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Research Article

Digital Lusternik-Schnirelmann category of digital functions

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Abstract

Roughly speaking, the digital Lusternik-Schnirelmann category of digital images studies how far a digital image is away from being digitally contractible. The digital Lusternik-Schnirelmann category (digital LS category, for short) is defined in [A. Borat and T. Vergili, Digital Lusternik-Schnirelmann category, Turkish J. Math. 2018]. In this paper, we introduce the digital LS category of digital functions. We will give some basic properties and discuss how this new concept will behave if we change the adjacency relation in the domain and in the image of the digital function and discuss its relation with the digital LS category of a digital image.

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1. Introduction

The Lusternik-Schnirelmann category of a space X, first introduced in [21], is the least number n such that there is an open cover $\{U_1, \dots, U_{n+1}\}$ of X with the property that the inclusion $i_j : U_j \hookrightarrow X$ is nullhomotopic in X for each j. For more details, see [11]. The Lusternik-Schnirelmann category of maps is first introduced in [14] and is studied in detail in [1]. For more details see [11, 18, 24] and for a simplicial analog see [23].

The digital LS category of a digital image is introduced in [3] and its close relative, digital topological complexity, is introduced in [19]. For the definition of usual topological complexity and its basic properties, see [13].

This paper is organized as follows. In Section 2, we will recall some basic definitions and theorems from digital topology that we will use throughout this paper. We will also recall the definition of the digital LS category of a digital image and discuss digital LS category of simple closed κ-curves.

In Section 3, we will introduce the definition of the digital LS category of a digital function and introduce the main theorems of this paper, answering questions such as

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how digitally homotopic functions and digital homotopy equivalence affect the digital Lusternik-Schnirelmann category of the given functions or the digital images. Moreover, relations between the digital LS category of a digital function and the digital LS category of a digital image are studied in this section and Section 4.

2. Background

In this section we will recall some basic definitions and theorems from digital topology and recall the digital Lusternik-Schnirelmann category of digital images. At the end of this section we will give a discussion on simple closed κ-curves and compute their digital LS category.

A *digital image* X is a subset of $\mathbb{Z}^n$ for a positive integer n. We impose *an adjacency relation* on $\mathbb{Z}^n$ in order to work on a digital image $X \subset \mathbb{Z}^n$ as follows [7]: Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ be two points in $\mathbb{Z}^n$. Then for $1 \leq \ell \leq n$, p and q are said to be c_ℓ*-adjacent* whenever

- there are at most ℓ indices i such that $|p_i - q_i| = 1$ and
- $p_j = q_j$ for all other indices j satisfying $|p_i - q_i| \neq 1$.

Note that c_ℓ indicates the number of adjacent points in $\mathbb{Z}^n$, according to this adjacency. For instance, we have $c_1 = 2$ in $\mathbb{Z}$, $c_1 = 4$ and $c_2 = 8$ in $\mathbb{Z}^2$, and $c_1 = 6$, $c_2 = 18$, $c_3 = 26$ in $\mathbb{Z}^3$. We usually denote an adjacency relation by Greek letters such as κ, λ, etc., and a digital image by a pair (X, κ) where $X \subseteq \mathbb{Z}^n$ and κ is an adjacency relation inherited from $\mathbb{Z}^n$.

A digital image (X, κ) is said to be κ*-connected* if for any pair of elements x and x' in X, there exists a sequence $\{x_i\}_{i=0}^n \subset X$ such that $x = x_0$, $x' = x_n$ and x_i and x_{i+1} are κ-adjacent for $0 \leq i < n$ [17].

In [4], *a digital interval* is defined as a subset of $\mathbb{Z}$ of the form

$$[a, b]_{\mathbb{Z}} = \{n \in \mathbb{Z} \mid a \leq n \leq b\}.$$

where 2-adjacency is assumed.

Definition 2.1. ([5]) Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)*-continuous* if $f(x)$ and $f(x')$ are λ-adjacent or equal in Y whenever x and x' are κ-adjacent in X.

Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is called (κ, λ)-*isomorphism* [8] (called a *homeomorphism* rather than isomorphism in [4,5]) if f is (κ, λ)-continuous and bijective and further its inverse $f^{-1} : Y \to X$ is (λ, κ)-continuous.

Theorem 2.2. ([5]) *Let $f : X \to Y$ and $g : Y \to Z$ be (κ_0, κ_1)-continuous and (κ_1, κ_2)-continuous functions respectively. Then the composite function $g \circ f : X \to Z$ is (κ_0, κ_2)-continuous.*

Definition 2.3. ([5,20]) Let $f, g : X \to Y$ be (κ, λ)-*continuous* functions. If there exist a positive integer m and a function

$$F : X \times [0, m]_{\mathbb{Z}} \to Y$$

with the following conditions, then F is said to be a (κ, λ)-*homotopy*, and f and g are called (κ, λ)-*homotopic* in Y (denoted by $f \simeq_{\kappa,\lambda} g$).

(i) For all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$.

(ii) For all $x \in X$, the induced function $F_x : [0, m]_{\mathbb{Z}} \to Y$ defined by $F_x(t) = F(x, t)$ is $(2, \lambda)$-continuous.

(iii) For all $t \in [0, m]_{\mathbb{Z}}$, the induced function $F_t : X \to Y$ defined by $F_t(x) = F(x, t)$ is (κ, λ)-continuous.

Definition 2.4. ([6]) A (κ, λ)-continuous map $f : X \to Y$ is said to be a (κ, λ)-homotopy equivalence if there exists a (λ, κ)-continuous map $g : Y \to X$ such that $g \circ f \simeq_{\kappa,\kappa} \mathrm{Id}_X$ and $f \circ g \simeq_{\lambda,\lambda} \mathrm{Id}_Y$ where Id_X and Id_Y are the identity maps on X and Y respectively. We say X and Y are (κ, λ)-homotopy equivalent if there is a (κ, λ)-homotopy equivalence from X to Y.

We call a (κ, λ)-continuous map $f : X \to Y$ (κ, λ)-*nullhomotopic* if it is (κ, λ)-homotopic to a constant function $c : X \to Y$, $c(x) = c_0$ for some $c_0 \in Y$ [4].

Throughout the paper, a *cover* of a digital image (X, κ) means a collection of subsets $\{U_i\}$ of X whose union equals X.

Definition 2.5. ([3]) The *digital LS category* of a digital image (X, κ) is the least integer n such that there exists a cover $\{U_1, U_2, \dots, U_{n+1}\}$ of X where each inclusion map $i_i : U_i \hookrightarrow X$ for $i = 1, \dots, n+1$ is (κ, κ)-nullhomotopic (or κ-nullhomotopic for short) in X. This will be denoted by $\mathrm{cat}_\kappa(X) = n$.

Notice that $\mathrm{cat}_\kappa(X)$ can be at most the number of lattice points in X.

The following definition is given in [10]. Suppose κ_1 and κ_2 are two adjacency relations on a set X. Then we say that κ_1 *dominates* κ_2, $\kappa_1 \geq_d \kappa_2$, if for $x, x' \in X$, if x and x' are κ_1-adjacent then x and x' are κ_2-adjacent.

Remark 2.6. Suppose $u \leq v$. In [3], the authors used the notations "$\kappa \leq \lambda$" or "$\lambda \geq \kappa$" in such a way that "$c_u \leq c_v$"; however, by the definition of these notations in [10] we have "$c_v \leq c_u$". For consistency in the literature, we have chosen to use the definition of [10].

As in the traditional algebraic topology setting, the digital LS category is a homotopy invariant in the digital sense.

Theorem 2.7. ([3]) *If the digital images (X, κ) and (Y, λ) are (κ, λ)-homotopy equivalent, then $\mathrm{cat}_\kappa(X) = \mathrm{cat}_\lambda(Y)$.*

A *simple closed κ-curve* S in a digital image (X, κ) is a sequence $\{x_i\}_{i=0}^{m-1}$ for $m \geq 4$ in X such that x_i and x_j are κ-adjacent if and only if $j = (i \pm 1) \bmod m$ [5].

