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RISK PREMIUM FOR DEPENDENT RISKS USING UTILITY COPULAS AND RISK AVERSION

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Abstract: In order to explain the dependency structure of random variables, copula functions are frequently used in areas such as insurance, actuarial and risk. In addition, the concept of risk aversion can be considered as a decision making parameter and insurance companies can calculate the risk premium by taking advantage of this parameter. In this study, risk aversion coefficient and risk premium based on utility copulas were calculated for dependent bivariate risks. For this, bivariate risk aversion coefficient and risk premium vector of the utility copula defined in Kettler (2007) [16] were found. Numerical results are presented in tables and graphs for various dependency parameter values.

Key words: Bivariate risk aversion, utility copula, dependence, bivariate utility function, risk premium vector.
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1. Introduction

In this study, it is aimed to find bivariate risk aversion functions and risk premiums for utility copulas based on utility functions under dependency conditions [16]. First of all, bivariate risk aversion and risk premium functions based on this utility copula model were obtained. Then, the changes of the risk premiums according to different dependency parameters were examined. Finally, the numerical results are presented and interpreted with tables and graphs.

The utility function is often used in the modelling of risky alternatives in many areas such as economics [8], finance [12] and insurance [9, 10]. It is also utilized as a tool for decision making in portfolio preference, fund preference and securities investments [27]. Thus, each decision maker may be considered to have a utility function which expresses his or her own preferences [7]. Eeckhoudt et al. examined the future state of wealth and health conditions associated with the utility function by looking at the current savings of individuals [14]. Denuit et al. provided optimal solutions for the bivariate utility function and related associated financial risks [11].

The utility function forms the basis of the expected utility theory. If a decision maker's preferences related to risky alternatives provide the axioms of the expected utility theory (completeness, reflexivity, transitivity, monotonicity, continuity, independence), the choices of this decision maker can be represented by a utility function.

However, Tasdemir [26], particularly asserts that this theory is not sufficient to provide some axioms of interrelated risky preferences showed on Ellsberg and Allais paradox examples. In this case, for the first time, Kettler [16] stated that utility copulas could be used to explain the dependency structure of the related risky preferences. Then, in the literature, Archimedean utility copulas [1] and utility copula construction methods [3] related studies were conducted.

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In the field of risk management and insurance, it is very important to determine the risk premiums and risk aversion to make the right decision among the related risky choices [15]. Risk aversion is a basic parameter that determines how much benefit is provided from a property or money [28]. On the other hand, the risk aversion measure helps to determine the least risky choice for a decision-maker who has to choose among risky preferences.

The measure of risk aversion can be determined by the utility function and utility copula. The first studies in the literature were performed according to the utility function. Arrow [5, 6] and Pratt [22] described a measure of the univariate risk aversion according to the univariate utility function. Duncan [13] obtained the univariate risk premium by using the bivariate risk aversion function based on the bivariate utility function. In the literature, there are also other studies related on univariate and bivariate risk aversion coefficients [7, 20, 21, 23, 25, 28].

In the literature, there are very few studies on risk aversion measures based on utility copulas. In one of these studies Kettler [16] gave the methods of construction utility copula. Then, for the first time, Abbas [1] referred to the risk aversion measures based on utility copulas for dependent risks. There are also his other studies on this subject [2, 3, 4].

On the other hand, in reality, there is a need to calculate the correct risk premiums for dependent risks. Therefore, in this paper, we study bivariate risk aversion and risk premium inferences using the utility copula model introduced by Kettler [16] for the dependency structure of the risks.

The definitions of utility function and utility copula are given in Section 2 of the paper. The bivariate risk aversion and risk premiums based on the utility copula are also defined in Section 2. Section 3 concludes the inferences of bivariate risk aversion and risk premium for logarithmic utility copula function in the literature. In the same section, numerical results obtained for various dependency parameter values are presented and interpreted with some tables and graphs. Finally, a brief summary of the paper and some evaluations are included in Section 4.

2. Preliminaries

In this section, firstly, the methods of constructions of the utility copulas are introduced by utility functions. Then the definitions related to bivariate risk aversion measures and risk premiums using utility copula are given.

Copulas are often used to model dependence among dependent random variables in actuarial and finance areas [18]. When the marginal distributions don't have normal distribution, it is an important statistical tool to identify the relationship between the dependence risk groups. In addition, when the dependency is modelled with copula functions, it is not necessary to know the joint distributions of the variables. Because of these advantages, it has many real applications [17].

Copula function was first defined by Sklar [24] as follows: $H(x, y)$, a two-dimensional continuous distribution function with marginals F and G, can be expressed by $H(x, y) = C(F(x), G(y))$ and there exists a unique bivariate copula $C : [0, 1]^2 \to [0, 1]$ [21].

Utility functions are often used in risk and actuarial applications to make appropriate decisions under uncertainty. On the other hand, it is an important tool for calculating some risk quantities such as risk aversion, risk seeking and risk premiums [11].

A set of bivariate utility functions is given as follows:

$$U = \left\{ \mathtt{u}\left(x, y\right) \in C^2 : A = [a, \infty) \times [b, \infty) \to R \right\}$$

when $\frac{\partial \mathtt{u}}{\partial x} > 0$, $\frac{\partial \mathtt{u}}{\partial y} > 0$, $\frac{\partial^2 \mathtt{u}}{\partial x^2} < 0$, $\frac{\partial^2 \mathtt{u}}{\partial y^2} < 0$, $\frac{\partial^2 \mathtt{u}}{\partial x \partial y} = \frac{\partial^2 \mathtt{u}}{\partial y \partial x} < 0$, $\frac{\partial^2 \mathtt{u}}{\partial x^2}\frac{\partial^2 \mathtt{u}}{\partial y^2} - \frac{\partial^2 \mathtt{u}}{\partial x \partial y}\frac{\partial^2 \mathtt{u}}{\partial y \partial x} > 0$.

