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LIE IDEALS AND JORDAN TRIPLE $(\alpha, \beta)-$DERIVATIONS IN RINGS

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Abstract. In this paper we prove that on a 2-torsion free semiprime ring R every Jordan triple (α, β)-derivation (resp. generalized Jordan triple (α, β)-derivation) on Lie ideal L is an (α, β)-derivation on L (resp. generalized (α, β)-derivation on L)

1. Introduction

Throughout the present paper R will denote an associative ring with center $Z(R)$. A ring R is n-torsion free, where $n > 1$ is an integer, in case $nx = 0$; $x \in R$, implies $x = 0$. For any $x, y \in R$, we denote the commutator $[x, y] = xy - yx$. Recall that R is prime if for $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$, and is semiprime if $aRa = \{0\}$ implies $a = 0$. An additive subgroup L of R is said to be a Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L is said to be square-closed if $a^2 \in L$ for all $a \in L$. Recall that a derivation of a ring R is an additive map $\delta : R \longrightarrow R$ such that $(xy)^\delta = (x)^\delta y + x(y)^\delta$ holds for all $x, y \in R$. On the other hand, $\delta : R \longrightarrow R$ an additive mapping is called a Jordan derivation if $\left(x^2\right)^\delta = (x)^\delta x + x(x)^\delta$ holds for all $x \in R$. A famous result due to Herstein [11, Theorem 3.3] shows that a Jordan derivation of a prime ring of characteristic not 2 must be a derivation. This result was extended to 2-torsion free semiprime rings by Cusack [10] and subsequently, by Bresar [7]. Following [6]. an additive mapping $\delta : R \to R$ is called a Jordan triple derivation if $(xyx)^\delta = (x)^\delta yx + x(y)^\delta x + xy(x)^\delta$ holds for all $x, y \in R$. One can easily prove that any Jordan derivation on an 2-torsion free ring is a Jordan triple derivation (see [11, Lemma 3.5]). Bresar has proved the following result.

Theorem 1.1. *([6, Theorem 4.3]) Let R be a 2-torsion free semiprime ring and $\delta : R \to R$ be a Jordan triple derivation. In this case δ is a derivation.*

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To understand our results it is better to review some generalizations of the notion of derivation. An additive mapping $F : R \to R$ is said to be generalized derivation (resp. a generalized Jordan derivation) on R if there exists a derivation $\delta : R \to R$ such that $(xy)^F = (x)^F y + x(y)^\delta$ (resp. $(x^2)^F = (x)^F x + x(x)^\delta$) holds for all $x, y \in R$. An additive mapping $F : R \to R$ is said to be generalized Jordan triple derivation on R if there exists a Jordan triple derivation $\delta : R \to R$ such that $(xyx)^F = (x)^F yx + x(y)^\delta x + xy(x)^\delta$ holds for all $x, y \in R$. In 2003, Jing and Lu [14, Theorem 3.5] proved that every generalized Jordan triple derivation on a 2-torsion free prime rings R is a generalized derivation. Recently, Vukman [20] extended Jing and Lu result for 2-torsion free semiprime rings.

If $\delta : R \longrightarrow R$ is a additive and if α and β are endomorphisms of R, then δ is said to be an (α,β)-derivation of R when for all $x, y \in R$, $(xy)^\delta = (x)^\delta \alpha(y) + \beta(x)(y)^\delta$. Note that for I, the identity map on R, an (I, I)-derivation is just a derivation. An example of (α,β)-derivation when R has a nontrivial central idempotent e is to let $\delta(x) = ex$, $\alpha(x) = (1 - e)x$, and $\beta = I$ (or δ) (formally). Here, δ is not a derivation because $(ee)^\delta = eee \neq 2eee = (ee)e + e(ee) = (e)^\delta e + e(e)^\delta$. In any ring with endomorphism α, if we let $d = I - \alpha$, then d is an (α, I)-derivation, but not a derivation when R is semiprime, unless $\alpha = I$. An additive mapping $\delta : R \to R$ is called Jordan triple (α,β)-derivation if $(xyx)^\delta = (x)^\delta \alpha(yx) + \beta(x)(y)^\delta \alpha(x) + \alpha(xy)(x)^\delta$ for all $x, y \in R$. Obviously, every (α,β)-derivation on a 2-torsion free ring is a Jordan triple (α,β)-derivation, but converse need not be true in general. In 2007, Liu and Shiue [15, Theorem 2] show that the converse is true for 2-torsion free semiprime rings R and probed the following result:

