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Research Article

# Oscillation criteria for first-order dynamic equations with nonmonotone delays

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## Abstract

In this paper, we consider the first-order dynamic equation as the following:

$$x^{\Delta}(t)+\sum_{i=1}^{m}p_i(t)x\left(\tau_i(t)\right)=0,\ t\in[t_0,\infty)_{\mathbb{T}}$$

where $p_i \in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+\right)$, $\tau_i \in C_{rd}\left([t_0,\infty)_{\mathbb{T}},\mathbb{T}\right)$ $(i=1,2,\ldots,m)$ and $\tau_i(t)\le t$, $\lim_{t\to\infty}\tau_i(t)=\infty$. When the delay terms $\tau_i(t)$ $(i=1,2,\ldots,m)$ are not necessarily monotone, we present new sufficient conditions for the oscillation of first-order delay dynamic equations on time scales.

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**Keywords.** dynamic equations, nonmonotone delay, oscillation, time scales

## 1. Introduction

As is well known, after Stefan Hilger [15], [16] introduced the theory of dynamic equations on time scales (or measure chain) in his Ph.D. thesis in 1988, a lot of papers have been devoted to this subject field. Especially, the oscillatory behaviour of solutions of differential/difference and dynamic equations has been studied by many authors. See, for example, [1–34] and the references cited therein. Consider the first-order delay dynamic equation

$$x^{\Delta}(t)+\sum_{i=1}^{m}p_i(t)x\left(\tau_i(t)\right)=0,\quad t\in[t_0,\infty)_{\mathbb{T}},\tag{1.1}$$

where $\mathbb{T}$ is a time scale unbounded above with $t_0\in\mathbb{T}$, $p_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$, $\tau_i\in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ $(i=1,2,\ldots,m)$ are not necessarily monotone such that

$$\tau_i(t)\le t\ \text{ for all } t\in\mathbb{T},\quad \lim_{t\to\infty}\tau_i(t)=\infty.\tag{1.2}$$

A function $p:\mathbb{T}\to\mathbb{R}$ is called positively regressive (we write $p\in\mathcal{R}^+$) if it is rd-continuous and satisfies $1+\mu(t)p(t)>0$ for all $t\in\mathbb{T}$, where $\mu:\mathbb{T}\to\mathbb{R}_0^+$ is the graininess function defined by $\mu(t):=\sigma(t)-t$ with the forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ defined by $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ for $t\in\mathbb{T}$. A point $t\in\mathbb{T}$ is called right-dense if $\sigma(t)=t$ (or equivalently

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$\mu(t) = 0$) holds, otherwise it is called right-scattered. The readers are referred to Bohner and Peterson [2] for further details concerning the time scales calculus.

A function $x : \mathbb{T} \to \mathbb{R}$ is called a solution of the equation (1.1), if $x(t)$ is delta differentiable for $t \in \mathbb{T}^\kappa$ and satisfies equation (1.1) for $t \in \mathbb{T}^\kappa$. We say that a solution $x$ of equation (1.1) has a generalized zero at $t$ if $x(t) = 0$ or if $\mu(t) > 0$ and $x(t)x(\sigma(t)) < 0$. Let $\sup \mathbb{T} = \infty$ and then a nontrivial solution $x$ of equation (1.1) is called oscillatory on $[t, \infty)$ if it has arbitrarily large generalized zeros in $[t, \infty)$.

For $m = 1$, equation (1.1) reduces to

$$x^{\Delta}(t) + p(t)x\left(\tau(t)\right) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.3}$$

Now, we give some well-known tests on oscillatory behaviour of (1.3). In 2002, Zhang and Deng [32], using the cylinder transforms, proved that if $\tau(t)$ is eventually nondecreasing and

$$\limsup_{t\to\infty} \sup_{\lambda\in E} \left\{\lambda e_{-\lambda p}(t, \tau(t))\right\} < 1,$$

where $E = \{\lambda : \lambda > 0,\ 1 - \lambda p(t)\mu(t) > 0\}$ and in 2005, Bohner [4], using exponential functions notation, proved that if $\tau(t)$ is eventually nondecreasing and

$$\limsup_{t\to\infty} \sup_{-\lambda p\in\mathcal{R}^+} \left\{\lambda e_{-\lambda p}(t, \tau(t))\right\} < 1,$$

where

$$e_{-\lambda p}\left(t, \tau(t)\right) = \exp\left\{ \int_{\tau(t)}^{t} \xi_{\mu(s)}(-\lambda p(s))\Delta s \right\},$$

and

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h} & , \text{ if } h \neq 0 \\ z & , \text{ if } h = 0 \end{cases},$$

then all solutions of equation (1.3) are oscillatory.