Depending on the utility function, bivariate utility copula $C(u, v)$ is defined by Kettler [16] as follows:

$$C\left(u, v\right) = -\mathtt{u}\left(\mathtt{u}_1^{-1}\left(u + k\right), \mathtt{u}_2^{-1}\left(v + k\right)\right) + \left(u + k\right) + k > 0, \qquad \mathtt{u}\left(x, y\right) \in R^2 \tag{2.1}$$

where $\mathtt{u}_1(x) = \mathtt{u}(x,b)$ and $\mathtt{u}_2(y) = \mathtt{u}(a,y)$ are marginal utility functions and $\mathtt{u}(a,b) = \mathtt{u}_1(a) = \mathtt{u}_2(b) = k$.

Bivariate risk aversion matrix is defined by the equation (2.2) depending on the utility copula given in equation (2.1) [1].

$$RA = r^{ij} = -\begin{bmatrix} \frac{r_{11}}{r_1} & \frac{r_{12}}{r_1} \\ \frac{r_{21}}{r_2} & \frac{r_{22}}{r_2} \end{bmatrix}, \qquad i,j = 1,2 \tag{2.2}$$

where components of RA are found as $r_{ij} = \partial^2 C(u,v)/\partial u \partial v$, $r_{ji} = \partial^2 C(u,v)/\partial v \partial u$ for $i \neq j$ and $r_{ii} = \partial^2 C(u,v)/\partial u^2$, $r_{jj} = \partial^2 C(u,v)/\partial v^2$ for $i = j$. Also marginal copula functions are found as $r_1 = \frac{d}{du} C(u,v)$ and $r_2 = \frac{d}{dv} C(u,v)$.

Also in RA, the partial derivatives r_1 and r_2 are positive and the sign of RA is based on the sign of the second derivatives that is r_{ij} , $i = 1, 2$, $j = 1, 2$. If the copula function is a concave function of u and v, the risk aversion function RA is positive, so the decision maker is a risk averse. If the copula function is a convex, risk aversion function RA is negative and decision maker is a risk seeking [1].

Risk premium refers to the net premium amount calculated to cover the possible loss amount and the number of losses. The approximately bivariate risk premium vector is given by Duncan [13] as follows:

$$\pi = \frac{1}{2} diag[(RA)\,\Sigma] \tag{2.3}$$

where $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ is variance-covariance matrix of risks. If a decision maker is risk averse ($RA > 0$), he will agree to pay more premiums ($\pi > 0$). If a decision maker is a risk seeking ($RA < 0$), he will keen to pay less premiums ($\pi < 0$).

In other words, the statistical premium is an increasing function of risk aversion [19]. That is, the more a decision maker wants to be risk aversion, the more risk premium should agree to pay.

3. Risk aversion matrix and risk premium vector for utility copula

In this section of the study, the bivariate risk aversion matrix and risk premium vector are obtained for logarithmic substitution utility copula function given in Kettler [16]. Accordingly, here the logarithmic substitution utility function, $\mathtt{u}(x,y)$ and utility copula, $C(u,v)$ are defined as following, respectively:

$$\mathtt{u}(x,y) = \ln(x+y-1), \qquad (x,y) \in [1,\ \infty)^2 \tag{3.1}$$

and

$$C(u,v) = \ (u+v-(\ln(e^u+e^v-1)), (u,v) \in (0,\infty)^2\ . \tag{3.2}$$

Here, utility copula model given in equation (3.2) is derived using equation (2.1) for logarithmic utility function, $\mathtt{u}(x,y)$ in equation (3.1). In addition, the copula function defined in equation (3.2) is extended to a one-parameter copula family as follows:

$$C_\theta(u,v) = \ \left(u^\theta + v^\theta - \ln(e^{u^\theta} + e^{v^\theta} - 1)\right)^{1/\theta}, (u,v) \in (0,\infty)^2, \quad \theta \in (0,\infty)\,. \tag{3.3}$$

In this study, we obtain the risk aversion matrix (RA) and risk premium vector (π) for the copula model given in equation (3.3) as follows:

$$RA_\theta = \begin{bmatrix} \frac{\theta e^{u^\theta} u^{\theta-1}}{k_1} + \frac{(\theta-1)u^{\theta-1}\left(e^{v^\theta}-1\right)}{k_1[C_\theta(u,v)]^\theta} - \frac{(\theta-1)}{u} & \frac{v^{\theta-1}}{k_1}\left(\frac{\theta e^{u^\theta} e^{v^\theta}}{\left(e^{v^\theta}-1\right)} - \frac{(\theta-1)\left(e^{u^\theta}-1\right)}{[C_\theta(u,v)]^\theta}\right) \\ \frac{u^{\theta-1}}{k_1}\left(\frac{\theta e^{u^\theta} e^{v^\theta}}{\left(e^{u^\theta}-1\right)} - \frac{(\theta-1)\left(e^{v^\theta}-1\right)}{[C_\theta(u,v)]^\theta}\right) & \frac{e^{v^\theta} v^\theta\left(-\theta\left([C_\theta(u,v)]^\theta-1\right)-1\right)}{vk_1[C_\theta(u,v)]^\theta} + \frac{(\theta-1)k_1\left([C_\theta(u,v)]^\theta - v^\theta\right) - \theta e^{v^\theta} v^{2\theta}}{vk_1[C_\theta(u,v)]^\theta} \end{bmatrix}$$

