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On Some Generalized Deferred Cesàro Means-II

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Abstract: In this study, using the genealized difference operator Δ^m, we introduce some new sequence spaces and investigate some topological properties of these sequence spaces
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1 Introduction

Let w be the set of all sequences of real or complex numbers and ℓ_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x = (x_k)$ with the usual norm $\|x\|_\infty = \sup |x_k|$, where $k \in \mathbb{N} = \{1, 2, \ldots\}$, the set of positive integers. Also by bs, cs, ℓ_1 and ℓ_p; we denote the spaces of all bounded, convergent, absolutely summable and $p-$absolutely summable sequences, respectively.

A sequence space X with a linear topology is called a $K-$space provided each of the maps $p_i : X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for each $i \in \mathbb{N}$, where $\mathbb{C}$ denotes the complex field. A $K-$space X is called an $FK-$space provided X is a complete linear metric space. An $FK-$space whose topology is normable is called a $BK-$space. We say that an $FK-$space X has AK (or has the AK property), if (e_k) (the sequence of unit vectors) is a Schauder bases for X.

The notion of difference sequence spaces was introduced by Kızmaz [1] and the notion was generalized by Et and Çolak [2]. Later on Et and Nuray [3] generalized these sequence spaces to the following sequence spaces:

Let X be any sequence space and let m be a non-negative integer. Then,

$$\Delta^m(X) = \{x = (x_k) : (\Delta^m x_k) \in X\}$$

$\Delta^0 x = (x_k)$, $\Delta^m x = \left(\Delta^{m-1}x_k - \Delta^{m-1}x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i}$. is a Banach space normed by

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \|\Delta^m x_k\|_\infty .$$

If $x \in X(\Delta^m)$ then there exists one and only one $y = (y_k) \in X$ such that

$$x_k = \sum_{i=1}^{k-m} (-1)^m \binom{k-i-1}{m-1} y_i = \sum_{i=1}^{k} (-1)^m \binom{k+m-i-1}{m-1} y_{i-m}, \quad y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. Recently, a large amount of work has been carried out by many mathematicians regarding various generalizations of sequence spaces. For a detailed account of sequence spaces one may refer to ([2-13]).

In 1932, Agnew [4] introduced the concept of deferred Cesaro mean of real (or complex) valued sequences $x = (x_k)$ defined by

$$(D_{p,q}x)_n = \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} x_k,\ n = 1, 2, 3, \ldots,$$

where $p = \{p(n)\}$ and $q = \{q(n)\}$ are the sequences of non-negative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty. \tag{1}$$

2 Topological Properties of $X(\Delta^m)$

In this section we prove some results involving the sequence spaces $C_0^d(\Delta^m), C_1^d(\Delta^m)$ and $C_\infty^d(\Delta^m)$.

Definition 1. *Let m be a fixed non-negative integer and let $\{p(n)\}$ and $\{q(n)\}$ be two sequences of non-negative integers satisfying the condition (1). We define the following sequence spaces:*

$$C_0^d(\Delta^m) = \left\{ x = (x_k) : \lim_n \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k = 0 \right\},$$

$$C_1^d(\Delta^m) = \left\{ x = (x_k) : \lim_n \frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \big(\Delta^m x_k - L\big) = 0 \right\},$$

$$C_\infty^d(\Delta^m) = \left\{ x = (x_k) : \sup_n \left(\frac{1}{(q(n) - p(n))} \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \right) < \infty \right\}.$$

The above sequence spaces contain some unbounded sequences for $m \geq 1$, for example let $x = (k^m)$, then $x \in C_\infty^d(\Delta^m)$, but $x \notin \ell_\infty$.

Theorem 1. *The sequence spaces $C_0^d(\Delta^m), C_1^d(\Delta^m)$ and $C_\infty^d(\Delta^m)$ are Banach spaces normed by*

$$\|x\|_\Delta = \sum_{i=1}^{m} |x_i| + \sup_n \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \right|.$$

Proof: Proof follows from Theorem 1 of Et and Nuray [3]. □

Theorem 2. $X(\Delta^{m-1}) \subset X(\Delta^m)$ *and the inclusion is strict for* $X = C_0^d, C_1^d$ *and* C_∞^d.

Proof: The inclusions part of the proof are esay. To see that the inclusions are strict, let $m = 2$ and $q(n) = n, p(n) = 0$ and consider a sequence defined by $x = \left(k^2\right)$, then $x \in C_1^d(\Delta^2)$, but $x \notin C_1^d(\Delta)$ (If $x = \left(k^2\right)$, then $\left(\Delta^2 x_k\right) = (2, 2, 2, ...)$. □

Theorem 3. *The inclusions $C_0^d(\Delta^m) \subset C_1^d(\Delta^m) \subset C_\infty^d(\Delta^m)$ are strict.*

Proof: First inclusion is esay. Second inclusion follows from the following inequality

$$\begin{aligned} \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k \right| &\leq \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k - L \right| + \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} L \right| \\ &\leq \frac{1}{(q(n) - p(n))} \left| \sum_{k=p(n)+1}^{q(n)} \Delta^m x_k - L \right| + L \end{aligned}$$

For strict the inclusion, observe that $x = (1, 0, 1, 0, ...) \in C_\infty^d(\Delta^m)$, but $x \notin C_1^d(\Delta^m)$, (If $x = (1, 0, 1, 0, ...)$, then $(\Delta^m x_k) = \left((-1)^{m+1}\, 2^{m+1}\right)$). □

Theorem 4. $C_1^d(\Delta^m)$ *is a closed subspace of* $C_\infty^d(\Delta^m)$.

Proof: Proof follows from Theorem 4 of Et and Nuray [3]. □

Theorem 5. *$C_1^d(\Delta^m)$ is a nowhere dense subset of $C_\infty^d(\Delta^m)$.*

Proof: Proof follows from the fact that $C_1^d(\Delta^m)$ is a proper and complete subspace of $C_\infty^d(\Delta^m)$. □

Theorem 6. *$C_\infty^d(\Delta^m)$ is not separable, in general.*

Proof: Suppose that $C_\infty^d(\Delta^m)$ is separable for some $m \geq 1$, for example let $m = 2$ and $q(n) = n, p(n) = 0$. In this case $C_\infty(\Delta^2)$ is separable. In Theorem 5, Bhardwaj et al. [5] show that $C_\infty(\Delta^2)$ is not separable. So $C_\infty^d(\Delta^m)$ is not separable, in general. □

Theorem 7. *$C_\infty^d(\Delta^m)$ does not have Schauder basis. separable, in general.*

Proof: Proof follows from the fact that if a normed space has a Schauder basis, then it is separable. □

Theorem 8. *$C_1^d(\Delta^m)$ is separable.*

Proof: Proof follows from Theorem 5 of Et and Nuray [3]. □

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