Theorem 1.2. *Let R be a 2-torsion free semiprime rings and let α, β be automorphisms of R. If $\delta : R \to R$ is a Jordan triple (α,β)-derivation, then δ is an (α,β)-derivation.*

An additive map $F : R \longrightarrow R$ is called a generalized (α,β)-derivation, for α and β endomorphisms of R, if there exists an (α,β)-derivation $\delta : R \longrightarrow R$ such that $(xy)^F = (x)^F \alpha(y) + \beta(x)(y)^\delta$ holds for all $x, y \in R$. Clearly, this notion include those of (α,β)-derivation when $F = \delta$, of derivation when $F = \delta$ and $\alpha = \beta = I$, and of generalized derivation, which is the case when $\alpha = \beta = I$. Maps of the form $(x)^F = ax + xb$ for $a, b \in R$ with $(x)^\delta = xb - bx$ and $\alpha = \beta = I$ are generalized derivations, and more generally, maps $(x)^\delta = a\alpha(x) + \beta(x)b$ are generalized (α,β)-derivation. To see this observe that $(xy)^F = a\alpha(x)\alpha(y) + \beta(x)\beta(y)b = (a\alpha(x) + \beta(x)b)\alpha(x) + \beta(x)(\beta(y)b - b\alpha(y))$, and as we have just seen above, $(x)^\delta = b\alpha(x) - \beta(x)b$ is an (α,β)-derivation of R. As for derivation, a generalized Jordan (α,β)-derivation F assumes $x = y$ in the definition above; that is, we assume only that $(x^2)^F = (x)^F \alpha(x) + \beta(x)(x)^\delta$, holds for all $x \in$. An additive map $F : R \longrightarrow R$ is called generalized Jordan triple (α,β)-derivation, for α and β endomorphisms of R, if there exists a Jordan triple (α,β)-derivation $\delta : R \longrightarrow R$ such that $(xyx)^F = (x)^F \alpha(yx) + \beta(x)(y)^\delta \alpha(x) + \beta(xy)(x)^\delta$, holds for all $x, y \in R$.

Clearly, this notion includes those of triple (α,β)-derivation when $F=\delta$, of triple derivation when $F=\delta$ and $\alpha=\beta=I$, and of generalized triple derivation which is the case $\alpha=\beta=I$. In 2007, Liu and Shiue [15, Theorem 3] proved the following generalization of all above results:

Theorem 1.3. *Let R be a 2-torsion free semiprime rings and α,β be automorphisms of R. If $F:R\to R$ is a generalized Jordan triple (α,β)-derivation, then F is a generalized (α,β)-derivation.*

The present paper is motivated by the previous results and we here continue this line of investigation to generalize Theorem 1.2 and Theorem 1.3 on Lie ideal of R.

2. Jordan Triple Derivations

It is obvious to see that every derivation is a Jordan triple derivation, but the converse need not to be true in general. In [6], Bresar proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation. Motivated by the result due to Bresar, in the present section it is shown that on a 2-torsion free semiprime ring R every Jordan triple $(\alpha,\beta)-$derivation on Lie ideal L is an $(\alpha,\beta)-$derivation on L. More precisely, we prove the following:

Theorem 2.1. *Let R be a 2-torsion free semiprime ring, α,β be automorphisms of R and $L\nsubseteq Z(R)$ be a nonzero square-closed Lie ideal of R. If $\delta:R\longrightarrow L$ satisfying*

$$(aba)^{\delta}=a^{\delta}\alpha(ba)+\beta(a)b^{\delta}\alpha(a)+\beta(ab)a^{\delta} \text{ for all } a,b\in L$$

and $a^{\delta},\beta(a)\in L$, then δ is a $(\alpha,\beta)-$derivation on L.

Corollary 2.1. *Let R be a 2-torsion free semiprime ring, α,β be automorphisms of R and $L\nsubseteq Z(R)$ be a nonzero square-closed Lie ideal of R. If $\delta:R\longrightarrow L$ satisfying*

$$(a^2)^{\delta}=a^{\delta}\alpha(a)+\beta(a)a^{\delta} \text{ for all } a\in L$$

and $a^{\delta},\beta(a)\in L$, then δ is a $(\alpha,\beta)-$derivation on L.