$$\pi = \frac{1}{2k_1}\begin{bmatrix} \frac{\sigma_{11}}{u}\left(e^{u^\theta}u^\theta + (\theta-1)\left(\frac{u^\theta\left(e^{v^\theta}-1\right)}{[C_\theta(u,v)]^\theta} - k_1\right)\right) + \sigma_{21}v^{\theta-1}\left(\frac{(\theta-1)\left(e^{u^\theta}-1\right)}{[C_\theta(u,v)]^\theta} - \frac{\theta e^{u^\theta} e^{v^\theta}}{\left(e^{v^\theta}-1\right)}\right) \\ -\frac{\sigma_{22}}{v}\left(\frac{(\theta-1)\left(e^{u^\theta}-1\right)\left([C_\theta(u,v)]^\theta - v^\theta\right)}{[C_\theta(u,v)]^\theta} + e^{v^\theta}\left(\theta\left(1-v^\theta\right)-1\right)\right) + \sigma_{12}u^{\theta-1}\left(\frac{\theta e^{u^\theta} e^{v^\theta}}{\left(e^{u^\theta}-1\right)} - \frac{(\theta-1)\left(e^{v^\theta}-1\right)}{[C_\theta(u,v)]^\theta}\right) \end{bmatrix}$$

Here, $k_1 = e^{u^\theta} + e^{v^\theta} - 1$ abbreviations are used. Now, we examine risk aversion matrix and risk premium vector for two case by $\theta = 1$ and $\theta = 2$. In addition, we give the samples for independent ($\sigma_{12} = 0$, $\sigma_{21} = 0$), semi-dependent ($\sigma_{12} = 0.5$, $\sigma_{21} = 0.5$) and fully-dependent ($\sigma_{12} = 1$, $\sigma_{21} = 1$).

Case 1: Risk aversion matrix and risk premium vector for $\boldsymbol{\theta = 1}$

$$RA_{\theta=1} = \begin{bmatrix} e^u/(e^u+e^v-1) & -e^ue^v/(e^v-1)(e^u+e^v-1) \\ -e^ue^v/(e^u-1)(e^u+e^v-1) & e^v/(e^u+e^v-1) \end{bmatrix}$$

$$\pi_{\theta=1} = -\frac{1}{2}\begin{bmatrix} (e^u/(e^v-1))((\sigma_{21}e^v - \sigma_{11}(e^v-1))/(e^u+e^v-1)) \\ (e^v/(e^u-1))((\sigma_{12}e^u - \sigma_{22}(e^u-1))/(e^u+e^v-1)) \end{bmatrix}$$

TABLE 1. Risk premium vector values of $C_\theta(u,v)$ copula with $\theta=1$

$\theta=1$		$\sigma_{12}=0,\sigma_{21}=0$		$\sigma_{12}=.5,\sigma_{21}=.5$		$\sigma_{12}=1,\sigma_{21}=1$	
u	v	π_{11}	π_{21}	π_{11}	π_{21}	π_{11}	π_{21}
0.1	0.1	0.4566	0.4566	−1.9423	−1.9423	−4.3411	−4.3411
	0.3	0.3798	0.4639	−0.3529	−1.9733	−1.0855	−4.4105
	0.5	0.3151	0.4700	−0.0853	−1.9995	−0.4857	−4.4691
	0.7	0.2608	0.4752	0.0018	−2.0215	−0.2572	−4.5182
	0.9	0.2155	0.4795	0.0339	−2.0399	−0.1476	−4.5592
0.3	0.1	0.4639	0.3798	−1.9733	−0.3529	−4.4105	−1.0855
	0.3	0.3971	0.3971	−0.3689	−0.3689	−1.1350	−1.1350
	0.5	0.3377	0.4125	−0.0914	−0.3832	−0.5206	−1.1790
	0.7	0.2856	0.4260	0.0019	−0.3958	−0.2817	−1.2176
	0.9	0.2402	0.4377	0.0378	−0.4067	−0.1646	−1.2512
0.5	0.1	0.4700	0.3151	−1.9995	−0.0853	−4.4691	−0.4857
	0.3	0.4125	0.3377	−0.3832	−0.0914	−1.1790	−0.5206
	0.5	0.3588	0.3588	−0.0971	−0.0971	−0.5531	−0.5531
	0.7	0.3096	0.3782	0.0021	−0.1024	−0.3054	−0.5830
	0.9	0.2652	0.3956	0.0418	−0.1071	−0.1817	−0.6099
0.7	0.1	0.4752	0.2608	−2.0215	0.0018	−4.5182	−0.2572
	0.3	0.4260	0.2856	−0.3958	0.0019	−1.2176	−0.2817
	0.5	0.3782	0.3096	−0.1024	0.0021	−0.5830	−0.3054
	0.7	0.3326	0.3326	0.0023	0.0023	−0.3281	−0.3281
	0.9	0.2899	0.3541	0.0456	0.0024	−0.1986	−0.3493
0.9	0.1	0.4795	0.2155	−2.0399	0.0339	−4.5592	−0.1476
	0.3	0.4377	0.2402	−0.4067	0.0378	−1.2512	−0.1646
	0.5	0.3956	0.2652	−0.1071	0.0418	−0.6099	−0.1817
	0.7	0.3541	0.2899	0.0024	0.0456	−0.3493	−0.1986
	0.9	0.3138	0.3138	0.0494	0.0494	−0.2150	−0.2150

