)f(s,y(s))ds: y\in V\right\}\right)\leq\int_J\|G(t,s)\|p(s)\mu(V(s))ds.$$

3 Main results

First of all, we define what we mean by a solution of the BVP (1.1)-(1.2).

Definition 3.1. A function $y \in C_{1-\gamma}(J,E)$ is said to be a solution of the problem (1.1)- (1.2) if y satisfies the equation $D^{\alpha,\beta}_{a^+}y(t) = f(t,y(t))$ on J , and the conditions $I^{1-\gamma}y(0) = y_0, I^{3-\gamma-2\beta}y'(0) = y_1$, and $I^{1-\gamma}y(\eta) = \lambda(I^{1-\gamma}y(T))$.

Lemma 3.2. Let $f : J \times E \times E \times E \times E \to E$ be a function such that $f \in C_{1-\gamma}(J,E)$ for any $y \in C_{1-\gamma}(J,E)$. A function $y \in C^{\gamma}_{1-\gamma}(J,E)$ is a solution of the integral equation

$$\begin{aligned} y(t) &= I^{\alpha}f(t,y(t)) + \frac{y_0}{\Gamma(\gamma)}t^{\gamma-1} + \frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2} + \zeta(\beta,\gamma,\eta,\lambda) \\ &\left[y_0(\lambda-1) + \frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1 + \lambda I^{\alpha-\gamma+1}f(T,y(T)) - I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]t^{\gamma+2(2\beta)-3} \end{aligned} \tag{3.1}$$

if and only if y is a solution of the Hilfer fractional BVP

$$D^{\alpha,\beta}_{a^+}y(t) = f(t,y(t)), t \in J := [0,T], \tag{3.2}$$

with the fractional boundary conditions

$$\begin{aligned} &I^{1-\gamma}y(0) = y_0, \ \ I^{3-\gamma-2\beta}y'(0) = y_1, \\ &I^{1-\gamma}y(\eta) = \lambda(I^{1-\gamma}y(T)), \gamma = \alpha+\beta-\alpha\beta. \end{aligned} \tag{3.3}$$

Proof. Assume y satisfies (3.1). Then Lemma 2.18 implies that

$$y(t) = c_0t^{\gamma-1} + c_1t^{\gamma+2\beta-2} + c_2t^{\gamma+2(2\beta)-3} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s,y(s))ds.$$

for some constants $c_0, c_1, c_2 \in \mathbb{R}$.
From (3.3), by Lemma 2.16 (2.6) , we have

- $I^{1-\gamma}y(0) = y_0$ implies that $c_0 = \frac{y_0}{\Gamma(\gamma)}$
- $I^{3-\gamma-2\beta}y'(0) = y_1$ implies that $c_1 = \frac{y_1}{\Gamma(\gamma+2\beta-1)}$
- $I^{1-\gamma}y(1) = \lambda(I^{1-\gamma}y(T))$ implies that

$$\begin{aligned} \left(I^{1-\gamma}y\right)(\eta) &= (I^{1-\gamma}\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1})(\eta) + (I^{1-\gamma}\frac{y_1}{\Gamma(\gamma)}t^{\gamma+2\beta-2})(\eta) + c_2\left(I^{1-\gamma}t^{\gamma+2(2\beta)-3}\right)(\eta) + I^{\alpha-\gamma+1}f(\eta,y(\eta)) \\ &= y_0 + \frac{y_1}{\Gamma(2\beta)}\eta^{2\beta-1} + c_2\frac{\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}\eta^{4\beta-2} + I^{\alpha-\gamma+1}f(\eta,y(\eta)) \\ \left(I^{1-\gamma}y\right)(T) &= (I^{1-\gamma}\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1})(T) + (I^{1-\gamma}\frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2})(T) + c_2\left(I^{1-\gamma}t^{\gamma+2(2\beta)-3}\right)(T) \\ &+ I^{\alpha-\gamma+1}f(T,y(T)) \\ &= y_0 + \frac{y_1}{\Gamma(2\beta)}T^{2\beta-1} + c_2\frac{\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}T^{4\beta-2} + I^{\alpha-\gamma+1}f(T,y(T)) \\ \lambda\left(I^{1-\gamma}y\right)(T) &= \lambda y_0 + \frac{\lambda y_1}{\Gamma(2\beta)}T^{2\beta-1} + c_2\frac{\lambda\Gamma(\gamma+2(2\beta)-2)}{\Gamma(4\beta-1)}T^{4\beta-2} + \lambda I^{\alpha-\gamma+1}f(T,y(T)) \end{aligned}$$