To facilitate our discussion, we shall begin with the following lemmas:

Lemma 2.1 ([4], Lemma 4)**.** *If $L\nsubseteq Z(R)$ is a Lie ideal of a 2-torsion free prime ring R and $a,b\in R$ such that $aLb=\{0\}$, then $a=0$ or $b=0$.*

Lemma 2.2 ([19], Lemma 2.4)**.** *Let R be a 2-torsion free semiprime ring, L be a Lie ideal of R and $a\in L$ such that $L\nsubseteq Z(R)$. If $aLa=0$, then $a^2=0$ and there exists a nonzero ideal $K=R[L,L]R$ of R generated by $[L,L]$ such that $[K,R]\subseteq L$ and $Ka=aK=0$.*

Corollary 2.2 ([12], Corollary 2.1)**.** *Let R be a 2-torsion free semiprime ring, L a Lie ideal of R such that $L\nsubseteq Z(R)$ and let $a,b\in L$.*

(1) *if* $aLa = 0$, *then* $a = 0$.
(2) *If* $aL = 0$(*or* $La = 0$), *then* $a = 0$
(3) *If* L *is square-closed and* $aLb = 0$, *then* $ab = 0$ *and* $ba = 0$.

Lemma 2.3. *Let* R *be a* 2*-torsion free semiprime ring,* L *be a noncentral Lie ideal of* R, β *be a homomorphisms of* R *and* $a, b \in L$. *If* $aub + \beta(bu)a = 0$, *for all* $u \in L$ *then* $aub = 0$.

Proof. If

$$aub + \beta(bu)a = 0, \text{ for all } u \in L. \tag{2.1}$$

Then replacing u by ubv in (2.1), we get

$$a(ubv)b + \beta(bu)\beta(bv)a = 0. \tag{2.2}$$

Now application of (2.1), yields that

$$-\beta(bu)avb + \beta(bu)\beta(bv)a = 0. \tag{2.3}$$

Again, by (2.1), we obtain $-\beta(bu)avb - \beta(bu)avb = 0$ that is $\beta(bu)avb = 0$. Again by (2.1) $aubvb = 0$. Hence $aubLb = 0$, so $aub = 0$ for all $u \in L$.

Lemma 2.4 ([19], Lemma 2.7)**.** *Let* $G_1, G_2, \cdots, G_n$ *be additive groups and* R *be a* 2*-torsion free semiprime ring and* $L \not\subseteq Z(R)$ *is a Lie ideal of* R. *Suppose that mappings* $S : G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ *and* $T : G_1 \times G_2 \times \cdots \times G_n \longrightarrow R$ *are additive in each argument. If* $S(a_1, a_2, \cdots, a_n)xT(a_1, a_2, \cdots, a_n) = 0$ *for all* $x \in L$, $a_i \in G_i$ $i = 1, 2, \cdots n$, *then* $S(a_1, a_2, \cdots, a_n)xT(b_1, b_2, \cdots, b_n) = 0$ *for all* $x \in L$, $a_i, b_i \in G_i$ $i = 1, 2, \cdots n$.

Lemma 2.5. *Let* R *be a ring,* L *be a Lie ideal of* R *and* $\delta : R \to R$ *be a Jordan triple* $(1, \beta)-$*derivation. For arbitrary* $a, b, c \in L$, *we have*

$$(abc + cba)^{\delta} = a^{\delta}(bc) + \beta(a)b^{\delta}(c) + \beta(ab)c^{\delta} + c^{\delta}(ba) + \beta(c)b^{\delta}(a) + \beta(cb)a^{\delta}.$$

Proof. We have

$$(aba)^{\delta} = a^{\delta}(ba) + \beta(a)b^{\delta}(a) + \beta(ab)a^{\delta}, \quad \text{for all } a, b \in L. \tag{2.4}$$

We compute, $W = ((a + c)b(a + c))^{\delta}$ in two different ways. On one hand, we find that $W = (a + c)^{\delta}b(a + c) + \beta(a + c)b^{\delta}(a + c) + \beta((a + c)b)(a + c)^{\delta}$, and on the other hand $W = (aba)^{\delta} + (abc + cba)^{\delta} + (cbc)^{\delta}$. Comparing two expressions we obtain the required result.

Remark 2.1. It is easy to see that every Jordan $(1, \beta)-$derivation of a 2-torsion free ring satisfies (2.4) (see [1] for reference).