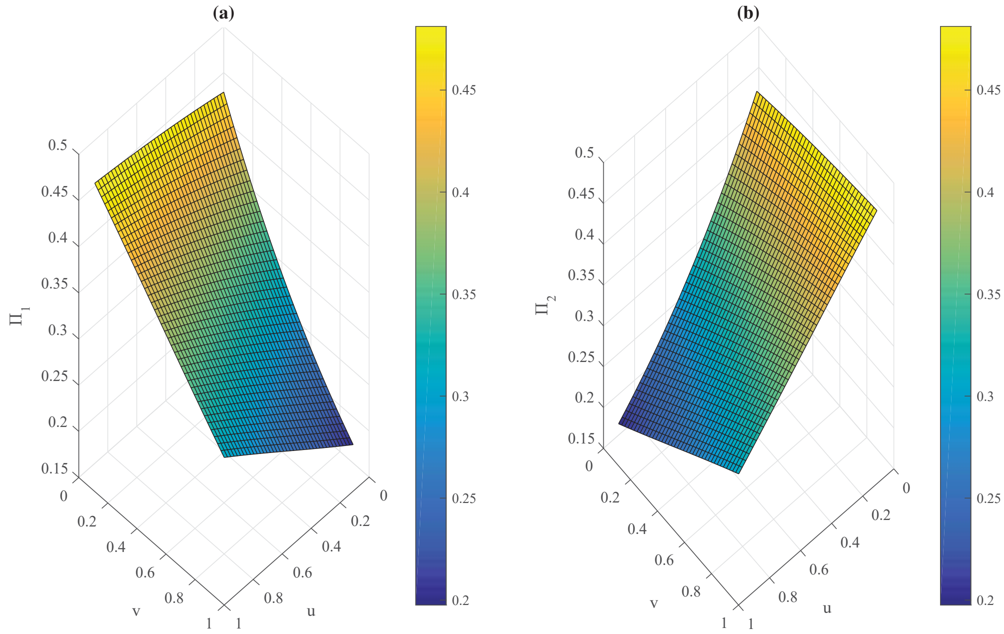

FIGURE 1. Risk premium graphics of $C_\theta(u,v)$ copula with $\theta = 1$ and $\sigma_{12} = 0, \sigma_{21} = 0$

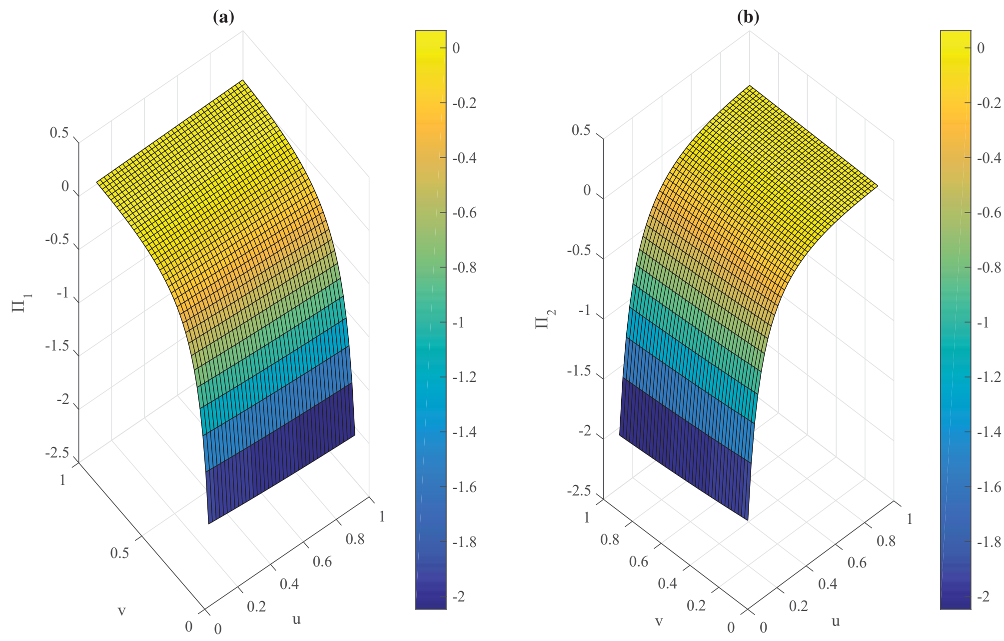

FIGURE 2. Risk premium graphics of $C_\theta(u,v)$ copula with $\theta = 1$ and $\sigma_{12} = .5, \sigma_{21} = .5$