that is,

$$c_2 = \zeta(\beta,\gamma,\eta,\lambda)\left[y_0(\lambda-1) + \frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1 + \lambda I^{\alpha-\gamma+1}f(T,y(T)) - I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]$$

With

$$\zeta(\beta,\gamma,\eta,\lambda)=\frac{\Gamma(4\beta-1)}{\Gamma(\gamma+4\beta-2)(\eta^{4\beta-2}-\lambda T^{4\beta-2})}$$

The following hypotheses will be used in the sequel.
(H1) $f: J\times E\to E$ satisfies the Caratheodory conditions;
(H2) There exists $p\in L^1(J,\mathbb{R}^+)\cap C(J,\mathbb{R}^+)$, such that,

$$\|f(t,y)\|\le p(t)\|y\|,\ \text{for } t\in J \text{ and each } y\in E;$$

(H3) For each $t\in J$ and each bounded set $B\subset E$, we have

$$\lim_{h\to 0^+}\mu(f(J_{t,h}\times B))\le t^{1-\gamma}p(t)\mu(B);\ \text{here } J_{t,h}=[t-h,t]\cap J.$$

Theorem 3.3. Assume that conditions (H1)-(H3) hold. Let

$$p^*=\sup_{t\in J}p(t).$$

If

$$p^*\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left[|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right]T^{2(2\beta)-2}\right]<1 \tag{3.4}$$

then the BVP (1.1)-(1.2) has at least one solution.

Proof. We transform the problem (1.1)-(1.2) into a fixed point problem, then we consider the operator $N: C_{1-\gamma}(J,E)\to C_{1-\gamma}(J,E)$ defined by

$$N(y)(t)=I^\alpha f(t,y(t))+\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1}+\frac{y_1}{\Gamma(\gamma+2\beta-1)}t^{\gamma+2\beta-2}+\zeta(\beta,\gamma,\eta,\lambda)$$
$$\left[y_0(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+\lambda I^{\alpha-\gamma+1}f(T,y(T))-I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]t^{\gamma+2(2\beta)-3}$$

Clearly, the fixed points of the operator N are solutions of the problem (1.1)-(1.2). Let

$$R\ge\frac{\frac{y_0}{\Gamma(\gamma)}+\frac{y_1T^{2\beta-1}}{\Gamma(\gamma+2\beta-1)}+|\zeta(\beta,\gamma,\eta,\lambda)|\left(\|y_0\||\lambda-1|+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right)}{1-p^*\left(\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}-\frac{|\zeta(\beta,\gamma,\eta,\lambda)|T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})\right)} \tag{3.5}$$

and consider

$$D=\{y\in C_{1-\gamma}(J,E):\|y\|\le R\}.$$

The subset D is closed, bounded and convex. We shall show that the assumptions of Theorem 2.4 are satisfied. The proof will be given in three steps.