Recall that if the identity map $\mathrm{Id}_X : X \to X$ is (κ, κ)-homotopic to a constant function $c : X \to X$, $c(x) = c_0$ for some c_0 in X, then (X, κ) is called κ-*contractible* [4, 20].

Theorem 2.8. ([9]) *A simple closed κ-curve S which contains more than four points is not κ-contractible.*

However, if we remove a point from a simple closed κ-curve S with $|S| > 4$, we get a κ-contractible digital image.

Proposition 2.9. *Let $S = \{x_i\}_{i=0}^{m}$ be a simple closed κ-curve with $m \geq 4$. Then $S \setminus \{x_m\}$ is κ-contractible.*

Proof. The desired digital κ-homotopy function between the identity map and a constant map on $S \setminus \{x_m\}$ as follows.

$$H : S \setminus \{x_m\} \times [0, m-1]_{\mathbb{Z}} \to S \setminus \{x_m\}$$

$$(x_i, t) \mapsto H(x_i, t) = \begin{cases} x_{i-t} & 0 \leq t \leq i \\ x_0 & \text{otherwise.} \end{cases}$$

$\square$

Theorem 2.10. *Let S be a simple closed κ-curve with $|S| > 4$. Then $\mathrm{cat}_\kappa(S) = 1$.*

Proof. Let $S = \{x_i\}_{i=0}^{m}$ with $m \geq 4$. Then $\text{cat}_\kappa(S) > 0$ since S is not κ-contractible. Consider the subsets of S, $U_1 = \{x_m\}$ and $U_2 = S \setminus \{x_m\}$. Then the inclusion map $i_i : U_i \to S$ for $i = 1, 2$ is κ-nullhomotopic since U_1 is a singleton set and U_2 is κ-contractible by Proposition 2.9. □

3. Digital Lusternik-Schnirelmann category of digital functions

Definition 3.1. The digital LS category of a (κ, λ)-continuous function $f : X \to Y$ is defined to be the least integer n such that there is a cover $\{U_1, \ldots, U_{n+1}\}$ of X such that $f|_{U_j}$ is (κ, λ)-nullhomotopic for each j.

We will denote it by $\text{cat}_{\kappa,\lambda}(f)$.

Notice that $\text{cat}_{\kappa,\lambda}(f)$ can be no more than the number of lattice points of X.

Remark 3.2. It is obvious that $\text{cat}_\kappa(X) = \text{cat}_{\kappa,\kappa}(\text{Id}_X)$ (it follows from the definition of digital LS category).

Proposition 3.3. *Suppose κ_1 and κ_2 are two adjacency relations on a set X with $\kappa_1 \geq_d \kappa_2$ and λ is an adjacency relation on a set Y. If f is (κ_1, λ)-continuous then $\text{cat}_{\kappa_1,\lambda}(f) \geq \text{cat}_{\kappa_2,\lambda}(f)$.*

Proof. Since $\kappa_1 \geq_d \kappa_2$, a function that is (κ_1, λ)-continuous is (κ_2, λ)-continuous. Therefore, f is (κ_2, λ)-continuous, and (κ_1, λ)-homotopies are (κ_2, λ)-homotopies. The assertion follows easily from Definition 3.1. □

Proposition 3.4. *Suppose λ_1 and λ_2 are two adjacency relations on a set Y with $\lambda_2 \geq_d \lambda_1$ and κ is an adjacency relation on a set X. If f is (κ, λ_2)-continuous then $\text{cat}_{\kappa,\lambda_1}(f) \leq \text{cat}_{\kappa,\lambda_2}(f)$*