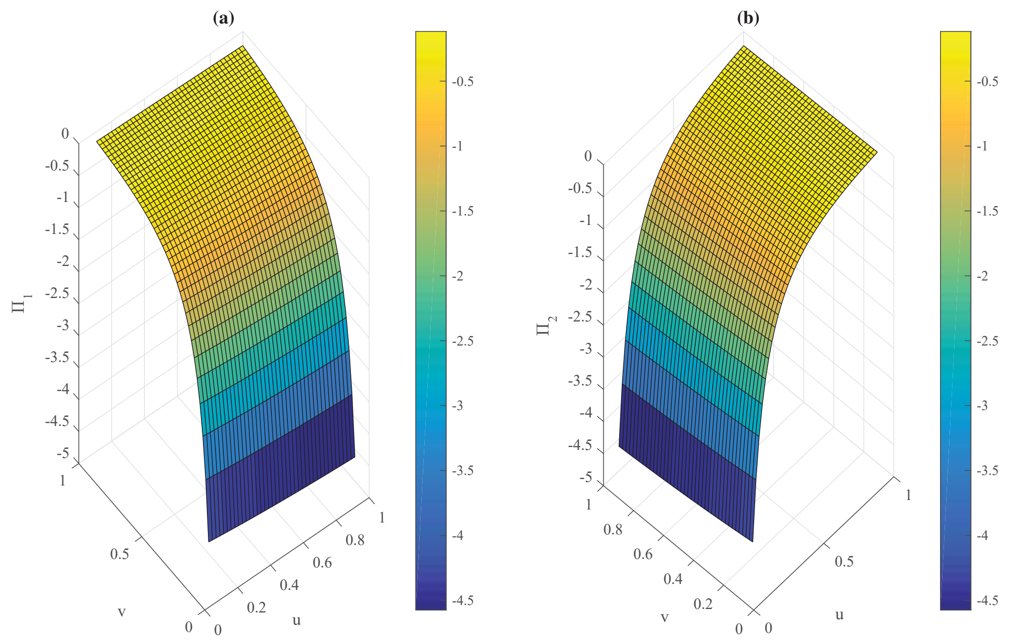

FIGURE 3. Risk premium graphics of $C_\theta(u,v)$ copula with $\theta=1$ and $\sigma_{12}=1,\sigma_{21}=1$

Case 2: Risk aversion matrix and risk premium vector for $\theta=2$

$$RA_{\theta=2}=\begin{bmatrix} \frac{1}{u}-\frac{2e^{u^2}u}{k_{1:\theta=2}}-\frac{u\left(e^{v^2}-1\right)}{k_{1:\theta=2}[C_2(u,v)]^2} & \frac{v}{k_{1:\theta=2}}\left(\frac{2e^{u^2}e^{v^2}}{\left(e^{v^2}-1\right)}-\frac{e^{u^2}-1}{[C_2(u,v)]^2}\right) \\ \frac{u}{k_{1:\theta=2}}\left(\frac{2e^{u^2}e^{v^2}}{\left(e^{u^2}-1\right)}-\frac{e^{v^2}-1}{[C_2(u,v)]^2}\right) & \frac{e^{v^2}\left(1-2\left([C_2(u,v)]^2-v^2\right)\right)-2e^{v^2}v^3}{k_{1:\theta=2}[C_2(u,v)]^2}+\frac{[C_2(u,v)]^2-v^2}{v[C_2(u,v)]^2} \end{bmatrix}$$

$$\pi_{\theta=2}=\frac{1}{2k_{1:\theta=2}}\begin{bmatrix} \frac{\sigma_{11}}{u}\left(e^{u^2}u^2+\left(\frac{u^2\left(e^{v^2}-1\right)}{[C_2(u,v)]^2}-k_{1:\theta=2}\right)\right)+\sigma_{21}v\left(\frac{e^{u^2}-1}{[C_2(u,v)]^2}-\frac{\theta e^{u^2}e^{v^2}}{\left(e^{v^2}-1\right)}\right) \\ -\frac{\sigma_{22}}{v}\left(\frac{\left(e^{u^2}-1\right)\left([C_2(u,v)]^2-v^2\right)}{[C_2(u,v)]^2}+e^{v^2}\left(2\left(1-v^2\right)-1\right)\right)+\sigma_{12}u\left(\frac{2e^{u^2}e^{v^2}}{\left(e^{u^2}-1\right)}-\frac{e^{v^2}-1}{[C_2(u,v)]^2}\right) \end{bmatrix}$$

Here $k_{1:\theta=2}=e^{u^2}+e^{v^2}-1$, this means that $\theta=2$ is written in the $k_1=e^{u^\theta}+e^{v^\theta}-1$.

Table 2. Risk premium vector values of $C_\theta(u,v)$ copula with $\theta=2$

$\theta=1$		$\sigma_{12}=0,\sigma_{21}=0$		$\sigma_{12}=.5,\sigma_{21}=.5$		$\sigma_{12}=1,\sigma_{21}=1$	
u	v	π_{11}	π_{21}	π_{11}	π_{21}	π_{11}	π_{21}
0.1	0.1	0.0743	0.0743	−2.4136	−2.4136	−4.9015	−4.9015
	0.3	0.0686	0.2241	−0.7288	−2.2666	−1.5262	−4.7574
	0.5	0.0586	0.3773	−0.3830	−2.1185	−0.8245	−4.6142
	0.7	0.0462	0.5360	−0.2322	−1.9659	−0.5106	−4.4679
	0.9	0.0336	0.7020	−0.1482	−1.8062	−0.3300	−4.3145
0.3	0.1	0.2241	0.0686	−2.2666	−0.7288	−4.7574	−1.5262
	0.3	0.2083	0.2083	−0.5970	−0.5970	−1.4023	−1.4023
	0.5	0.1797	0.3546	−0.2732	−0.4651	−0.7261	−1.2847
	0.7	0.1436	0.5105	−0.1475	−0.3271	−0.4386	−1.1648
	0.9	0.1059	0.6778	−0.0880	−0.1787	−0.2819	−1.0352
0.5	0.1	0.3773	0.0586	−2.1185	−0.3830	−4.6142	−0.8245
	0.3	0.3546	0.1797	−0.4651	−0.2732	−1.2847	−0.7261
	0.5	0.3121	0.3121	−0.1621	−0.1621	−0.6362	−0.6362
	0.7	0.2559	0.4610	−0.0600	−0.0412	−0.3759	−0.5435
	0.9	0.1940	0.6288	−0.0244	0.0956	−0.2428	−0.4376
0.7	0.1	0.5360	0.0462	−1.9659	−0.2322	−4.4679	−0.5106
	0.3	0.5105	0.1436	−0.3271	−0.1475	−1.1648	−0.4386
	0.5	0.4610	0.2559	−0.0412	−0.0600	−0.5435	−0.3759
	0.7	0.3913	0.3913	0.0406	0.0406	−0.3101	−0.3101
	0.9	0.3087	0.5549	0.0533	0.1628	−0.2021	−0.2292
0.9	0.1	0.7020	0.0336	−1.8062	−0.1482	−4.3145	−0.3300
	0.3	0.6778	0.1059	−0.1787	−0.0880	−1.0352	−0.2819
	0.5	0.6288	0.1940	0.0956	−0.0244	−0.4376	−0.2428
	0.7	0.5549	0.3087	0.1628	0.0533	−0.2292	−0.2021
	0.9	0.4592	0.4592	0.1559	0.1559	−0.1473	−0.1473