For the purpose of this section we shall write; $\Delta(a, b, c) = (abc)^{\delta} - a^{\delta}(bc) - \beta(a)b^{\delta}(c) - \beta(ab)c^{\delta}$, and $\Lambda(a, b, c) = abc - cba$. We list a few elementary properties of δ and Λ:

(i) $\Delta(a, b, c) + \Delta(c, b, a) = 0$

(*ii*) $\Delta((a+b),c,d) = \Delta(a,c,d) + \Delta(b,c,d)$ and $\Lambda((a+b),c,d) = \Lambda(a,c,d) + \Lambda(b,c,d)$

(*iii*) $\Delta(a,(b+c),d) = \Delta(a,b,d) + \Delta(a,c,d)$ and $\Lambda(a,(b+c),d) = \Lambda(a,b,d) + \Lambda(a,c,d)$

(*iv*) $\Delta(a,b,(c+d)) = \Delta(a,b,c) + \Delta(a,b,d)$ and $\Lambda(a,b,(c+d)) = \Lambda(a,b,c) + \Lambda(a,b,d)$.

Proposition 2.1. *Let R be a semiprime ring and $L \not\subseteq Z(R)$ be a square-closed Lie ideal of R. If $\Delta(a,b,c) = 0$ holds for all $a,b,c \in L$, then δ is an $(1,\beta)-$derivation of L.*

Proof. We have $\Delta(a,b,c) = 0$ for all $a,b,c \in L$, that is,

$$(abc)^\delta = a^\delta(bc) + \beta(a)b^\delta(c) + \beta(ab)c^\delta.$$

Let $M = abxab$. We have

$$M^\delta = (a(bxa)b)^\delta = a^\delta(bxab) + \beta(a)b^\delta(xab) + \beta(ab)x^\delta(ab) + \beta(abx)a^\delta(b) + \beta(abxa)b^\delta \text{ for all } x,a,b \in L. \quad (2.5)$$

On the other hand,

$$M^\delta = ((ab)x(ab))^\delta = (ab)^\delta(xab) + \beta(ab)x^\delta(ab) + \beta(abx)(ab)^\delta. \quad (2.6)$$

Comparing (2.5) with (2.6) we get

$$\{(ab)^\delta - a^\delta(b) - \beta(a)b^\delta\}(xab) + \beta(abx)\{(ab)^\delta - a^\delta(b) - \beta(a)b^\delta\} = 0$$

that is, $a^b(xab) + \beta(abx)a^b = 0$, where a^b stands for $(ab)^\delta - a^\delta(b) - \beta(a)b^\delta$. Thus by Lemma 2.3 we find that $a^b(xab) = 0$, for all $a,b,x \in L$. Now by Lemma 2.4, we get $a^b(xcd) = 0$, for all $a,b,c,d,x \in L$. Hence, by using Corollary 2.2, we obtain $a^b = 0$ for all $a,b \in L$ that is δ is a $(1,\beta)-$derivation on L.

Lemma 2.6. *Let R be a ring and L be a Lie ideal of R. For any $a,b,c,x \in L$, we have*

$$\Delta(a,b,c)x\Lambda(a,b,c) + \beta(\Lambda(a,b,c))\beta(x)\Delta(a,b,c) = 0.$$

Proof. For any $a,b,c,x \in L$, suppose that $N = abcxcba + cbaxabc$. Now we find

$$\begin{aligned} N^\delta &= (a(bcxcb)a + c(baxab)c)^\delta = (a(bcxcb)a)^\delta + (c(baxab)c)^\delta \\ &= a^\delta(bcxcba) + \beta(a)b^\delta(cxcba) + \beta(ab)c^\delta(xcba) \\ &\quad + \beta(abc)x^\delta(cba) + \beta(abcx)c^\delta(ba) + \beta(abcxc)b^\delta(a) \\ &\quad + \beta(abcxcb)a^\delta + c^\delta(baxabc) + \beta(c)b^\delta(axabc) \\ &\quad + \beta(cb)a^\delta(xabc) + \beta(cba)x^\delta(abc) + \beta(cbax)a^\delta(bc) \\ &\quad + \beta(cbaxa)b^\delta(c) + \beta(cbaxab)c^\delta. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} N^\delta &= ((abc)x(cba) + (cba)x(abc))^\delta \\ &= (abc)^\delta(xcba) + \beta(abc)x^\delta(cba) + \beta(abcx)(cba)^\delta \\ &\quad + (cba)^\delta(xabc) + \beta(cba)x^\delta(abc) + \beta(cbax)(abc)^\delta. \end{aligned}$$

On comparing last two expressions we get

$$-\Delta(c,b,a)(xcba)+\Delta(c,b,a)(xabc)+\beta(abcx)\Delta(c,b,a)-\beta(cbax)\Delta(c,b,a)=0.$$

This implies that $\Delta(a,b,c)x\Lambda(a,b,c)+\beta(\Lambda(a,b,c))\beta(x)\Delta(a,b,c)=0$ for all $a,b,c\in L$.