Proof. Since f is (κ, λ_2)-continuous and $\lambda_2 \geq_d \lambda_1$, f is also (κ, λ_1)-continuous. Let $\text{cat}_{\kappa,\lambda_2}(f) = n$. Then there are $U_1, \ldots, U_{n+1}$ subsets of X covering X such that $f|_{U_j} : U_j \to X$ is (κ, λ_2)-nullhomotopic, for each j. That is, there is a (κ, λ_2)-homotopy $H^j : U_j \times [0, m]_{\mathbb{Z}} \to Y$ between $f|_{U_j}$ and a constant function $c_j : (U_j, \kappa) \to (Y, \lambda_2)$. If we consider λ_1-adjacency on Y and consider $G^j : U_j \times [0, m]_{\mathbb{Z}} \to Y$ defined by $G^j(x, t) = H^j(x, t)$, the function G^j becomes a (κ, λ_1)-homotopy which follows from the idea that two λ_2-adjacent points are λ_1-adjacent since $\lambda_2 \geq_d \lambda_1$. Thus $\text{cat}_{\kappa,\lambda_1}(f) \leq n$. □

Example 3.5. Let (X, κ) and (Y, λ) be digital images. If $f : X \to Y$ is (κ, λ)-nullhomotopic, $\text{cat}_{\kappa,\lambda}(f) = 0$.

Theorem 3.6. *Suppose X is κ-connected and Y is λ-connected. If $f : X \to Y$ is a (κ, λ)-continuous function, then $\text{cat}_{\kappa,\lambda}(f) \leq \min\{\text{cat}_\kappa(X), \text{cat}_\lambda(Y)\}$.*

Proof. Let $\text{cat}_\kappa(X) = n$. Then there is a cover $\{U_1, \ldots, U_{n+1}\}$ of X such that the inclusion $i_j : U_j \hookrightarrow X$ is κ-nullhomotopic in X for each j. So for each j, we can write a (κ, κ)-homotopy

$$H^j : U_j \times [0, m]_{\mathbb{Z}} \to X \quad \text{satisfying}$$

(1a) For all $x \in U_j$, $H^j(x, 0) = i_j(x)$ and $H^j(x, m) = c_j$ for some $c_j \in X$.
(1b) For all $x \in U_j$, $H^j_x : [0, m]_{\mathbb{Z}} \to X$ defined by $H^j_x(t) = H^j(x, t)$ is $(2, \kappa)$-continuous.
(1c) For all $t \in [0, m]_{\mathbb{Z}}$, $H^j_t : U_j \to X$ defined by $H^j_t(x) = H^j(x, t)$ is (κ, κ)-continuous.

Define functions as follows.

$$F^j : U_j \times [0, m]_{\mathbb{Z}} \to Y$$
$$(x, t) \to F^j(x, t) = f(H^j(x, t))$$

If we verify that the following three conditions hold (i.e., F^j is a (κ, λ)-homotopy), then it follows that $\mathrm{cat}_{\kappa,\lambda}(f) \leq \mathrm{cat}_\kappa(X)$.

(2a) For all $x \in U_j$, $F^j(x,0) = f(H^j(x,0)) = f(i_j(x)) = f|_{U_j}(x)$ and $F^j(x,m) = f(H^j(x,1)) = f(c_j)$ is constant in Y.

(2b) For all $x \in U_j$, $F^j_x : [0,m]_\mathbb{Z} \to Y$ defined by $F^j_x(t) = F^j(x,t) = f(H^j_x(t))$ is $(2,\lambda)$-continuous by Theorem 2.2 since it is a composition of a (κ,λ)-continuous map f and a $(2,\kappa)$-continuous map H^j_x.

(2c) For all $t \in [0,m]_\mathbb{Z}$, $F^j_t : U_j \to Y$ defined by $F^j_t(x) = F^j(x,t) = f(H^j_t(x))$ is (κ,λ)-continuous by Theorem 2.2 since it is a composition of a (κ,λ)-continuous map f and a (κ,κ)-continuous map H^j_t.

For the second half of the proof, suppose that $\mathrm{cat}_\lambda(Y) = k$. Then there is a cover $\{V_1, \ldots, V_{k+1}\}$ of Y such that the inclusion $i_j : V_j \hookrightarrow Y$ is λ-nullhomotopic in Y for each j. So for each j, we can write a (λ,λ)-homotopy

$$H^j : V_j \times [0,m]_\mathbb{Z} \to Y \quad \text{satisfying}$$

(3a) For all $y \in V_j$, $H^j(y,0) = i_j(y)$ and $H^j(y,m) = \bar{d}_j(y) = d_j$ where $\bar{d}_j : V_j \to Y$ defined by $\bar{d}_j(y) = d_j$ is a constant function for some $d_j \in Y$.