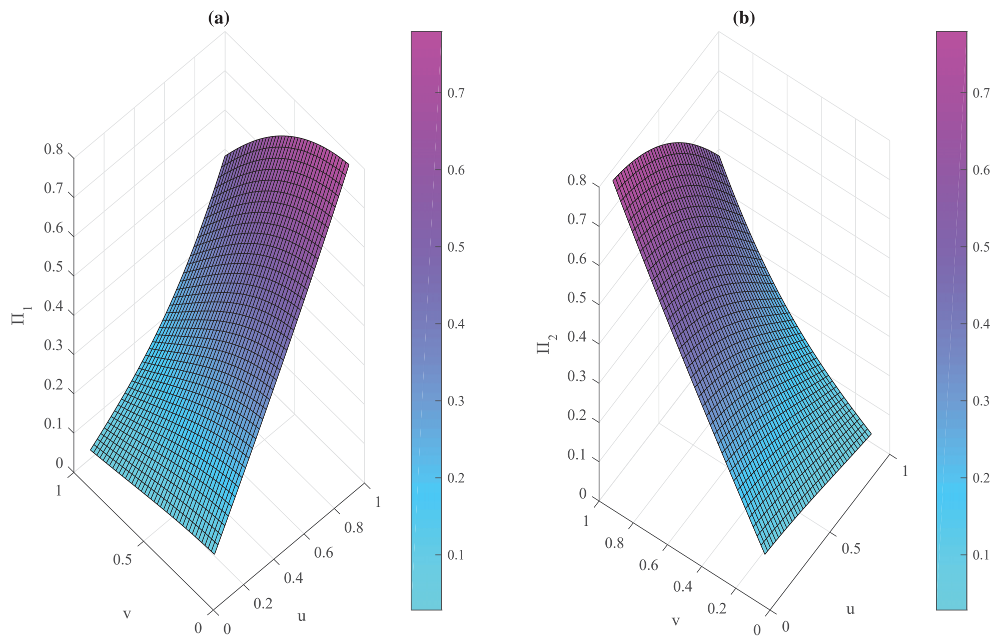

FIGURE 4. Risk premium graphics of $C_\theta(u,v)$ copula with $\theta = 2$ and $\sigma_{12} = 0, \sigma_{21} = 0$

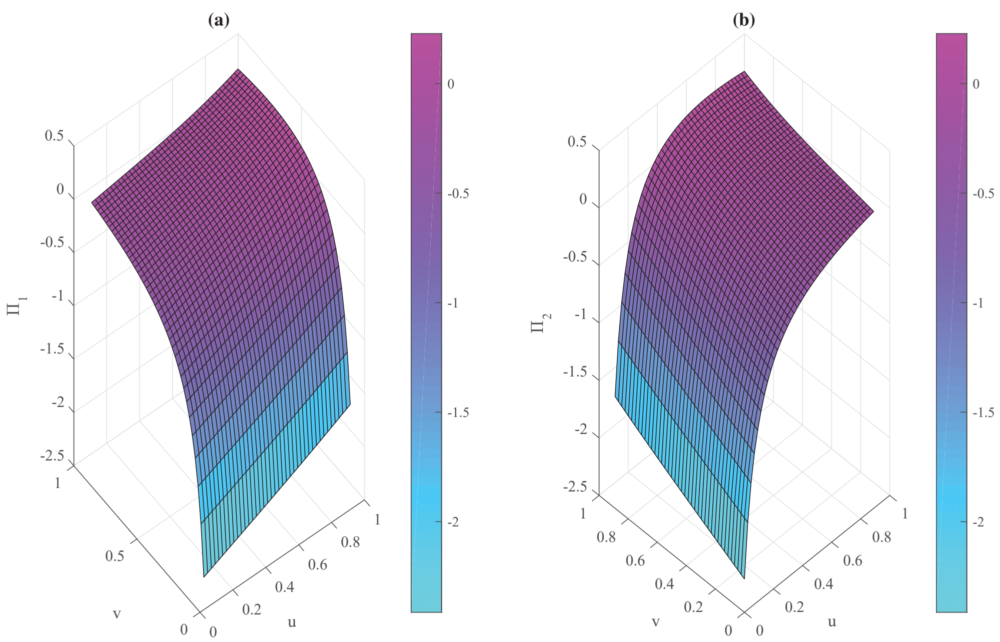

FIGURE 5. Risk premium graphics of $C_\theta(u,v)$ copula with $\theta = 2$ and $\sigma_{12} = .5, \sigma_{21} = .5$