Lemma 2.7. *Let R be a semiprime ring and $L\nsubseteq Z(R)$ be a square-closed Lie ideal of R. Then $\Delta(a,b,c)x\Lambda(r,s,t)=0$ holds for all $a,b,c,r,s,t,x\in L$.*

Proof. By Lemma 2.6, we have $\Delta(a,b,c)x\Lambda(a,b,c)+\beta(\Lambda(a,b,c))\beta(x)\Delta(a,b,c)=0$ for all $a,b,c\in L$. Thus we get $\Delta(a,b,c)x\Lambda(a,b,c)=0$ by Lemma 2.3. Now by Lemma 2.4 we find that$\Delta(a,b,c)x\Lambda(r,s,t)=0$, for all $a,b,c,r,s,t\in L$.

For an arbitrary ring R, we set $S=\{a\in C(L)\mid aL\subseteq C(L)\}$, where $C(L)$ is center of L.

Lemma 2.8. *Let R be a semiprime ring, L be a square-closed Lie ideal of R and $a\in L$. If $axy=yxa$ holds for all $x,y\in L$, then $a\in S$.*

Proof: Let $x,y,z,w\in L$. We get

$$a(wz)yx=yx(wz)a=ya(wz)x=y(awz)x=yzwax=(yzwa)x=awyzx.$$

This implies that

$$aw(zy-yz)x=0,\ \text{for all } x,y,z,w\in L.$$

That is,

$$aw\,[z,y]\,Law\,[z,y]=0,\ \text{for all } y,z,w\in L.$$

By Corollary 2.2, we have

$$aw\,[z,y]=0,\ \text{for all } y,z,w\in L.$$

Replacing z by a in this equation, we get

$$aw\,[a,y]=0,\ \text{for all } y,w\in L.$$

Hence $ayw\,[a,y]=0=yaw\,[a,y]$ for all $y,w\in L$, and so $[a,y]\ L\,[a,y]=0$, for all $y\in L$. By Corollary 2.2, we have $[a,y]=0$, for all $y\in L$. Therefore, $axy=yxa=yax$ for all $x,y\in L$. That is $aL\subseteq C(L)$. Thus, $a\in S$.

Lemma 2.9. *Let R be a semiprime ring, L be a square-closed Lie ideal of R, $a\in C(L)$, $c\in L$, β be a homomorphisms of R and $\beta(L)\subseteq L$. If $(\beta(ab)-ab)c=0$ holds for all $b\in L$, then $a(\beta(b)-b)c=0$.*

Proof: Replacing b by bx, $x \in L$ in the hypothesis and using $a \in C(L)$, we have

$$\begin{aligned} 0 &= (\beta(abx) - abx)c = \beta(ab)\beta(x)c - abxc \\ &= \beta(ba)\beta(x)c - abxc = \beta(b)\beta(ax)c - abxc \\ &= \beta(b)axc - abxc = a\beta(b)xc - abxc \\ &= a(\beta(b) - b)xc. \end{aligned}$$

That is,

$$a(\beta(b) - b)xc = 0, \text{for all } b, x \in L.$$

Using $\beta(L) \subseteq L$ and replacing x by $cxa(\beta(b) - b)$, we obtain that

$$a(\beta(b) - b)cxa(\beta(b) - b)c = 0, \text{for all } b, x \in L.$$

This implies that

$$a(\beta(b) - b)cLa(\beta(b) - b)c = 0, \text{for all } b \in L.$$

By Corollary 2.2, we have

$$a(\beta(b) - b)c = 0, \text{for all } b \in L.$$

Lemma 2.10. *Let R be a 2-torsion free semiprime ring and L be a square-closed Lie ideal of R. If $\Lambda(a, b, c) = 0$ for all $a, b, c \in L$, then $L \subseteq Z(R)$.*

Proof. Assume that $L \nsubseteq Z(R)$. We have $\Lambda(a, b, c) = 0$ for all $a, b, c \in L$ that is, $abc = cba$. Replacing b by $2tb$, we get $2atbc = 2ctba$ for all $a, b, c, t \in L$. Again replacing t by $2tw$ and using the fact that R is 2-torsion free to get, $atwbc = ctwba$ and hence $a(tw)bc = bc(tw)a = ba(tw)c = awtbc$. Thus we find that $a[t, w]bc = 0$ for all $a, b, c, t, w \in L$. By Corollary 2.2, we get $[t, w] = 0$ for all $t, w \in L$, that is L is a commutative Lie ideal of R. And so, we have $[a, [a, t]] = 0$ for all $t \in R$ and hence by Sublemma on page 5 of [11], $a \in Z(R)$. Hence $L \subseteq Z(R)$, a contradiction. This completes the proof of the theorem.