(3b) For all $y \in V_j$, $H^j_y : [0,m]_\mathbb{Z} \to Y$ defined by $H^j_y(t) = H^j(y,t)$ is $(2,\lambda)$-continuous.

(3c) For all $t \in [0,m]_\mathbb{Z}$, $H^j_t : V_j \to Y$ defined by $H^j_t(y) = H^j(y,t)$ is (λ,λ)-continuous.

Define functions as follows.

$$F^j : f^{-1}(V_j) \times [0,m]_\mathbb{Z} \to Y$$
$$(x,t) \to F^j(x,t) = H^j(f(x),t)$$

If we verify that it is a (κ,λ)-homotopy, then it follows that $\mathrm{cat}_{\kappa,\lambda}(f) \leq \mathrm{cat}_\lambda(Y)$ and this completes the proof.

(4a) For all $x \in f^{-1}(V_j)$, $F^j(x,0) = H^j(f(x),0) = i_j(f(x)) = f|_{f^{-1}(V_j)}(x)$ and $F^j(x,m) = H^j(f(x),m) = \bar{d}_j(f(x)) = d_j$ is constant in Y.

(4b) For all $x \in f^{-1}(V_j)$, $F^j_x : [0,m]_\mathbb{Z} \to Y$ defined by $F^j_x(t) = F^j(x,t) = H^j_{f(x)}(t)$ is $(2,\lambda)$-continuous from (3b).

(4c) For all $t \in [0,m]_\mathbb{Z}$, $F^j_t : f^{-1}(V_j) \to Y$ defined by $F^j_t(x) = F^j(x,t) = H^j_t(f(x))$ is (κ,λ)-continuous by Theorem 2.2 since it is a composition of a (κ,λ) continuous map f and a (λ,λ)-continuous map H^j_t.

□

Proposition 3.7. *If $f : X \to Y$ is (κ,λ)-continuous and $g : Y \to Z$ is (λ,η)-continuous then* $\mathrm{cat}_{\kappa,\eta}(g \circ f) \leq \min\{\mathrm{cat}_{\kappa,\lambda}(f), \mathrm{cat}_{\lambda,\eta}(g)\}$.

Proof. In the first half of the proof we show that $\mathrm{cat}_{\kappa,\eta}(g \circ f) \leq \mathrm{cat}_{\kappa,\lambda}(f)$. Let $\mathrm{cat}_{\kappa,\lambda}(f) = n$. Then there is a cover $\{U_1, \ldots, U_{n+1}\}$ of X such that $f|_{U_j} : U_j \to Y$ is (κ,λ)-nullhomotopic (i.e., it is (κ,λ)-homotopic to a constant function $c_j : (U_j,\kappa) \to (Y,\lambda)$) for each j.

$(g \circ f)|_{U_j} = g \circ (f|_{U_j}) \simeq_{\kappa,\eta} g \circ c_j \simeq_{\kappa,\eta} \bar{c}_j$ where $\bar{c}_j : (U_j,\kappa) \to (Z,\eta)$ is a constant function. So $\mathrm{cat}_{\kappa,\lambda}(g \circ f) \leq n$.

In the second half of the proof, we show that $\mathrm{cat}_{\kappa,\eta}(g \circ f) \leq \mathrm{cat}_{\lambda,\eta}(g)$. Let $\mathrm{cat}_{\lambda,\eta}(g) = m$. Then there is a cover $\{V_1, \ldots, V_{m+1}\}$ of Y such that $g|_{V_j} : V_j \to Y$ is (λ,η)-nullhomotopic (i.e., it is (λ,η)-homotopic to a constant function $d_j : (V_j,\lambda) \to (Z,\eta)$) for each j.

Take $U_j := f^{-1}(V_j)$ for each j. Notice that the union of U_j's is X.