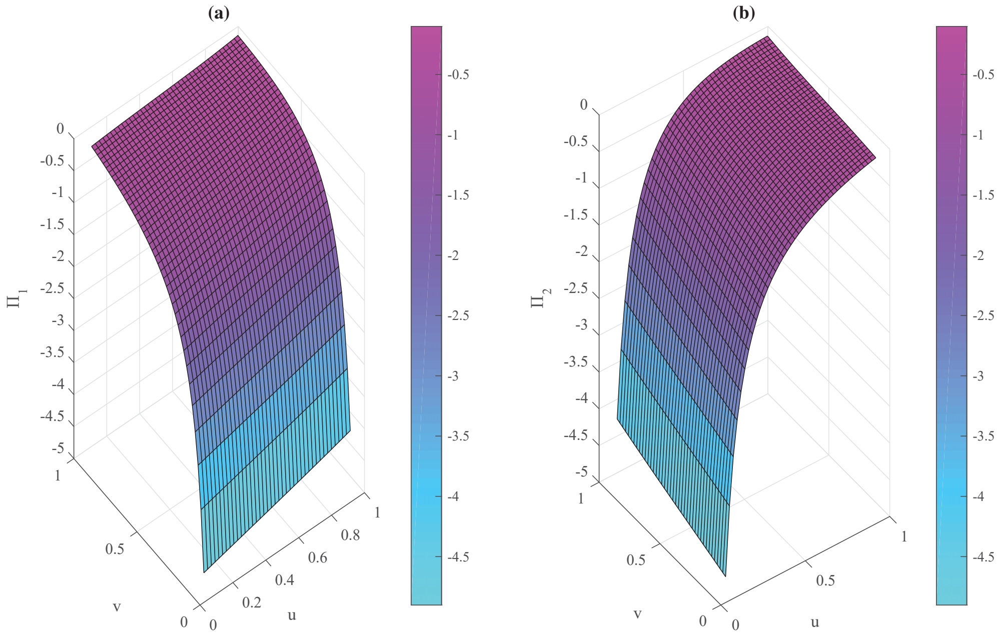

FIGURE 6. Risk premium graphics of $C_\theta(u,v)$ copula with $\theta=2$ and $\sigma_{12}=1, \sigma_{21}=1$

The calculated risk premium vector values for case 1 ($\theta=1$) and case 2 ($\theta=2$) are presented in Table 1 and Table 2, respectively.

In Table 1 and Table 2, the premium coefficients are greater than 0 for the two independent risks ($\sigma_{12}=0$, $\sigma_{21}=0$). This shows that the insurer has a risk-averse attitude towards independent risks.

In cases the risks are semi-dependent and fully dependent, while the dependence of the risks increases, there is a decrease in the premium coefficients. This situation shows that insurance of interdependent risks together will be more advantageous in terms of premium payments. On the other hand, as the value of the dependency parameter θ increases, in both cases a significant decrease in the coefficients of premium to be determined for the risks are observed. These results mean that as the risk dependency increases, the insurance company, who does not wish to take risks, wants to receive more premiums than the risk holders as much as the determined premium coefficient.

4. Conclusions

The selection of utility functions and copula models are important in risk management decisions. Accordingly, the bivariate risk aversion and risk premium can be considered as a decision-maker in terms of dependent risk groups. In cases the risks are dependent, the bivariate risk aversion functions created by the utility copula models also become important to decide under uncertainty.

In this study, it is emphasized to create the utility copula by using a known utility function and then to obtain the risk aversion matrix and risk premium vector by using the utility copula. For this purpose, risk premium coefficients were calculated for different dependency parameters of the utility copula, which was selected under the risk of dependent insurance. Accordingly, if an insurance company has dependent risks, it can determine the premiums to be received from the risks according to the obtained premium coefficients.

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References

[1] Abbas, A.E. (2009). Multiattribute utility copulas. *Operations Research,* 57(6), 1367-1383.

[2] Abbas, A.E. (2010). General decompositions of multiattribute utility functions with partial utility independence. *Journal of Multi-Criteria Decision Analysis J. Multi-Crit. Decis. Anal.,* 17, 37–59.

[3] Abbas, A.E. (2013). Utility copula functions matching all boundary assessments. *Operations Research,* 61(2), 359-371.

[4] Abbas, A.E. (2014). Constructing multiattribute utility functions for decision analysis. *In INFORMS Tutorials in Operations Research,* 62-98.

[5] Arrow, K. J. (1965). Aspects of the theory of risk-bearing (Yrjo Jahnsson Lectures). Yrjo Jahnssonin Saatio, Helsinki.

[6] Arrow, K.J. (1971). Essays in the Theory of Risk Bearing, Chicago: Markham.

[7] Cardin, M. and Ferretti, P. (2004). *Bivariate Risk Aversion and Concordance Aversion: Similarities and Differences.* Working paper, Department of Applied Mathematics, University Ca'Foscari, Venice, Italy.

[8] Cipu, C. and Gheorghe, C. (2015). Some applications in economy for utility functions involving risk theory. *Procedia Economics and Finance,* 22, 595-600.