1-First we show that N is continuous:

Let y_n be a sequence such that $y_n\to y$ in $C_{1-\gamma}(J,E)$. Then for each $t\in J$,

$$\|t^{1-\gamma}(N(y_n)(t)-N(y)(t))\|\le\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\|f(s,y_n(s))-f(s,y(s))\|ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|t^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}$$
$$\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}\|f(s,y_n(s))-f(s,y(s))\|ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}\|f(s,y_n(s))-f(s,y(s))\|ds\right]$$
$$\le\left(\frac{t^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|t^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})\right)\|f(s,y_n(s))-f(s,y(s))\|$$
$$\le\left(\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})\right)\|f(s,y_n(s))-f(s,y(s))\|$$

Since f is of Caratheodory type, then by the Lebesgue dominated convergence theorem we have

$$\|N(y_n) - N(y)\|_\infty \to 0 \text{ as } n \to \infty.$$

2-Second we show that N maps D into itself:

Take $y \in D$, by (H2), we have, for each $t \in J$ and assume that $Ny(t) \neq 0$.

$$\begin{aligned}
\|t^{1-\gamma}N(y)(t)\| \leq & \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,y(s))\|ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}t^{2\beta-1} \\
& + |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]t^{4\beta-2} \\
& + \frac{|\zeta(\beta,\gamma,\eta,\lambda)|t^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}\|f(s,y(s))\|ds + \int_0^\eta (\eta-s)^{\alpha-\gamma}\|f(s,y(s))\|ds\right] \\
\leq & \frac{T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\|y\|ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
& + |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
& + \frac{T^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}|\zeta(\beta,\gamma,\eta,\lambda)|\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}p(s)\|y\|ds + \int_0^1 (1-s)^{\alpha-\gamma}p(s)\|y\|ds\right] \\
\leq & \frac{RT^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
& + |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
& + \frac{RT^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}|\zeta(\beta,\gamma,\eta,\lambda)|\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}p(s)ds + \int_0^\eta (\eta-s)^{\alpha-\gamma}p(s)ds\right] \\
\leq & \frac{Rp^*T^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
& + |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
& + \frac{Rp^*T^{4\beta-2}}{\Gamma(\alpha-\gamma+1)}|\zeta(\beta,\gamma,\eta,\lambda)|\left[|\lambda|\int_0^T (T-s)^{\alpha-\gamma}ds + \int_0^\eta (\eta-s)^{\alpha-\gamma}ds\right] \\
\leq & \frac{Rp^*T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{\|y_0\|}{\Gamma(\gamma)} + \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}T^{2\beta-1} \\
& + |\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1| + \frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right]T^{4\beta-2} \\
& + |\zeta(\beta,\gamma,\eta,\lambda)|\left[\frac{|\lambda|Rp^*T^{\alpha-\gamma+4\beta-1}}{\Gamma(\alpha-\gamma+2)} + \frac{Rp^*\eta^{\alpha-\gamma+1}T^{4\beta-2}}{\Gamma(\alpha-\gamma+2)}\right] \\
\leq & R.
\end{aligned}$$

3-Finally we show that $N(D)$ is bounded and equicontinuous:

By Step 2, it is obvious that $N(D) \subset C_{1-\gamma}(J,E)$ is bounded. For the equicontinuity of $N(D)$, let $t_1, t_2 \in J$, $t_1 < t_2$ and $y \in D$, so $t_2^{1-\gamma}Ny(t_2) - t_1^{1-\gamma}Ny(t_1) \neq 0$. Then

