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Research Article

Structure of weighted Hardy spaces on finitely connected domains

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Abstract

We give a complete characterization of a certain class of Hardy type spaces on finitely connected planar domains. In particular, we provide a decomposition result and give a description of such functions through their boundary values. As an application, we describe an isomorphism from the weighted Hardy space onto the classical Hardy-Smirnov space. This allows us to identify the multiplier space of the mentioned Hardy type spaces as the space of bounded holomorphic functions on the domain.

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1. Introduction

Recently, E. A. Poletsky and M. Stessin in [9] introduced a scale of Hardy and Bergman type spaces which consists of holomorphic functions on the underlying hyperconvex domain in the n-dimensional complex Euclidean space $\mathbb{C}^n$, where $n \geq 1$ is an integer. The class of hyperconvex domains contain a wide range of classical domains, in particular the n-dimensional unit ball and the polydisk in $\mathbb{C}^n$ are hyperconvex. This way, the theory of Hardy and Bergman spaces in the classical settings have been unified and generalized in [9]. Further studies in this direction will help us to construct a more unified system of methods to attack the problems of Hardy and Bergman spaces. Before the works of Poletsky and Stessin, M. A. Alan has already defined using a similar construction Hardy type spaces in hyperconvex domains in $\mathbb{C}^n$ in [1].

In this paper we concentrate on the $n = 1$ case for the Hardy space construction of Poletsky and Stessin. By definition to each subharmonic function u continuous near the boundary of G corresponds a space, which is denoted by H^p_u of holomorphic functions in G. Here G is a bounded regular domain in $\mathbb{C}$. Throughout the paper these spaces will be called Poletsky-Stessin Hardy spaces. Following the motivating work of Poletsky and Stessin, the structure and first examples on the unit disk of Poletsky-Stessin Hardy spaces were further investigated in [2], [11] and [12].

Among these recent work, the author and M. A. Alan gave a complete characterization of H^p_u spaces that live in the plane domains through the boundary values of the functions in this class or through a growth description of their harmonic majorants. As an application, a Beurling's type theorem was proved in [2]. This states roughly that if G is the unit disk

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$\mathbb{D}$, and u is such a subharmonic exhaustion on $\mathbb{D}$, there exists a bounded outer function φ so that the space H^p_u isometrically equals to $\mathcal{M}_{\varphi,p}$, $p > 0$. Moreover, φ belongs to the class H^p_u. Here we denote by $\mathcal{M}_{\varphi,p}$ the space $\varphi^{2/p}H^p$ together with the norm defined by

$$\|f\|_{\mathcal{M}_{\varphi,p}} := \|f/\varphi^{2/p}\|_p, \quad f \in \mathcal{M}_{\varphi,p}.$$

Using this result, we were able to construct several examples of holomorphic spaces with certain properties. Let us remark that the space $\mathcal{M}_{\varphi,2}$ is a useful tool in the study of sub-Hardy Hilbert spaces (see [10]).

The problems and results addressed in this paper can be summarized as follows:

When G is finitely connected:

(1) We provide a decomposition result of functions which belong to $H^p_u(G)$. (Theorem 4.1).
(2) We completely describe $H^p_u(G)$ in terms of their boundary values.
(3) We describe an isomorphism from the classical Hardy space $H^p(G)$ onto the space $H^p_u(G)$.

The last two results extend the research in [2], [11] and [12] from the disk to finitely connected planar domains. This paper is organized as follows: Section 2 is a brief summary of the previous work which are related and will be used in the paper. In section 3 we solve the inverse balayage problem for a given weight function. The main results of the paper are stated and proved in section 4.

2. Poletsky-Stessin-Hardy spaces

In this section we will give the basic definitions to be used throughout the paper and recall some earlier results. Let $G \subset \mathbb{C}$ be a bounded domain. We say that a function $u \le 0$ defined on G is an exhaustion on G if the level set

$$B_{c,u} := \{z \in G : u(z) < c\}$$

is relatively compact in G for any $c < 0$. Let u be an exhaustion and $c < 0$, we set

$$u_c := \max\{u, c\}, \quad S_{c,u} := \{z \in G : u(z) = c\}.$$

Let $u \in sh(G)$ be an exhaustion function. We assume that u is continuous taking values in $\mathbb{R} \cup \{-\infty\}$ with the extended topology on this set. Following Demailly [4] we define

$$\mu_{c,u} := \Delta u_c - \chi_{G \setminus B_{c,u}} \Delta u.$$

Here χ_ω denotes the characteristic function for a given set $\omega \subset G$. We will denote the class of negative subharmonic exhaustion functions on G by $\mathcal{E}(G)$. Also we denote by $\mathcal{E}_0(G)$ the class of all functions $u \in \mathcal{E}(G)$ such that $\int \Delta u < \infty$.