For all $x \in f^{-1}(V_j)$, we have

$$(g \circ f)(x) = g(f|_{f^{-1}(V_j)}(x)) = g|_{V_j}(f|_{f^{-1}(V_j)}(x)) = (g|_{V_j} \circ f|_{f^{-1}(V_j)})(x).$$

In other words, $(g \circ f)|_{U_j} = g|_{V_j} \circ f|_{U_j}$.

From the assumption, $g|_{V_j} \simeq_{\lambda,\eta} d_j$. Hence $(g \circ f)|_{U_j} = g|_{V_j} \circ f|_{U_j} \simeq_{\kappa,\eta} d_j \circ f|_{U_j} \simeq_{\kappa,\eta} \bar{d}_j$ where $\bar{d}_j : (U_j, \kappa) \to (Z, \eta)$ is some constant function. Hence $\mathrm{cat}_{\kappa,\eta}(g \circ f) \leq m$. □

Proposition 3.8. *If $f, g : X \to Y$ are (κ, λ)-homotopic, then $\mathrm{cat}_{\kappa,\lambda}(f) = \mathrm{cat}_{\kappa,\lambda}(g)$.*

Proof. It suffices to show that $\mathrm{cat}_{\kappa,\lambda}(f) \leq \mathrm{cat}_{\kappa,\lambda}(g)$. Let $\mathrm{cat}_{\kappa,\lambda}(g) = n$. Then there is a cover $\{U_1, \ldots, U_{n+1}\}$ of X such that $g|_{U_j} : U_j \to Y$ is (κ, λ)-nullhomotopic (that is, it is (κ, λ)-homotopic to a constant function $c_j : (U_j, \kappa) \to (Y, \lambda)$ for each j). Since $f \simeq_{\kappa,\lambda} g$, we have $f|_{U_j} \simeq_{\kappa,\lambda} g|_{U_j}$ for each j. Hence $f|_{U_j}$ is (κ, λ)-nullhomotopic. □

Corollary 3.9. *Let $f : X \to Y$ be a (κ, λ)-continuous function. Then $\mathrm{cat}_{\kappa,\lambda}(f) = 0$ if and only if $f \simeq_{\kappa,\lambda} c$ where $c : X \to Y$ is a constant function.*

Proof. This follows from Definition 3.1 and Proposition 3.8. □

Theorem 3.10. *If $f : X \to Y$ is (κ, λ)-homotopy equivalence, then $\mathrm{cat}_{\kappa,\lambda}(f) = \mathrm{cat}_{\kappa}(X) = \mathrm{cat}_{\lambda}(Y)$.*

Proof. By Remark 3.2, $\mathrm{cat}_{\kappa}(X) = \mathrm{cat}_{\kappa,\kappa}(\mathrm{Id}_X)$. Let g be a homotopy inverse of f. That is, $g : Y \to X$ is a (λ, κ)-continuous function such that $g \circ f \simeq_{\kappa,\kappa} \mathrm{Id}_X$ and $f \circ g \simeq_{\lambda,\lambda} \mathrm{Id}_Y$. Then we have

$$\mathrm{cat}_{\kappa}(X) = \mathrm{cat}_{\kappa,\kappa}(\mathrm{Id}_X) = \mathrm{cat}_{\kappa,\kappa}(g \circ f) \leq \mathrm{cat}_{\kappa,\lambda}(f) \leq \mathrm{cat}_{\kappa}(X).$$

Note that the second equality and the first and the second inequalities follow from Proposition 3.8, Proposition 3.7, and Theorem 3.6 respectively.

The assertion follows from Theorem 2.7. □

Corollary 3.11. *Let $f : X \to Y$ be a (κ, λ)-homotopy equivalence from a κ-contractible digital image (or to a λ-contractible digital image). Then $\mathrm{cat}_{\kappa,\lambda}(f) = 0$.*

Proof. This follows from Theorem 3.10. □

Note that a set X in $\mathbb{Z}^n$ is *symmetric with respect to the origin* if X has the property that x is an element in X if and only if $-x$ is an element in X. If $X \subset \mathbb{Z}^n$ and $Y \subset \mathbb{Z}^m$ are two digital images and X is symmetric, a map $f : X \to Y$ which satisfies $f(-x) = -f(x)$ for all $x \in X$ is called an *antipodal map* [9]. For the definition of a symmetric subset of an Euclidean space with respect to the origin and an antipodal map on it, see [12], pp. 261.