$$
\begin{aligned}
\|t_2^{1-\gamma}Ny(t_2)-t_1^{1-\gamma}Ny(t_1)\| \leq\ & \frac{1}{\Gamma(\gamma+2\beta-1)}\|y_1t_2^{2\beta-1}-y_1t_1^{2\beta-1}\|+|\zeta(\beta,\gamma,\eta,\lambda)| \\
& \left\|\left[y_0|\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+|\lambda|I^{\alpha-\gamma+1}f(T,y(T))-I^{\alpha-\gamma+1}f(\eta,y(\eta))\right]\right\| \\
& (t_2^{2(2\beta)-2}-t_1^{2(2\beta)-2}) \\
& +\left\|\frac{t_2^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y(s))ds-\frac{t_1^{1-\gamma}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y(s))ds\right\| \\
& \leq \frac{1}{\Gamma(\gamma+2\beta-1)}\|y_1\|(t_2^{2\beta-1}-t_1^{2\beta-1})+|\zeta(\beta,\gamma,\eta,\lambda)| \\
& \left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right](t_2^{2(2\beta)-2}-t_1^{2(2\beta)-2}) \\
& +\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}\|f(s,y(s))\|ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}\|f(s,y(s))\|ds\right] \\
& (t_2^{2(2\beta)-2}-t_1^{2(2\beta)-2}) \\
& +\frac{1}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}\|f(s,y(s))\|ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}\|f(s,y(s))\|ds\right. \\
& \left.+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\|f(s,y(s))\|ds\right] \\
& \leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1})+|\zeta(\beta,\gamma,\eta,\lambda)| \\
& \left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|+\frac{1}{\Gamma(\alpha-\gamma+1)}\right. \\
& \left.\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)\|y\|ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)\|y\|ds\right]\right](t_2^{2(2\beta)-2}-t_1^{2(2\beta)-2}) \\
& +\frac{1}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}p(s)\|y\|ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}p(s)\|y\|ds\right. \\
& \left.+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}p(s)\|y\|ds\right] \\
& \leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1})+|\zeta(\beta,\gamma,\eta,\lambda)| \\
& \left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|+\frac{R}{\Gamma(\alpha-\gamma+1)}\right. \\
& \left.\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)ds\right]\right](t_2^{2(2\beta)-2}-t_1^{2(2\beta)-2}) \\
& +\frac{R}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}p(s)ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}p(s)ds\right. \\
& \left.+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}p(s)ds\right] \\
& \leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1}) \\
& +|\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|\right. \\
& \left.+\frac{Rp^*}{\Gamma(\alpha-\gamma+1)}\left[|\lambda|\int_0^T(T-s)^{\alpha-\gamma}ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}ds\right]\right](t_2^{2(2\beta)-2}-t_1^{2(2\beta)-2}) \\
& +\frac{Rp^*}{\Gamma(\alpha)}\left[t_2^{1-\gamma}\int_0^{t_1}(t_2-s)^{\alpha-1}ds-t_1^{1-\gamma}\int_0^{t_1}(t_1-s)^{\alpha-1}ds+t_2^{1-\gamma}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right]
\end{aligned}
$$

$$\leq \frac{\|y_1\|}{\Gamma(\gamma+2\beta-1)}(t_2^{2\beta-1}-t_1^{2\beta-1})$$
$$+|\zeta(\beta,\gamma,\eta,\lambda)|\left[\|y_0\||\lambda-1|+\frac{|\lambda|T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}\|y_1\|+\frac{Rp^*(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1})}{\Gamma(\alpha-\gamma+2)}\right]$$
$$(t_2^{2(2\beta)-2)}-t_1^{2(2\beta)-2)})+\frac{Rp^*}{\Gamma(\alpha+1)}(t_2^{\alpha-\gamma+1}-t_1^{\alpha-\gamma+1}).$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero.
Hence $N(D) \subset D$.

Now we show that the implication holds:

Let $V \subset D$ such that $V = \overline{conv}(N(V) \cup \{0\})$.

We have $V(t) \subset \overline{conv}(N(V) \cup \{0\})$ for all $t \in J$. $NV(t) \subset ND(t)$, $t \in J$ is bounded and equicontinuous in E, the function $t \to v(t) = \mu(V(t))$ is continuous on J .
By assumption (H2), and the properties of the measure μ we have for each $t \in J$.