Following [9] we set

$$sh_u(G) := sh_u := \left\{ v \in sh(G) : v \ge 0, \sup_{c<0} \int_{S_{c,u}} v\, d\mu_{c,u} < \infty \right\},$$

and we define

$$H^p_u(G) := H^p_u := \{f \in hol(G) : |f|^p \in sh_u\}$$

for $p > 0$. Let us write

$$\|v\|_u := \sup_{c<0} \int_{S_{c,u}} v\, d\mu_{c,u} = \int_G (v\Delta u - u\Delta v) \tag{2.1}$$

if v is a nonnegative subharmonic function. We set

$$\|f\|_{u,p} := \sup_{c<0} \left(\int_{S_{c,u}} |f|^p\, d\mu_{c,u} \right)^{1/p} \tag{2.2}$$

if f is a holomorphic function on G. We list relavant recollections of basic facts below:

- H_u^p is a Banach space when $p \geq 1$.
- The Demailly measure μ_u has finite mass if and only if the constant function (equivalently, all polynomials) $f \equiv 1$ belongs to H_u^p.
- Let G be a bounded regular domain and let $w \in G$. The classical Green function $v(z) = g_G(z, w)$ is a subharmonic exhaustion for G.
- Let $G = \mathbb{D}$ be the unit disk and let $v(z) = \log|z|$. Then $H_v^p(\mathbb{D})$ is the usual Hardy space $H^p(\mathbb{D})$.

We denote by $P_G(z, w)$ the Poisson kernel for the domain G. The space of all holomorphic functions f in G so that $|f|^p$ admits a harmonic majorant in G is denoted by $H^p(G)$ (see for example [5]). By [9], $H_u^p \subset H^p$. It follows by a well-known fact that when G has a sufficiently smooth boundary, then a function f from the space $H_u^p(G)$ possesses boundary values (almost everywhere non-tangential limits), which we denote by f^*. The usual normalized arclength measure on ∂G is denoted by ν. Hence, $\nu(\partial G) = 1$.

Theorem 2.1 ([2, Theorem 2.10])**.** Suppose that G is a Jordan domain with rectifiable boundary or a bounded domain with C^2 boundary. Let $p > 1$, $f \in H^p(G)$ and let $u \in \mathcal{E}(G)$. Then the following statements are equivalent:

i. $f \in H_u^p(G)$.

ii. $f^* \in L^p(V_u \nu)$, where

$$V_u(\zeta) := \int_G P_G(z, \zeta) \Delta u(z), \quad \zeta \in \partial G. \tag{2.3}$$

iii. There exists a measure $\widetilde{\mu_u}$ on ∂G so that $f^* \in L^p(\widetilde{\mu_u})$. Moreover, if E is any Borel subset of ∂G with measure $\nu(E) = 0$, then $\widetilde{\mu_u}(E) = 0$ and we have the equality

$$\int_{\partial G} \gamma \, d\widetilde{\mu_u} = \int_G P_G(\gamma) \Delta u \tag{2.4}$$

for every $\gamma \in L^1(\nu)$.

In addition, if $f \in H_u^p(G)$, then $\|f\|_{u,p} = \|f^*\|_{L^p(\widetilde{\mu_u})}$ and $d\widetilde{\mu_u} = V_u d\nu$.

Remark 2.2. i. If G is the unit disk or a Jordan domain with rectifiable boundary, then the statements in Theorem 2.1 hold true for any $p > 0$.

ii. Take a compact set $K \subset G$ so that $\Delta u(K) > r > 0$. Let $m := \min_{\zeta \in \partial G} \min_{z \in K} P_G(z, \zeta)$. Then $V_u(\zeta) \geq rm > 0$ for every $\zeta \in \partial G$. Hence, replacing u by $u/(rm)$, we may assume without loss of generality that $V_u \geq 1$ on ∂G.