Example 3.12. Let MSC_4 be a digital image in $\mathbb{Z}^2$ 4-isomorphic to

$$\{c_0 = (1, -1), c_1 = (1, 0), c_2 = (1, 1), c_3 = (0, 1), c_4 = (-1, 1), c_5 = (-1, 0), c_6 = (-1, -1), c_7 = (0, -1)\}$$

(see Figure 1). Then $\mathrm{cat}_4(\mathrm{MSC}_4) = 1$. Note that this assertion appeared in [3], but the argument offered as proof in [3] only establishes that $\mathrm{cat}_4(\mathrm{MSC}_4) \in \{0, 1\}$. To complete the proof, it suffices to show that $\mathrm{cat}_4(\mathrm{MSC}_4) \neq 0$ and this follows from the observation that MSC_4 is not 4-contractible [9]; Corollary 3.9, which shows that $\mathrm{cat}_{4,4}(\mathrm{Id}_{\mathrm{MSC}_4}) \neq 0$; and Remark 3.2.

Remark 3.13. The proof of $\mathrm{cat}_4(\mathrm{MSC}_4) = 1$ also follows from Theorem 2.10 since MSC_4 is a simple closed 4-curve.

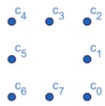

Figure 1. MSC_4

The definition of wedge in digital topology was introduced in [16]. We refer to [8] for a corrected definition. Let (X, κ) be a digital image such that $X = X_0 \cup X_1$ where $X_0 \cap X_1 = \{x_0\}$ and if x and y are κ-adjacent for $x \in X_0$ and $y \in X_1$ then $x_0 \in \{x, y\}$. In this case, X is called the *wedge* of X_0 and X_1 and denoted $X = X_0 \vee X_1$ and x_0 is called the *wedge point*.

Lemma 3.14. *Let $X_0 = X_1 = \mathrm{MSC}_4$ and consider $X = X_0 \vee X_1$ where x_0 is the wedge point of X. Then X is not 4-contractible.*

Proof. Assume that X is 4-contractible. Then there exists a digital homotopy $H : X \times [0, m]_{\mathbb{Z}} \to X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$. Consider the digital $(4, 4)$-continuous function $r : X \to X_0$ where $r(x) = x$ for $x \in X_0$ and $r(x) = x_0$ otherwise. Then the digital map $G : X_0 \times [0, m]_{\mathbb{Z}} \to X_0$ defined by $G(x, t) = r \circ H(x, t)$ gives a digital $(4, 4)$-homotopy between an identity map on X_0 and a constant map at x_0 which is a contradiction with the fact that MSC_4 is not 4-contractible [9]. $\square$

Example 3.15. Consider the digital image $\mathrm{MSC}_4 \vee \mathrm{MSC}_4$ with the wedge point c_0 as shown in Figure 2. It can easily be seen that the subsets $U = \{c_4, c_3, c_2, c_1, c_0, d_1, d_2, d_3, d_4\}$ and $V = \{c_5, c_6, c_7, c_0, d_7, d_6, d_5\}$ are 4-contractible and cover $\mathrm{MSC}_4 \vee \mathrm{MSC}_4$ so that the inclusion maps $i_U : U \to \mathrm{MSC}_4 \vee \mathrm{MSC}_4$ and $i_V : V \to \mathrm{MSC}_4 \vee \mathrm{MSC}_4$ are both 4-nullhomotopic. Hence $\mathrm{cat}_4(\mathrm{MSC}_4 \vee \mathrm{MSC}_4) \le 1$. By Lemma 3.14, the identity function on $\mathrm{MSC}_4 \vee \mathrm{MSC}_4$ cannot have LS category value 0 for 4-adjacency and it follows from Remark 3.2 that $\mathrm{cat}_4(\mathrm{MSC}_4 \vee \mathrm{MSC}_4) \neq 0$. Hence $\mathrm{cat}_4(\mathrm{MSC}_4 \vee \mathrm{MSC}_4) = 1$.