In 2005, Zhang et al. [33] and in 2006, Şahiner and Stavroulakis [27], using different technique, obtained that if $\tau(t)$ is eventually nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{1.4}$$

then all solutions of equation (1.3) are oscillatory. In 2005, Zhang et al. [33] (See also Agarwal and Bohner [1, Theorem 1]) established the following result. Assume that $\tau(t)$ is eventually nondecreasing and

$$m := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > \frac{1}{e}, \tag{1.5}$$

then all solutions of (1.3) oscillate.

In 2006, Şahiner and Stavroulakis [27] found out that if $\tau(t)$ is eventually nondecreasing,

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > c \quad \text{and} \quad \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\Delta s > 1 - \frac{c^2}{4}, \tag{1.6}$$

where $c \in (0, 1)_{\mathbb{R}}$, then every solution of equation (1.3) oscillates. Furthermore, Agarwal and Bohner [1] improved the condition (1.6) as follows:

If $\tau(t)$ is eventually nondecreasing,

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\Delta s > c \quad\text{and}\quad \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)\Delta s > 1-\left(1-\sqrt{1-c}\right)^2 \tag{1.7}$$

where $c \in (0,1)_{\mathbb{R}}$, then every solution of equation (1.3) oscillates.

Also, in 2016, Karpuz and Öcalan [19] enhanced the condition (1.7) by extending the second integral condition to the larger interval $[\tau(t), t]_{\mathbb{T}}$ as the following:
Assume that $\tau(t)$ is eventually nondecreasing and

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t} p(s)\Delta s > c \quad\text{and}\quad \limsup_{t\to\infty}\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1-\left(1-\sqrt{1-c}\right)^2, \tag{1.8}$$

where $c \in (0,1)_{\mathbb{R}}$. Then every solution of equation (1.3) oscillates.

Zhang et al. [33] established the following result. Assume that $\tau(t)$ is eventually nondecreasing and $m \in [0, \frac{1}{e}]$ (where $m$ is defined by (1.5)). Moreover, if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1-m-\sqrt{1-2m-m^2}}{2}, \tag{1.9}$$

where $\lambda_1 \in [1, e]$ is the unique root of the equation $\lambda = e^{m\lambda}$, then all solutions of equation (1.3) are oscillatory. It is obvious that, since

$$\frac{1+\ln\lambda_1}{\lambda_1} \le 1 \quad\text{for}\quad \lambda_1 \in [1, e],$$

the condition (1.9) implies

$$\limsup_{t\to\infty}\int_{\tau(t)}^{\sigma(t)} p(s)\Delta s > 1 - \frac{1-m-\sqrt{1-2m-m^2}}{2}. \tag{1.10}$$

Clearly, when $0 < c \le \frac{1}{e}$, it is easy to verify that

$$\frac{1-c-\sqrt{1-2c-c^2}}{2} > \left(1-\sqrt{1-c}\right)^2 > \frac{c^2}{4}$$

and therefore the condition (1.10) is weaker than the conditions (1.6) and (1.8).

Now, we assume that $\tau(t)$ is not necessarily monotone. Set

$$h(t) = \sup_{s\le t}\tau(s), \quad t\in\mathbb{T},\ t\ge 0. \tag{1.11}$$

Clearly, $h(t)$ is nondecreasing and $\tau(t) \le h(t)$ for all $t \ge 0$.

In 2017, Öcalan, Özkan and Yıldız [24, Theorem 2.2] studied the equation (1.3) when $\tau(t)$ is not necessarily monotone and obtained the following result.

**Theorem A.** If

$$\limsup_{t\to\infty}\int_{h(t)}^{\sigma(t)} p(s)\Delta s > 1, \tag{1.12}$$

where $h(t)$ is defined by (1.11), then every solution of (1.3) is oscillatory.

Finally, Öcalan [25, Corollary 2.4] established the following result when $\tau(t)$ is not necessarily monotone.

**Theorem B.** If

$$\liminf_{t\to\infty}\int\limits_{h(t)}^{t} p(s)\Delta s=\liminf_{t\to\infty}\int\limits_{\tau(t)}^{t} p(s)\Delta s>\frac{1}{e}, \tag{1.13}$$

where $h(t)$ is defined by (1.11), then all solutions of (1.3) oscillate.

A slight modification in the proofs of Theorems $A$ and $B$ leads to the following result.