iii. The weight function V_u is lower semicontinuous. In fact, by Fatou's lemma

$$\liminf_j V_u(\zeta_j) = \liminf_j \int_G P_G(z, \zeta_j) \Delta u(z) \geq \int_G P_G(z, \zeta) \Delta u(z) = V_u(\zeta).$$

Note that V_u is the *balayage* of the measure Δu on ∂G.

iv. Suppose G is a bounded domain with C^2 boundary or a Jordan domain with rectifiable boundary. Let $u \in \mathcal{E}_0(G)$. Then

$$u(z) = \int_G g_G(z, w) \Delta u(w), \quad z \in G.$$

Since

$$\frac{\partial g_G(\zeta, w)}{\partial n} = P_G(w, \zeta)$$

when $\zeta \in \partial G$ and $w \in G$, $\frac{\partial u}{\partial n}(\zeta)$ exists for every $\zeta \in \partial G$. Here $\frac{\partial}{\partial n}$ denotes the normal derivative in the outward direction on ∂G and

$$\frac{\partial u(\zeta)}{\partial n} = V_u(\zeta) = \int_G P_G(w, \zeta) \Delta u(w), \quad \zeta \in \partial G.$$

Using the property (2.4) in Theorem 2.1 we see that

$$\int_{\partial G} V_u(\zeta)d\nu(\zeta) = \int_G \Delta u = \int_{\partial G} \frac{\partial u}{\partial n}(\zeta)d\nu(\zeta).$$

Let φ be a nonzero holomorphic function on the unit disk. Let $\mathcal{M}_{\varphi,p}$ denote the space $\varphi^{2/p}H^p$ which we endow with the norm

$$\|f\|_{\mathcal{M}_{\varphi,p}} := \|f/\varphi^{2/p}\|_p, \quad f \in \mathcal{M}_{\varphi,p}.$$

A function $\varphi \in H^2_u$ is called a u-inner function if $|\varphi^*(\zeta)|^2 V_u(\zeta) = 1$ for almost every $\zeta \in \partial\mathbb{D}$. If, moreover, $\varphi(z)$ is zero-free, we say that the function φ is singular u-inner. The next result is proved in Theorem 3.2 and Corollary 3.3 in [2].

Theorem 2.3. Let $u \in \mathcal{E}(\mathbb{D})$. There exists a bounded u-inner and outer function $\varphi \in H^2_u$ so that $H^2_u = \mathcal{M}_{\varphi,2}$. Moreover, these spaces are isometric.

The function φ is uniquely determined (up to a unit constant). We have

$$V_u(e^{i\theta}) = \frac{1}{|\varphi(e^{i\theta})|^2} = \frac{1}{\varphi^2(e^{i\theta})} \operatorname{sgn} \frac{1}{\varphi^2(e^{i\theta})}. \tag{2.5}$$

Here we set $sgn\alpha := |\alpha|/\alpha$ for any complex number $\alpha \neq 0$ and $sgn 0 := 0$.

Theorem 2.4. The set $L^p(V_u d\theta)$ coincides with $\varphi^{2/p}L^p(d\theta)$ and the map $f \mapsto \varphi^{-2/p}f$ is an isometric isomorphism from the space $L^p(V_u d\theta)$ onto $L^p(d\theta)$.

The next result reveals a complete factorization of functions belonging to the space $H^p_u(\mathbb{D})$.

Theorem 2.5 ([2, Theorem 3.4])**.** Let $0 < p < \infty$, $f \in H^p_u(\mathbb{D})$, $f \not\equiv 0$, and B be the Blaschke product formed with the zeros of f. There are zero-free $\varphi \in H^2_u \cap H^\infty$, $S \in H^\infty$ and $F \in H^p$ such that φ is singular u-inner and outer, S is singular inner, F is outer, and

$$f = BS\varphi^{2/p}F. \tag{2.6}$$

Moreover, $\|f\|_{p,u} = \|F\|_p$ and $H^p_u(\mathbb{D}) = \mathcal{M}_{\varphi,p}$.

Rephrasing the last statement above we have a concrete isomorphism.

Corollary 2.6. The map $f \mapsto \varphi^{-2/p}f$ is an isometric isomorphism from the space H^p_u onto H^p.

The following Lemma will be useful in the next section. Its proof is a simple calculation and we outline it here.