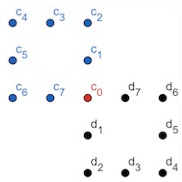

Figure 2. $\mathrm{MSC}_4 \vee \mathrm{MSC}_4$

Example 3.16. Define the antipodal map $f : (\mathrm{MSC}_4, \kappa_1) \to (\mathrm{MSC}_4, \kappa_2)$, $f(c_i) = c_{i+4(\mathrm{mod}\, 8)}$ (see Figure 3). Note that f is (κ_1, κ_2)-continuous whenever $\kappa_2 \ge \kappa_1$ or $\kappa_1 = \kappa_2$ where $\kappa_1, \kappa_2 \in \{4, 8\}$.

- Take $\kappa_1 = \kappa_2 = 8$. Since MSC_4 is 8-contractible [15, 16], $1 \simeq_{8,8} c$ where 1 and c are the identity map and constant map respectively. Then $f \simeq_{8,8} f \circ c$, that is f is also homotopic to a constant map. Hence $\mathrm{cat}_{8,8}(f) = 0$.
- Take $\kappa_1 = 4$ and $\kappa_2 = 8$. By Theorem 3.6 and Example 2.7 in [3], we have
$$\mathrm{cat}_{4,8}(f) \le \min\{\mathrm{cat}_4(\mathrm{MSC}_4) = 1, \mathrm{cat}_8(\mathrm{MSC}_4) = 0\}$$
and hence $\mathrm{cat}_{4,8}(f) = 0$.
- Take $\kappa_1 = \kappa_2 = 4$. Since $f \circ f$ is the identity function, it follows from Theorem 3.10 that $\mathrm{cat}_{4,4}(f) = 1$.

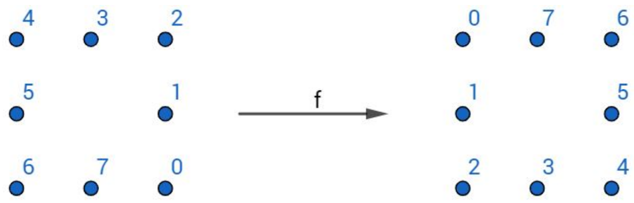

Figure 3. The antipodal map f from MSC_4 to itself.

4. Digital LS category of digital diagonal map

Definition 4.1. ([2,22]) For (X,κ) and (Y,λ), the normal (or strong) product adjacency, $NP(\kappa,\lambda)$, on $X\times Y$ is defined as follows. Two elements (x_1,y_1) and (x_2,y_2) in $X\times Y$ are $NP(\kappa,\lambda)$-adjacent if either

(i) $x_1=x_2$ and y_1, y_2 are λ-adjacent or
(ii) $y_1=y_2$ and x_1, x_2 are κ-adjacent or
(iii) x_1, x_2 are κ-adjacent and y_1, y_2 are λ-adjacent.

It is an easy exercise to see that the digital diagonal map $\Delta_X : X\to X\times X$, $\Delta_X(x)=(x,x)$, is $(\kappa, NP(\kappa,\kappa))$-continuous.

Theorem 4.2. $\mathrm{cat}_{\kappa,NP(\kappa,\kappa)}(\Delta_X)=\mathrm{cat}_\kappa(X)$.

Proof. We can regard the function Δ_X as from X to $\Delta_X(X)$. Let $\mathrm{pr}_1:\Delta_X\to X$ be the projection map $\mathrm{pr}_1(x,x)=x$. We see easily that pr_1 is $(NP(\kappa,\kappa),\kappa)$-continuous and is the inverse of the function Δ_X. Therefore, Δ_X is a $(\kappa,NP(\kappa,\kappa))$-isomorphism. The assertion follows from Theorem 3.10. □

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