[9] Cherubini, U., Luciano, E. and Vecchiato, W. (2004). *Copula Methods in Finance.* John Wiley & Sons Ltd, England.

[10] Courbage, C. and Rey, B. (2007). Precautionary saving in the presence of other risks. *Economic Theory,* 32, 417–424.

[11] Denuit, M. et al. (2006).*Actuarial theory for dependent risks: measures, orders and models.* John Wiley & Sons.

[12] Denuit, M.M., Eeckhoudt, L. and Menegatti, M. (2011). Correlated risks, bivariate utility and optimal choices. *Econ Theory,* 46, 39–54.

[13] Duncan, G.T. (1977). A matrix measure of multivariate local risk aversion. *Econometrica: Journal of the Econometric Society,* 895-903.

[14] Eeckhoudt, L., Rey B. and Schlesinger H. (2007).*A good sign for multivariate risk taking.* Management Science, 53(1), 117-124.

[15] Goovaerts, M., Linders D., Van Weert K. and Tank F. (2012). On the in-terplay between distortion, mean value and Haezendonck-Goovaerts risk measures. *Insurance: Mathematics and Economics,* 51(1):10–18.

[16] Kettler, P. C. (2007). Utility Copulas. Preprint series. Pure mathematics http://urn. nb. no/URN: NBN: no-8076.

[17] Kizilok Kara, E. and Gebizlioglu, O.L. (2014). Measurement of bivariate risks by the north-south quantile points approach. *Journal of Computational and Applied Mathematics,* 255, 208-215.

[18] Kizilok Kara, E. and Acik Kemaloglu, S. (2016). Portfolio Optimization Of Dynamic Copula Models For Dependent Financial Data Using Change Point Approach. *Communications Faculty of Sciences University of Ankara-Series A1 Mathematics and Statistics* 64(2), 175-188.

[19] Lai, L.H. (2015). Statistical premium in correlated losses of insurance. *Economic Modelling,* 49, 248.253.

[20] Li, J., Liu, D. and Wang J. (2016). Risk aversion with two risks: A theoretical extension. *Journal of Mathematical Economics*, 63, 100–105.

[21] Nelsen, R.B. (2006). *An Introduction to Copulas.* Springer Science & Business Media, New York.

[22] Pratt, J.W. (1964). Risk aversion in the small and in the large. *Econometrica: Journal of the Econometric Society,* 122-136.

[23] Sengupta, J.K. (1983). Multivariate risk aversion with applications. *Mathematical Modelling,* 4(4), 307-322.

[24] Sklar, A. (1959). Functions de repartition an dimensions et leurs marges. *Publ. Inst. Statist. Univ. Paris,* 8, 229-231.

[25] Tank, F., Gebizlioglu, O.L. (2004). Sarmanov distribution class for dependent risks and its applications. *Belgian Actuarial Bulletin*, 4(1), 50–52.

[26] Tasdemir, M. (2007). Belirsizlik altında tercihler ve beklenen fayda modelinin yetersizlikleri.

[27] Tekin, B. (2016). Traditional finance-behavioral finance distinction in the context of expected utility and prospect theories. *Journal of Accounting, Finance and Auditing Studies*, 2(4), 75-107.

[28] Thomas, P.J. (2016). Measuring risk-aversion, The challenge. *Measurement,* 79, 285–301.

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DISCRETE TIME SHOCK MODEL WITH VARYING SUCCESS PROBABILITY

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Abstract: Suppose a system fails if the time between two consecutive shocks falls below a fixed threshold $\delta \in \mathbb{N}$ and the lifetime of the system is measured as the time to the occurrence of this event. In this paper, we consider the interarrival times between $(i-1)$-th and i-th successive shocks follow a geometric distribution with mean $1/p_i$,where $p_i = \theta p^{i-1}$, $i = 1, 2, \ldots,$ $0 < \theta < 1$, $0 < p \leq 1$. Under the above considerations, the distribution of system lifetime is obtained. Probability generating function and than also moments of system are derived. The proportion estimates of distribution parameters are studied. A numerical example is also presented by using real data.

Key words: $q-$distributions; $\delta-$Shock model; Probability; Generating function
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1. Introduction

Shock models have aroused great interest in reliability theory [1]-[9]. Shock models are systems that experience shocks of random magnitudes at random times. There are three modes of shock models which are run shock model, extreme shock model and cumulative shock model. In an run shock model, the amplitudes of a specified number of consecutive shocks are considered a failure criterion. See, e.g., [3]. For extreme shock and cumulative shock model please see [1]-[3].

Let us consider, a system collapses when the time between two consecutive shocks falls below a fixed threshold δ. Furthermore, the system's lifetime is measured as the time to the occurrence of that event. Such systems called as δ-shock model. Since the δ-shock model take into account the time between two consecutive shocks instead of magnitudes, it can be considered as a forth mode in shock models. δ-shock models have been studied by [6]-[9].

Recently, Eryilmaz [9] studied the discrete time release of the δ-shock model. In this model, he assumed that the shocks occur according to a binomial process at all times and the interarrival times between successive shocks have a geometric distribution with mean $1/p$.

In this paper, we assume that the interarrival times between $(i-1)$-th and i-th successive shocks follow a geometric distribution with mean $1/q_i$,where $q_i = 1-\theta q^{i-1}$, $i = 1, 2, \ldots, 0 < \theta < 1, 0 < q \leq 1$. Studying such a geometric model in the context of this delta-shock model can be motivated as follows: Consider a unit that is subject to a sequence of shocks. Assume that the unit degrades

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