Lemma 2.7. Let c be a number with $-1 < c < 0$. Then there exists a function $\kappa = \kappa_c$ defined on $(-\infty, 0]$ with the following properties:

i. $\kappa : (-\infty, 0] \to (-\infty, 0]$ is non-decreasing, convex and C^∞,
ii. κ is real-analytic in $(c, 0]$,
iii. $\kappa(t) \equiv c$ when $t \leq c$, $\kappa(0) = 0$, and $\kappa'(0) = 1$.

Proof. Let $a := -\frac{\ln(-c)}{e}$, $b := \frac{-1}{\ln(-c)}$, and

$$\kappa(t) := \begin{cases} c + e^{\frac{-a}{(t-c)^b}}, & t > c, \\ c, & t \leq c. \end{cases}$$

Then

$$\kappa'(t) = \frac{1}{e(t-c)^{b+1}} e^{\frac{-a}{(t-c)^b}}$$

and

$$\kappa''(t) = \frac{1}{e(t-c)^{2b+2}}(1/e - (b+1)(t-c)^{b+1})e^{\frac{-a}{(t-c)^b}}$$

for $t > c$. For $t \le c$, $\kappa'(t) = \kappa''(t) = 0$. It can be checked that $\kappa''(t) > 0$ for $c < t \le 0$, and κ satisfies all properties in i., ii. and iii. □

3. Constructing subharmonic exhaustion from a given weight function

Let G be a Jordan domain with rectifiable boundary and ψ be a given holomorphic function in $H^1(G)$. We consider in this section the problem of finding a subharmonic exhaustion u on G so that $V_u(\zeta) = |\psi(\zeta)|$ when $\zeta \in \partial G$. We can always suppose that $G = \mathbb{D}$ after a conformal map of G onto $\mathbb{D}$. The following result was obtained for the disk in [6] (see also [8]).

Theorem 3.1. Let G be a Jordan domain with rectifiable boundary, ψ be a lower semi-continuous function on ∂G so that $\psi \ge k$ for some constant $k > 0$. Then there exists a function $u \in \mathcal{E}(G)$ so that $\psi = V_u$.

a. u is the decreasing limit of functions in $\mathcal{E}_0 \cap C^\infty(\overline{G})$ which converge uniformly to u on $\overline{G}$.
b. $u \in \mathcal{E}_0(G)$ if and only if $\psi \in L^1(d\nu)$.

In this section we consider finitely connected domains. If Γ is the rectifiable boundary of a Jordan domain, let G^* and G be the unbounded and bounded components of $\mathbb{C}\backslash\Gamma$, respectively. By a conformal map Φ the curve Γ can be mapped onto the unit circle and G^* can be mapped onto $\mathbb{D}^*$, the compliment in $\mathbb{C}$ of the closed unit disk. We say that a function u^* defined on G^* belongs to $\mathcal{E}(G^*)$ if the corresponding function $u := u^* \circ \Phi^{-1}(1/z)$ belongs to $\mathcal{E}(\mathbb{D})$. A holomorphic function f on G^* belongs to $H^p_{u^*}(G^*)$ if the function $f \circ \Phi^{-1}(1/z)$ belongs to $H^p_u(\mathbb{D})$. Clearly $H^p_{u^*}(G^*) \subset H^p(G^*)$. We write $a \approx b$ if there exists an absolute constant $C > 0$ so that $C^{-1}b \le a \le Cb$.

Lemma 3.2. Let Γ be the $C^{1+\varepsilon}$ boundary of a Jordan domain for some $\varepsilon > 0$. Let G^* be the unbounded component of $\mathbb{C}\backslash\Gamma$ and let ψ be a lower semicontinuous function on Γ so that $\psi \ge k$ for some constant $k > 0$. Then there exists a function $v \in \mathcal{E}(G^*)$ so that $\frac{\partial v(\zeta)}{\partial n} \approx \psi(\zeta)$ on Γ.