Proof of Theorem 2.1. Since $\alpha^{-1}\delta$ is a Jordan triple $(1, \alpha^{-1}\beta)$-derivation, replacing δ by $\alpha^{-1}\delta$ we may assume that δ is a Jordan triple $(1, \beta)$-derivation. Then, our goal will be to show that δ is a $(1, \beta)-$derivation of associative triple systems. We have

$$\begin{aligned} \Lambda(\Delta(a, b, c), r, s)x\Lambda(\Delta(a, b, c), r, s) &= (\Delta(a, b, c)rs - sr\Delta(a, b, c))x\Lambda(\Delta(a, b, c), r, s) \\ &= \Delta(a, b, c)rsx\Lambda(\Delta(a, b, c), r, s) \\ &\quad -sr\Delta(a, b, c)x\Lambda(\Delta(a, b, c), r, s). \end{aligned}$$

By Lemma 2.7, the above relation reduces to

$$\Lambda(\Delta(a, b, c), r, s)L\Lambda(\Delta(a, b, c), r, s) = 0, \text{ for all } a, b, c, r, s \in L.$$

By Corollary 2.2, we have

$$\Lambda(\Delta(a, b, c), r, s) = 0, \text{ for all } a, b, c, r, s \in L.$$

We obtain that

$$\Delta(a,b,c)rs - sr\Delta(a,b,c) = 0, \text{ for all } a,b,c,r,s \in L.$$

Using $\Delta(a,b,c), r, s \in L$ and Lemma 2.8, we have $\Delta(a,b,c) \in S$. This implies that

$$rs\Delta(a,b,c) - sr\Delta(a,b,c) = 0, \text{ for all } a,b,c,r,s \in L.$$

That is,

$$[r,s]\Delta(a,b,c) = 0, \text{ for all } a,b,c,r,s \in L. \tag{2.7}$$

Similarly, we have

$$\Delta(a,b,c)[r,s] = 0, \text{ for all } a,b,c,r,s \in L. \tag{2.8}$$

Let $a \in S$ and $b, c \in L$. Thus, $a, ab, ac, abc \in C(L)$ and $abc = cba$. Consider $N = abcxcba$. We have

$$\begin{array}{rcl} N^\delta & = & (a(bcxcb)a)^\delta \\ & = & a^\delta(bcxcba) + \beta(a)b^\delta(cxcba) + \beta(ab)c^\delta(xcba) \\ & & +\beta(abc)x^\delta(cba) + \beta(abcx)c^\delta(ba) + \beta(abcxc)b^\delta(a) \\ & & +\beta(abcxcb)a^\delta. \end{array}$$

On the other hand, we have

$$\begin{array}{rcl} N^\delta & = & ((abc)x(cba))^\delta = ((abc)x(abc))^\delta \\ & = & (abc)^\delta(xabc) + \beta(abc)x^\delta(abc) + \beta(abcx)(abc)^\delta \end{array}$$

Comparing the last two equations and using $abc = cba$, we have

$$\Delta(a,b,c)xabc + \beta(abc)\beta(x)\Delta(c,b,a) = 0.$$

Using $\Delta(a,b,c) = -\Delta(c,b,a)$, we have

$$\Delta(a,b,c)xabc - \beta(abc)\beta(x)\Delta(a,b,c) = 0.$$

Since $abc \in C(L)$, we find that

$$-\Delta(a,b,c)abcx + \beta(abc)\beta(x)\Delta(a,b,c) = 0.$$

Using $abcx \in C(L)$, we have

$$-(abc)x\Delta(a,b,c) + \beta(abc)\beta(x)\Delta(a,b,c) = 0.$$

This implies that

$$(\beta(abc)\beta(x) - (abc)x)\Delta(a,b,c) = 0.$$

By Lemma 2.9, we have

$$(abc)(\beta(x) - x)\Delta(a,b,c) = 0, \text{ for all } a,b,c,x \in L.$$

Multiplying y form the right hand side, using $abc \in C(L)$ and $\Delta(a,b,c) \in S$, we have

$$(\beta(x) - x)(abc)y\Delta(a,b,c) = 0, \text{for all } a,b,c,x,y \in L.$$

By Lemma 2.4, we have

$$(\beta(x)-x)(srt)y\Delta(a,b,c)=0,\ \text{for all } a,s\in S \text{ and } x,r,t,b,c,y\in L.$$