**Theorem 1.1.** *Assume that all the conditions of Theorems A and B hold. Then*
*(i) the dynamic inequality*

$$x^{\Delta}(t)+p(t)x\left(\tau(t)\right)\leq 0,\quad t\in[t_0,\infty)_{\mathbb{T}}$$

*has no eventually positive solutions;*
*(ii) the dynamic inequality*

$$x^{\Delta}(t)+p(t)x\left(\tau(t)\right)\geq 0,\quad t\in[t_0,\infty)_{\mathbb{T}}$$

*has no eventually negative solutions.*

## 2. Main results

In this section, we present some new sufficient conditions for the oscillation of all solutions of (1.1), under the assumption that the arguments $\tau_i(t)$ $(i=1,2,\ldots,m)$ are not necessarily monotone. Set

$$h_i(t)=\sup_{s\leq t}\{\tau_i(s)\}\quad\text{and}\quad h(t)=\max_{1\leq i\leq m}\{h_i(t)\},\ t\in\mathbb{T},\ t\geq 0. \tag{2.1}$$

Clearly, $h_i(t)$ $(i=1,2,\ldots,m)$ are nondecreasing and $\tau_i(t)\leq h_i(t)\leq h(t)$ $(i=1,2,\ldots,m)$ for all $t\geq 0$.

The following lemma was given by Şahiner and Stavroulakis [27].

**Lemma 2.1.** *Assume that $f:\mathbb{T}\to\mathbb{R}$ is rd-continuous, $g:\mathbb{T}\to\mathbb{R}$ is nonincreasing and $\tau:\mathbb{T}\to\mathbb{T}$ is nondecreasing. If $b<u$, then*

$$\int\limits_{b}^{\sigma(u)} f(s)g(\tau(s))\Delta s\geq g(\tau(u))\int\limits_{b}^{\sigma(u)} f(s)\Delta s.$$

The following result is easily obtained by using the similar way in the proof of Lemma 2.3 in [24].

**Lemma 2.2.** *Assume that* (2.1) *holds and $\alpha>0$. Then, we have*

$$\alpha:=\liminf_{t\to\infty}\int\limits_{h(t)}^{t}\sum_{i=1}^{m}p_i(s)\Delta s=\liminf_{t\to\infty}\int\limits_{\tau(t)}^{t}\sum_{i=1}^{m}p_i(s)\Delta s,$$

*where $\tau(t)=\max_{1\leq i\leq m}\{\tau_i(t)\}$, $t\in\mathbb{T}$, $t\geq 0$.*

**Theorem 2.3.** *Assume that $-\sum\limits_{i=1}^{m}p_i\in\mathcal{R}^{+}$. If $\tau_i(t)$ $(i=1,2,\ldots,m)$ are not necessarily monotone and*

$$\limsup_{t\to\infty}\int\limits_{h(t)}^{\sigma(t)}\sum_{i=1}^{m}p_i(s)\Delta s>1 \tag{2.2}$$

*or*

$$\liminf_{t\to\infty}\int\limits_{\tau(t)}^{t}\sum_{i=1}^{m}p_i(s)\Delta s>\frac{1}{e}, \tag{2.3}$$

*where $h(t)$ is defined by* (2.1) *and* $\tau(t) = \max_{1\leq i\leq m}\{\tau_i(t)\}$. *Then all solutions of* (1.1) *oscillate.*

***Proof.*** Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1.1). Since $-x(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then, there exists $t_1 > t_0$ such that $x(t),\ x(\tau_i(t)) > 0$ $(i = 1, 2, \ldots, m)$, for all $t \geq t_1$. Thus, from (1.1) we have

$$x^{\Delta}(t) = -\sum_{i=1}^{m} p_i(t)x(\tau_i(t)) \leq 0 \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function. In view of this and $\tau_i(t) \leq \tau(t)$ $(i = 1, 2, \ldots, m)$, (1.1) gives

$$x^{\Delta}(t) + \left(\sum_{i=1}^{m} p_i(t)\right) x(\tau(t)) \leq 0, \quad t \geq t_1.$$

Comparing (2.2) and (2.3), we obtain a contradiction to Theorem 1.1. Thus, the proof of the theorem is completed. □

Now, we consider the case where $0 < \alpha \leq \frac{1}{e}$. Then, we will obtain new oscillatory condition for all solutions of (1.1). We need the following lemma to establish our result. When the case $\tau_i(t)$ $(i = 1, 2, \ldots, m)$ are not necessarily monotone, the following lemma can be easily obtained by using the similar process in [33, Lemma 2.4]. So, the proof of the following result is omitted here.

**Lemma 2.4.** *Assume that $\tau_i(t)$ $(i = 1, 2, \ldots, m)$ are not necessarily monotone. Let $0 \leq \alpha \leq \frac{1}{e}$ and $x(t)$ be an eventually positive solution of Eq.*(1.1)*. Then, we get*

$$\liminf_{t\to\infty} \frac{x(\sigma(t))}{x(h(t))} \geq \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.4}$$

*where $h(t)$ is defined by* (2.1) *and* $\tau(t) = \max_{1\leq i\leq m}\{\tau_i(t)\}$.