Proof. Suppose first that Γ is the unit circle. Let $u \in \mathcal{E}(\mathbb{D})$ be the function proved for ψ in Theorem 3.1. The function $v(z) = u(1/z)$ belongs to $\mathcal{E}(\mathbb{D}^*)$ and satisfies $\frac{\partial v(\zeta)}{\partial n} = \psi(\zeta)$ on Γ. If Γ is a Jordan curve with $C^{1+\varepsilon}$ boundary and if Φ is a conformal map of G onto $\mathbb{D}$, let $v(z) := u \circ \Phi(1/z)$, where $u \in \mathcal{E}(\mathbb{D})$ is so that $\frac{\partial u(\zeta)}{\partial n} = \psi \circ \Phi^{-1}(1/\zeta)$ for $|\zeta| = 1$. By a classical result of Painlevé (cf. [7, Theorem 5.2.4]) Φ extends to a C^1 map on the closed set $\overline{G}$. Thus, the estimate holds. □

Let G be a bounded domain so that $\partial G = \cup_{j=0}^N \Gamma_j$, where each Γ_j is a Jordan curve and $\Gamma_j \cap \Gamma_k = \emptyset$ when $j \ne k$. Let G_0 be the bounded component of $\mathbb{C}\backslash\Gamma_0$. We assume that each Γ_j, $j \ne 0$, is contained in G_0. Let G_j^* be the domain with boundary Γ_j which contains G. *For simplicity, in the following theorem and in the rest of the paper we will assume that all Γ_j are at least $C^{1+\varepsilon}$ for some $\varepsilon > 0$, for all $j = 0, 1, \dots, N$.* The following theorem gives a full description of those weight functions produced by a subharmonic exhaustion (see Remark 2.2).

Theorem 3.3. Let ψ be lower semicontinuous on ∂G so that $\psi \ge s$ for some $s > 0$. Then there exists a function $u \in \mathcal{E}(G)$ so that $\psi = V_u \approx \frac{\partial u}{\partial n}$. Moreover $u \in \mathcal{E}_0(G)$ if and only if $\psi \in L^1(\partial G, d\nu)$.

Proof. For any $j \in \{0, 1, \cdots, N\}$ let $\psi_j(\zeta) = \psi(\zeta)$ if $\zeta \in \Gamma_j$. Let $v_j \in \mathcal{E}(G_j^*)$ be the function for ψ_j given in Lemma 3.2 so that $V_{v_j} = \psi_j$ on ∂G_j. We can choose $c_j < 0$ close to 0 so that the level set B_{c_j, v_j} contains all Γ_k for $k \ne j$. Let $c = \max\{c_0, \cdots, c_N\}$. Set $u_j(z) := \kappa_c(v_j(z))$ and $u(z) := \sum_{j=0}^N u_j(z) - cN$ for $z \in G$. If $\zeta \in \Gamma_j$, then $u_j(\zeta) = 0$ while

$u_k(\zeta) = c$ when $k \neq j$. Hence $u(\zeta) = 0$ and $u \in \mathcal{E}(G)$. On the other hand if $k \neq j$, then $u_j(z) \equiv c$ on an open set containing Γ_k, therefore by Lemma 3.2 and Lemma 2.7

$$\frac{\partial u(\zeta)}{\partial n} = \frac{\partial u_j(\zeta)}{\partial n} = \frac{\partial v_j(\zeta)}{\partial n} \approx \psi(\zeta)$$

when $\zeta \in \Gamma_j$. Together we have that $V_u = \psi$ on ∂G. The equivalence of $u \in \mathcal{E}_0$ and $\psi \in L^1$ follows directly from the identity

$$\int_{\partial G} V_u(\zeta) d\nu(\zeta) = \int_G \Delta u.$$

□

4. Decomposition results and the multiplier algebra of $H^p_u(G)$

From the Cauchy integral formula if $f \in H(G)$, then f has a unique decomposition of the form

$$f(z) = f_0(z) + \ldots + f_N(z), \tag{4.1}$$

where $f_0 \in H(G_0)$, $f_j \in H_0(G^*_j)$ if $j \in \{1, \ldots, N\}$, and $H_0(G^*_j)$ denotes the class of holomorphic functions in $H(G^*_j)$ that vanish at infinity. This holds true for bounded holomorphic functions; from [3] we see that

$$H^\infty(G) = H^\infty(G_0) + H^\infty_0(G^*_1) + \ldots + H^\infty_0(G^*_N),$$

where $H^\infty_0(G^*_j) = H^\infty(G^*_j) \cap H_0(G^*_j)$.

The next theorem tells us that if G is finitely connected, the space $H^p_u(G)$ has a similar decomposition.