Using $\Delta(a,b,c)\in S$, we have

$$(\beta(x)-x)\Delta(a,b,c)^2L(\beta(x)-x)\Delta(a,b,c)^2=0,\ \text{for all } a\in S \text{ and } x,b,c\in L.$$

By Corollary 2.2 and using $abc=cba$, for all $b,c\in L$, we have

$$(\beta(x)-x)\Delta(a,b,c)^2=0,\ \text{for all } a\in S \text{ and } x,b,c\in L.$$

Using $\Delta(a,b,c)\in S$, we get

$$\Delta(a,b,c)^2(\beta(x)-x)=0,\ \text{for all } a\in S \text{ and } x,b,c\in L. \tag{2.9}$$

Using equations (2.8) and (2.9), we have

$$\begin{aligned}
2\Delta(a,b,c)^3 &= \Delta(a,b,c)^2\Delta(a,b,c)+\Delta(a,b,c)^2\Delta(a,b,c)\\
&= \Delta(a,b,c)^2\Delta(a,b,c)-\Delta(a,b,c)^2\Delta(c,b,a)\\
&= \Delta(a,b,c)^2(\Delta(a,b,c)-\Delta(c,b,a))\\
&= \Delta(a,b,c)^2((abc)^\delta-a^\delta(bc)-\beta(a)b^\delta c-\beta(ab)c^\delta\\
&\quad -(cba)^\delta+c^\delta(ba)+\beta(c)b^\delta(a)+\beta(cb)a^\delta)\\
&= \Delta(a,b,c)^2(-a^\delta(bc)-\beta(a)b^\delta c-\beta(ab)c^\delta+c^\delta(ba)\\
&\quad +\beta(c)b^\delta(a)+\beta(cb)a^\delta)\\
&= \Delta(a,b,c)^2(-a^\delta(bc)-\beta(a)b^\delta c-\beta(ab)c^\delta+c^\delta(ba)\\
&\quad +\beta(c)b^\delta(a)+\beta(cb)a^\delta\\
&\quad +a^\delta\beta(bc)-a^\delta\beta(bc)+a^\delta\beta(cb)-a^\delta\beta(cb)+ab^\delta c-ab^\delta c)\\
&= \Delta(a,b,c)^2(a^\delta(\beta(bc)-bc)-a^\delta(\beta(bc)-\beta(cb))+(\beta(cb)a^\delta-a^\delta\beta(cb))\\
&\quad -(\beta(a)-a)b^\delta c+(\beta(c)-c)b^\delta a+(ab-\beta(ab))c^\delta)\\
&= \Delta(a,b,c)^2(a^\delta(\beta(bc)-bc)-a^\delta[\beta(b),\beta(c)]\\
&\quad +[\beta(cb),a^\delta]-(\beta(a)-a)b^\delta c+(\beta(c)-c)b^\delta a+(ab-\beta(ab))c^\delta)\\
&= 0.
\end{aligned}$$

We have, $2\Delta(a,b,c)^3=0$. Since R is 2-torsion free, we have $\Delta(a,b,c)^3=0$. Using $\Delta(a,b,c)\in S$, we have $\Delta(a,b,c)^2x\Delta(a,b,c)^2=0$, for all $x\in L$. By Corollary 2.2, we have $\Delta(a,b,c)^2=0$. Similarly, we get $\Delta(a,b,c)=0$, for all $a\in S$ and $b,c\in L$. Also, if $a\in S$, then $aL\subseteq C(L)$ and $\beta(a),\beta^{-1}(a)\in S$. Let $a\in S$ and $x,y,b,c\in L$. Using the last equation, we have

$$\begin{aligned}
(ayxbc)^\delta &= ((ayx)bc)^\delta=(ayx)^\delta(bc)+\beta(ayx)b^\delta c+\beta((ayx)b)c^\delta\\
&= (a^\delta(yx)+\beta(a)y^\delta x+\beta(ay)x^\delta)(bc)+\beta(ayx)b^\delta c+\beta((ayx)b)c^\delta.
\end{aligned}$$

On the other hand,

$$(ayxbc)^{\delta} = a^{\delta}(yxbc) + \beta(a)y^{\delta}xbc + \beta(ay)(xbc)^{\delta}.$$

Comparing the last two equations, we have

$$ay\beta^{-1}(\Delta(x,b,c)) = 0, \text{ for all } a \in S \text{ and } x,b,c \in L.$$

Replacing a by $\beta^{-1}(\Delta(x,b,c))$, we have

$$\beta^{-1}(\Delta(x,b,c))L\beta^{-1}(\Delta(x,b,c)) = 0, \text{ for all } x,b,c \in L.$$

Corollary 2.2, we find that

$$\Delta(x,b,c) = 0, \text{ for all } x,b,c \in L.$$

By Proposition 2.1, we conclude that δ is an $(1,\beta)-$derivation of L. This completes the proof of the theorem.