**Theorem 2.5.** *Assume that $-\sum_{i=1}^{m} p_i \in \mathcal{R}^+$ and $0 \leq \alpha \leq \frac{1}{e}$. If $\tau_i(t)$ $(i = 1, 2, \ldots, m)$ are not necessarily monotone and*

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s)\Delta s > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.5}$$

*where $h(t)$ is defined by* (2.1)*. Then all solutions of* (1.1) *oscillate.*

***Proof.*** Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1.1). Since $-x(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then, there exists $t_1 > t_0$ such that $x(t),\ x(\tau_i(t)) > 0$ $(i = 1, 2, \ldots, m)$, for all $t \geq t_1$. Thus, from (1.1) we have

$$x^{\Delta}(t) = -\sum_{i=1}^{m} p_i(t)x(\tau_i(t)) \leq 0 \quad \text{for all } t \geq t_1,$$

which means that $x(t)$ is an eventually nonincreasing function. In view of this and $\tau_i(t) \leq h_i(t) \leq h(t)(i = 1, 2, \ldots, m)$, Eq.(1.1) gives

$$x^{\Delta}(t) + \sum_{i=1}^{m} p_i(t)x(h(t)) \leq 0, \quad t \geq t_1. \tag{2.6}$$

Integrating (2.6) from $h(t)$ to $\sigma(t)$ and taking into account the facts that the function $h(t)$ is nondecreasing and the function $x(t)$ is nonincreasing, we obtain

$$x(\sigma(t)) - x(h(t)) + \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) x\left(h(s)\right) \Delta s \leq 0.$$

Therefore, by using Lemma 2.1, we get

$$x(\sigma(t)) - x(h(t)) + x(h(t)) \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s \leq 0$$

or

$$x(\sigma(t)) + x(h(t)) \left[ \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s - 1 \right] \leq 0.$$

Consequently,

$$\int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s \leq 1 - \frac{x\left(\sigma(t)\right)}{x\left(h(t)\right)},$$

which gives

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s \leq 1 - \liminf_{t\to\infty} \frac{x\left(\sigma(t)\right)}{x\left(h(t)\right)} \tag{2.7}$$

and by (2.4), (2.7) leads to

$$\limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} \sum_{i=1}^{m} p_i(s) \Delta s \leq 1 - \frac{1 - \alpha - \sqrt{1 - 2\alpha - \alpha^2}}{2},$$

which contradicts with (2.5). The proof of the theorem is completed. □

**Example 2.6.** Let $m = 1$, $h \in \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, where $h > 0$. Then, we have

$$\sigma(t) = t + h, \quad \mu(t) = h \quad \text{and} \quad x^{\Delta}(t) = \frac{x(t+h) - x(t)}{h}$$

for $t \in \mathbb{T}$. Thus, Eq.(1.1) becomes

$$\frac{x(t+h) - x(t)}{h} + p(t) x\left(\tau(t)\right) = 0, \quad t \in \{hk : k \in \mathbb{Z}\}.$$

Let $\tau(t) = t - 2$ and $h = 2$. Since $p(t) \in \{hk : k \in \mathbb{Z}\}$, we assume

$$p(2t) = 0.18 \text{ and } p(2t+2) = 0.27, \quad t = 0, 2, 4, \ldots$$

When $\mathbb{T} = h\mathbb{Z}$, from (iii) in [2, Theorem 1.79], we have the following.

$$\int_{a}^{b} f(t) \Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh) h \quad \text{for} \quad a < b. \tag{2.8}$$

So, by using (2.8), we observe that, for $\tau(t)$, $p(t) \in \{hk : k \in \mathbb{Z}\}$.

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s) \Delta s = \liminf_{t\to\infty} \sum_{j=\frac{t-2}{2}}^{\frac{t}{2}-1} p(2j) 2 = \liminf_{t\to\infty} 2p(t-2) = 0.36 \not> \frac{1}{e}$$

and

$$\beta := \limsup_{t\to\infty} \int_{h(t)}^{\sigma(t)} p(s)\Delta s = \limsup_{t\to\infty} \sum_{j=\frac{t-2}{2}}^{\frac{t+2}{2}-1} p(2j)2 = \limsup_{t\to\infty} 2[p(t-2)+p(t)] = 0.9 \not> 1$$

shows that Theorem 2.3 fails. On the other hand,

$$\beta = 0.9 \not> 1 - \left(1 - \sqrt{1-0.36}\right)^2 = 0.96$$

demonstrates that the condition (1.8) doesn't hold. However, since

$$\beta = 0.9 > 1 - \frac{1 - 0.36 - \sqrt{1 - 2(0.36) - (0.36)^2}}{2} = 0.873\,91,$$

every solution oscillates by Theorem 2.5.

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