Theorem 4.1. Let G be a finitely connected domain whose boundary consists of disjoint Jordan curves Γ_j, $0 \leq j \leq N$, with rectifiable boundary so that Γ_0 surrounds G. Let G_0 be the bounded component of $\mathbb{C} \backslash \Gamma_0$ and for $j \in \{1, \ldots, N\}$, let G^*_j be the domain with boundary Γ_j which contains G. Let $u \in \mathcal{E}_0(G)$ and $p > 0$. Then there exists $u_0 \in \mathcal{E}_0(G_0)$, $u_j \in \mathcal{E}_0(G^*_j)$, $j = 1, \ldots, N$, so that every $f \in H^p_u(G)$ can be represented in the form

$$f(z) = f_0(z) + \cdots + f_N(z), \tag{4.2}$$

where $f_0 \in H^p_{u_0}(G_0)$, $f_j \in H^p_{u_j}(G^*_j) \cap H_0(G^*_j)$ for $j \in \{1, \ldots, N\}$.

Proof. Let $u \in \mathcal{E}_0(G)$ and $f \in H^p_u(G)$. Then the function V_u is lower semicontinuous and $V_u \geq s$ on ∂G for some $s > 0$ (see Remark 2.2). By the proof of Theorem 3.3 it is deduced that there exists $\omega \in \mathcal{E}_0(G)$ so that $V_\omega = V_u$ and ω is of the form

$$\omega = u_0 + \cdots + u_N - cN,$$

where $c < 0$, $u_j \in \mathcal{E}_0(G^*_j)$ and $V_{u_j} = V_u|_{\Gamma_j}$ for $0 \leq j \leq N$.

Since $H^p_u(G) \subset H^p(G)$ it is well-known that (see for example [5, Theorem 10.12]) $f(z) = f_0(z) + \cdots + f_N(z)$, where $f_j \in H^p(G^*_j)$. In fact, each f_j is of the form (after analytic continuation)

$$f_j(z) = \int_{\Gamma_j} \frac{f(\zeta)}{\zeta - z} d\nu(\zeta).$$

Hence for each j the functions $f_0, f_1, \cdots, \hat{f}_j, \cdots, f_N$ are bounded on Γ_j. Here $\hat{f}_j$ means that f_j is not in the list. Therefore, $|f_j(\zeta)|^p \leq |f(\zeta)|^p + m$ when $\zeta \in \Gamma_j$ for some constant m. The norm of f_j is estimated as

$$\begin{aligned} \|f_j\|^p_{H^p_{u_j}(G^*_j)} &= \int_{\Gamma_j} |f_j(\zeta)|^p V_u(\zeta) d\nu \leq \int_{\Gamma_j} |f(\zeta)|^p V_u(\zeta) d\nu + m\|V_u\|_{L^1} \\ &\leq \|f\|^p_{H^p_u(G)} + m\|V_u\|_{L^1}. \end{aligned}$$

Since $f \in H^p_u(G)$ and $u \in \mathcal{E}_0(G)$, $\|f\|_{H^p_u(G)} < \infty$ and $\|V_u\|_{L^1} < \infty$, hence $\|f_j\|_{H^p_{u_j}(G^*_j)} < \infty$ and $f_j \in H^p_{u_j}(G^*_j)$. □

Finally, we describe the boundary values of functions from $H^p_u(G)$ and describe an isomorphism from $H^p_u(G)$ onto $H^p(G)$. We use the same assumptions and notations of Theorem 4.1 in the next statement.

Theorem 4.2. Let $u \in \mathcal{E}_0(G)$ and $p > 0$. Then there exists $u_0 \in \mathcal{E}_0(G_0)$, $u_j \in \mathcal{E}_0(G^*_j)$, and zero-free functions $\varphi_0 \in H^\infty(G_0) \cap H^2_{u_0}(G_0)$ and $\varphi_j \in H^\infty_0(G_0) \cap H^2_{u_j}(G^*_j)$, $j = 1, \ldots, N$, so that

i. $H^p_u(G) = \varphi_0^{2/p} H^p(G_0) + \varphi_1^{2/p} H^p(G^*_1) + \ldots + \varphi_N^{2/p} H^p(G^*_N)$.
ii. If $f \in H^p_u(G)$, then $f^*|_{\Gamma_j} \in L^p(V_{u_j} d\nu)$ for $j = 0, 1, \ldots, N$.
iii. $\|f\|_{H^p_u(G)} \approx \|f_0\|_{H^p_{u_0}(G_0)} + \sum_{j=1}^N \|f_j\|_{H^p_{u_j}(G^*_j)}$ for $f \in H^p_u(G)$, where $f(z) = f_0(z) + \cdots + f_N(z)$ denotes the unique decomposition in (4.2).
iv. The mapping $T : H^p_u(G) \to H^p(G)$ given by

$$Tf = \sum_{j=0}^{N} f_j / \varphi_j^{2/p}$$

is a linear isomorphism of $H^p_u(G)$ onto $H^p(G)$.