$$t^{1-\gamma}v(t) \leq \mu(t^{1-\gamma}N(V)(t)\cup\{0\})) \leq \mu(t^{1-\gamma}(NV)(t))$$
$$\leq \mu\left[t^{1-\gamma}\left[I^\alpha f(t,y(t))+\frac{y_0}{\Gamma(\gamma)}t^{\gamma-1}+\frac{y_1}{\Gamma(\gamma)}t^{\gamma+2\beta-2}+\zeta(\beta,\gamma,\eta,\lambda)\right.\right.$$
$$\left.\left.\left(y_0(\lambda-1)+\frac{\lambda T^{2\beta-1}-\eta^{2\beta-1}}{\Gamma(2\beta)}y_1+\lambda I^{\alpha-\gamma+1}f(T,y(T))-I^{\alpha-\gamma+1}f(\eta,y(\eta))\right)t^{\gamma+2(2\beta)-3}\right]\right]$$
$$\leq \mu\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}f(s,y(s))ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}f(s,y(s))ds\right)t^{2(2\beta)-2}\right]$$
$$\leq \left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\mu(f(s,y(s)))ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}\mu(f(s,y(s)))ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}\mu(f(s,y(s)))ds\right)t^{2(2\beta)-2}\right]$$
$$\leq \left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)\mu(V(s))ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)\mu(V(s))ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)\mu(V(s))ds\right)t^{2(2\beta)-2}\right]$$
$$\leq \left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)v(s)ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)v(s)ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)v(s)ds\right)t^{2(2\beta)-2}\right]$$
$$\leq \|v\|\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\right.$$
$$\left.\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}p(s)ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}p(s)ds\right)t^{2(2\beta)-2}\right]$$
$$\leq p^*\|v\|\left[\frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}ds+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+1)}\left(|\lambda|\int_0^T(T-s)^{\alpha-\gamma}ds+\int_0^\eta(\eta-s)^{\alpha-\gamma}ds\right)t^{2(2\beta)-2}\right]$$
$$\leq p^*\|v\|\left[\frac{t^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right)t^{2(2\beta)-2}\right]$$

This means that

$$\|v\| \leq p^*\|v\|\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left(|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right)T^{2(2\beta)-2}\right]$$

By $p^*\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left[|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right]T^{2(2\beta)-2}\right]<1$ it follows that $\|v\|=0$, that is $v(t)=0$ for each $t\in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D. Applying now Theorem 2.16, we conclude that N has a fixed point which is a solution of the problem (1.1)-(1.2).

4 Example

We consider the problem for Hilfer fractional differential equations of the form:

$$\begin{cases} D^{\alpha,\beta}y(t)=f(t,y(t)),(t,y)\in([0,1],\mathbb{R}),\\ \\ I^{1-\gamma}y(0)=y_0, I^{3-\gamma-2\beta}y'(0)=y_1, I^{1-\gamma}y(\eta)=\lambda(I^{1-\gamma}y(T)) \end{cases} \tag{4.1}$$

Here

$$\alpha=\frac{1}{2},\qquad \beta=\frac{1}{2},\qquad \gamma=\frac{3}{4},$$
$$\lambda=\frac{1}{2},\qquad \eta=\frac{1}{4},\qquad T=1.$$

With

$$f(t,yt))=\frac{1}{4}+\frac{ct^2}{e^{t+4}}|y(t)|,\quad t\in[0,1]$$

and

$$c=\frac{e^3}{10}\sqrt{\pi}$$

Clearly, the function f is continuous. For each $y\in E$ and $t\in[0,1]$, we have

$$\|f(t,y(t))\|\leq\frac{ct^2}{e^{t+4}}\|y\|$$

Hence, the hypothesis (H2) is satisfied with $p^*=ce^{-3}$. We shall show that condition 3.4 holds with $T=1$. Indeed,

$$p^*\left[\frac{T^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\frac{|\zeta(\beta,\gamma,\eta,\lambda)|}{\Gamma(\alpha-\gamma+2)}\left[|\lambda|T^{\alpha-\gamma+1}+\eta^{\alpha-\gamma+1}\right]T^{2(2\beta)-2}\right]<1$$

Simple computations show that all conditions of Theorem 3.1 are satisfied. It follows that the problem 4.1 has a solution defined on [0,T].

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