Example 2.1. *Let S be any ring and let* $R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a,b \in S \right\}$ *and* $L = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b \in S \right\}$. *Define* $d : R \to R$ *by* $d\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, *and* $\beta : R \to R$ *by* $\beta\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -a & 0 & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *It is easy to check that R is a ring, L is a Lie ideal of R, β is an one to one, onto and d is a Jordan triple $(1,\beta)$-derivation on L but not an $(1,\beta)$-derivation.*

3. Generalized Jordan Triple $(\alpha,\beta)-$Derivations

An additive mapping $\mu : R \longrightarrow R$ is said to be a Jordan triple left centralizer on L if $(aba)^{\mu} = a^{\mu}ba$ for all $a,b \in L$ and called a Jordan left centralizer on L if $(a^2)^{\mu} = a^{\mu}a$.
To facilitate our discussion, we shall begin with the following lemma:

Lemma 3.1 ([12], Theorem 3.1)**.** *Let R be a 2-torsion free semiprime ring and $L \not\subseteq Z(R)$ be a square-closed Lie ideal. If $\mu : R \to R$ is Jordan triple left centralizer on L, then μ is a Jordan left centralizer on L.*

Theorem 3.1. *Let R be a $2-$torsion free semiprime ring, α,β be automorphisms of R and $L \not\subseteq Z(R)$ be a square-closed Lie ideal. If $F : R \to R$ is generalized Jordan triple $(\alpha,\beta)-$derivation on L such that $a^{\delta}, \beta(a) \in L$, then F is a generalized $(\alpha,\beta)-$derivation on L.*

Proof. We are given that F is a generalized Jordan triple (α, β)-derivation on L. Therefore we have

$$(aba)^F = a^F \alpha(ba) + \beta(a) b^\delta \alpha(a) + \beta(ab) a^\delta \text{ for all } a, b \in L. \qquad (3.1)$$

In (3.1), we take δ is a Jordan triple (α, β)-derivation on L. Since R is a $2-$torsion free semiprime ring, so in view of Theorem 2.1, δ is (α, β)-derivation on L. Now we write $\Gamma = F - \delta$. Then

$$\begin{aligned} \Gamma(aba) &= (aba)^{F-\delta} \\ &= (aba)^F - (aba)^\delta \\ &= (a^F - a^\delta)\alpha(ba) \text{ for all } a, b \in L. \end{aligned}$$

Then we have $\Gamma(aba) = \Gamma(a)\alpha(ba)$ for all $a, b \in L$. So, $\alpha^{-1}\Gamma$ becomes a Jordan triple left centralizer. In other words $\alpha^{-1}\Gamma$ is a Jordan triple left centralizer on L. Since R is a $2-$torsion free semiprime ring one can conclude that $\alpha^{-1}\Gamma$ is a Jordan left centralizer by Lemma 3.1. Hence

$$\alpha^{-1}\Gamma(ab) = \alpha^{-1}\Gamma(a)b \text{ for all } a, b \in L.$$

That is, $\Gamma(ab) = \Gamma(a)\alpha(b)$ and hence F is of the form $F = \Gamma + \delta$, where δ is an $(\alpha, \beta)-$ derivation and $\Gamma(ab) = \Gamma(a)\alpha(b)$. Therefore, F is a generalized Jordan $(\alpha, \beta)-$derivation on L.

Since every generalized (α, β)-derivation is also a generalized Jordan Triple (α, β) derivation, we immediately obtain

Corollary 3.1. *Let R be a $2-$torsion free semiprime ring, α, β be automorphisms of R and $L \nsubseteq Z(R)$ be a square-closed Lie ideal. If $F : R \to R$ is generalized Jordan $(\alpha, \beta)-$derivation on L such that $a^\delta, \beta(a) \in L$, then F is a generalized $(\alpha, \beta)-$derivation on L.*

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