Proof. Statements *i.* and *ii.* are direct consequences of Theorem 2.1, Theorem 2.5 and the discussions in this section. For the third statement, let X denote the vector space $H^p_u(G)$ endowed with the norm

$$\|f\|_X = \|f_0\|_{H^p_{u_0}(G_0)} + \sum_{j=1}^{N} \|f_j\|_{H^p_{u_j}(G^*_j)}.$$

For $p \geq 1$, both $H^p_u(G)$ (with the usual norm) and X are Banach spaces. For $0 < p < 1$, we don't have normed spaces, however, these are complete metric spaces. In fact, $H^p_u(G)$ and X are closed subspaces of $L^p(V_u d\nu)$ and several consequences of Baire category theorem, such as closed graph theorem, carry over to both of these spaces. Notice that as sets both $H^p_u(G)$ and X are the same. By closed graph theorem, we deduce that there exists a constant $C > 0$ such that

$$1/C \|f\|_X \leq \|f\|_{H^p_u(G)} \leq C \|f\|_X$$

for every $f \in H^p_u(G)$. This proves *iii.*

By Corollary 2.6, after a conformal mapping of G^*_j onto $\mathbb{D}$, any function $g \in H^p(G^*_j)$ is of the form $g = \varphi_j^{2/p} h$ for some $h \in H^p(G^*_j)$ and $\|g\|_{H^p(G^*_j)} \approx \|h\|_{H^p(G^*_j)}$. It follows from statements *i.* and *iii.* that T gives an isomorphism of $H^p_u(G)$ onto $H^p(G)$. This fisinishes the proof. □

Let X be a Banach space of holomorphic functions on G. The multiplier algebra $M(X)$ of X is defined to be the class of holomorphic functions g on G such that $gf \in X$ for every $f \in X$. By the closed graph theorem a function $g \in H(G)$ belongs to the multiplier algebra $M(H^p_u(G))$ if and only if the multiplication operator M_g which assigns to a function $f \in H(G)$ the product gf is a bounded operator on $H^p_u(G)$.

Corollary 4.3. For $p > 0$, $M(H^p_u(G)) = H^\infty(G)$.

Proof. It is well-known that $M(H^p(\mathbb{D})) = H^\infty(\mathbb{D})$. Hence, if Γ is the rectifiable boundary of a Jordan domain and if U is a connected component (bounded or unbounded) of Γ, then $M(H^p(U)) = H^\infty(U)$. This can be seen by considering a conformal mapping from

$\mathbb{D}$ onto U. By Theorem 4.2 (i), $H^p_u(G) = \varphi_0^{2/p} H^p(G_0) + \varphi_1^{2/p} H^p(G_1^*) + \ldots + \varphi_N^{2/p} H^p(G_N^*)$. Let $g \in M(H^p_u(G))$. Then from Theorem 4.2 (iii), if $f \in H^p_u(G)$, then

$$\|gf\|_{H^p_u(G)} \approx \|gf_0\|_{H^p_{u_0}(G_0)} + \sum_{j=1}^{N} \|gf_j\|_{H^p_{u_j}(G_j^*)} < \infty \tag{4.3}$$

where f_j's are as in (4.2). Let $g(z) = \sum_{j=0}^{N} g_j(z)$ be the unique decomposition of g as described in (4.1). Fix $j \in \{0, \ldots, N\}$. If $i \neq j$, then g_i is bounded near Γ_j, therefore, $\|g_i f_j\|_{H^p_{u_j}(G_j^*)} < \infty$. Combining with (4.3), this means that $\|g_j f_j\|_{H^p_{u_j}(G_j^*)} < \infty$. Since $g_j \in H(G_j^*)$ and f_j can be an arbitrary element of $H^p_{u_j}(G_j^*)$, we see that $g_j \in M(H^p_{u_j}(G_j^*))$, therefore it is bounded on G_j^*. Hence, $g \in H^\infty(G)$ and we have proved that $M(H^p_u(G)) \subset H^\infty(G)$. The reverse inclusion clearly holds from Theorem 4.2(iii). The proof is finished. □

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